ELEMENTS OF ISOANALYSIS ON $\widehat{\mathfrak{R}}^n$ ISOSTRUCTURES ON $\widehat{\mathfrak{R}}^n$

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Abstract

Let $\widehat{\mathfrak{R}}^n$ be a vector space on $\mathfrak{R}$.From this space we form a new vector space $\widehat{\mathfrak{R}}^n$ with new structures,that is: isovectorial, isotopological, isoaffine and isometric.

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1 Basic Definitions

We consider the isofield $\left(\widehat{\mathfrak{R}}, +, *\right)$ and we form a new isoplane $\left(\widehat{\mathfrak{R}}^2, +, *\right)$ as follows:

$$\widehat{\mathfrak{R}}^2 = \widehat{\mathfrak{R}} x \widehat{\mathfrak{R}} = \left\{ (\widehat{a}_1, \widehat{a}_2) / \widehat{a}_i = a_i \widehat{I}, a_i \in \mathfrak{R}, i = 1, 2 \right\} \tag{1}$$

This new isoplane $\widehat{\mathfrak{R}}^2$ is called Cartesian isoplane.

Generally, the isoset:

$$\widehat{\mathfrak{R}}^n = \widehat{\mathfrak{R}} x \dots x \widehat{\mathfrak{R}} = \{(\widehat{a}_1, \widehat{a}_2, \dots, \widehat{a}_n)/\widehat{a}_i = a_i \widehat{I}, a_i \in \mathfrak{R}, i = 1, 2, \dots, n \text{ and } \widehat{I} \text{ the isounit}\} \tag{2}$$

where $\widehat{\mathfrak{R}} x \dots x \widehat{\mathfrak{R}}$ is take by n times, is the Cartesian product of isofield $\widehat{\mathfrak{R}}$ n times and it is called real Cartesian isospace. On this isospace we define the follows isostructures:

2 Isovector and Isoaffine Spaces

From the vector space $V^n(\mathfrak{R})$, by means of the isofield $\widehat{\mathfrak{R}}$, we obtain an isovector space $V^n(\widehat{\mathfrak{R}})$, which is called real Cartesian isovector space.

The vectors:

$$e_1 = (1, 0, \dots, 0), e_2 = (0, 1, \dots, 0), \dots, e_n = (0, 0, \dots, 1) \tag{3}$$

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that constitute the canonical base of $V^n(\mathfrak{R})$, are a base of $V^n(\widehat{\mathfrak{R}})$, because every isovector $v \in V^n$ can be written:

$$v = \lambda_1 e_1 + \lambda_2 e_2 + \ldots + \lambda_n e_n,\ \lambda_i \in \widehat{\mathfrak{R}},\ i = 1, \ldots, n \tag{4}$$

We can associate to the isovector space $V^n(\widehat{\mathfrak{R}})$ the Affine space $A^n(\widehat{\mathfrak{R}})$ which as a set identified with $\widehat{\mathfrak{R}}^n$. Then $A^n(\widehat{\mathfrak{R}})$ is called real Cartesian isoaffine space of dimension n.

Let:

$$P_0 = (0, 0, \ldots, 0), P_1 = (0, 1, \ldots, 0), \ldots, P_n = (0, 0, \ldots, 1) \tag{5}$$

be $n + 1$ points of $A^n(\widehat{\mathfrak{R}})$. These points form an Affine base $A^n(\widehat{\mathfrak{R}})$, because the vectors:

$$P_0P_1,\ P_0P_2,\ \ldots,\ P_0P_n \tag{6}$$

constitute of a base of $V^n(\widehat{\mathfrak{R}})$, which is called fundamental Affine base of $V^n(\widehat{\mathfrak{R}})$.

If P is a point of $A^n(\widehat{\mathfrak{R}})$, then $P_0P \in V^n(\widehat{\mathfrak{R}})$ can be written:

$$P_0P = \beta_1 P_0P_1 + \beta_2 P_0P_2 + \ldots + \beta_n P_0P_n \tag{7}$$

where $\beta_i \in \widehat{\mathfrak{R}}$ are called isoaffine coordinates of point P in connection with fundamental Affine base of $V^n(\widehat{\mathfrak{R}})$.

On the Cartesian isoaffine space $A^n(\widehat{\mathfrak{R}})$ we consider the functions $\widehat{x}_i$, which are defined as follows:

$$\widehat{x}_i : A^n(\widehat{\mathfrak{R}}) \to \widehat{\mathfrak{R}} \tag{8}$$

$$\widehat{x}_i : P = (\widehat{p}_1, \widehat{p}_2, \ldots, \widehat{p}_n) \mapsto \widehat{x}_i(P) = p_i, i = 1, \ldots, n \tag{9}$$

The above functions $\widehat{x}_i$are calleed natural isoaffine coordinates.

3 Isotopological Space

On the set $\mathfrak{R}^n$ we consider a topology T:

$$T = \{\varnothing, \mathfrak{R}^n, \underset{i \in I}{\cup} B_i\} \tag{10}$$

where $B_i$is subset of $\mathfrak{R}^n$ defined as follows:

$$B_i = \{P = (p_1, p_2, \ldots, p_n)/a_i < p_i < \beta_i, a_i, \beta_i \in \mathfrak{R}, i = 1, \ldots, n\} \tag{11}$$

and it is called open rectangle of $\mathfrak{R}^n$.

The topology T can be considered as the Cartesian product of topology of the open intervals of straight line n times. Topological space $(\mathfrak{R}^n, T)$ is symbolized $T^n(\mathfrak{R})$ and is called real Cartesian topological space.

We consider the same topology on the set $\widehat{\mathfrak{R}}^n$, which coincides with set $\mathfrak{R}^n$. The set $\widehat{\mathfrak{R}}^n = \mathfrak{R}^n$ with topology T is called real Cartesian isotopological space and is symbolized $T^n(\widehat{\mathfrak{R}})$. It's evident that $T^n(\mathfrak{R}) = T^n(\widehat{\mathfrak{R}})$.

Let $\widehat{\mathfrak{R}}^n = \{(\widehat{x}_1, \widehat{x}_2, \dots, \widehat{x}_n)/\widehat{x}_i = \widehat{\mathfrak{R}}, i = 1, 2, \dots, n\}$be a real isocartesian space of dimension n. On this we obtain the previous structures, that is, the isovector structure, the isoaffine structure and the isotopological structure. We consider an isomapping, that is a mapping between two isosets:

$$f : \widehat{\mathfrak{R}}^n \to \widehat{\mathfrak{R}}^n,\ f : P \mapsto f(P) = P,\ \forall P \in \widehat{\mathfrak{R}}^n \tag{12}$$

which is the identity mapping on $\widehat{\mathfrak{R}}^n$. The set $\widehat{\mathfrak{R}}^n$ with three above structures and isomapping f is called Cartesian isomanifold of dimension n.

The isosubset $\widehat{U}$ of $\widehat{\mathfrak{R}}^n$defined as follows:

$$\widehat{U} = \{P = (\widehat{p}_1, \widehat{p}_2, \dots, \widehat{p}_n)/a_i < p_i < \beta_i, a_i, \beta_i \in \mathfrak{R}, i = 1, 2, \dots, n\} \tag{13}$$

is called open rectangle of $\widehat{\mathfrak{R}}^n$.

The isosubset:

$$\left[\widehat{a}_1, \widehat{\beta}_1\right] x \left[\widehat{a}_2, \widehat{\beta}_2\right] x \dots x \left[\widehat{a}_n, \widehat{\beta}_n\right] \tag{14}$$

of $\widehat{\mathfrak{R}}^n$, where $a_i,\ \beta_i \in \mathfrak{R},\ i = 1, \dots, n$ is called closet rectangle of $\widehat{\mathfrak{R}}^n$.

The isosubset $\widehat{U}$ of $\widehat{\mathfrak{R}}^n$ is called open, if and only if, for every point $P \in \widehat{U}$ there is an open rectangle $\widehat{A}$ of $\widehat{\mathfrak{R}}^n$, such that $P \in \widehat{A} \subset \widehat{U}$.

The isosubset $\widehat{U}$ of $\widehat{\mathfrak{R}}^n$is called closed, if and only if:

$$\widehat{\mathfrak{R}}^n - \widehat{U} = \widehat{U}^c \tag{15}$$

is open.

If $\widehat{A}$ is an isosubset of $\widehat{\mathfrak{R}}^n$, then a point $P \in \widehat{\mathfrak{R}}^n$is called interior point of $\widehat{A}$, if and only if, P belong to open isoset $\widehat{B}$ such that $\widehat{B} \subset \widehat{A}$.

The isoset of interior points of $\widehat{A}$ symbolized $\overset{\circ}{\widehat{A}}$ or $Int(\widehat{A}) = \widehat{Int}(A)$, is called interior of $\widehat{A}$.

Exterior of isosubset $\widehat{A}$ of $\widehat{\mathfrak{R}}^n$, written $Ext(\widehat{A})$, is the interior of the complement of $\widehat{A}$, that is:

$$Ext(\widehat{A}) = \widehat{Ext}(A) = Int(\widehat{A}^c) = \widehat{Int}(A^c) \tag{16}$$

The isoset of points which do not belong to the interior or the exterior of $\widehat{A} \subseteq \widehat{\mathfrak{R}}^n$ is called boundary of $\widehat{A}$, written $\partial\widehat{A} = \widehat{\partial}A$.

A collection $\widehat{O} = \{\widehat{A}_i\}_{i \in I}$ of open isosubsets of $\widehat{\mathfrak{R}}^n$, is called open cover or, simply cover of $\widehat{U}$, where $\widehat{U} \subseteq \widehat{\mathfrak{R}}^n$, if and only if, $\widehat{U} = \underset{i \in I}{\cup}\ \widehat{A}_i$, or equivalently, if and only if:

$$(\forall P \in \widehat{U})(\exists \widehat{A}_i \in \widehat{O} = \{\widehat{A}_i\}_{i \in I})[P \in \widehat{A}_i] \tag{17}$$

The isosubset $\widehat{U}$of $\widehat{\mathfrak{R}}^n$ is called compact, if and if, each open cover $\widehat{O}$ of $\widehat{U}$ contains a finite subcollection of open isosets, which constitutes a cover of $\widehat{U}$.

If $\widehat{A}$ is an isosubset of isotopological space $\widehat{T}$, then intersection of closed isosubsets, which contain the isosubset $\widehat{A}$, is called closure of $\widehat{A}$, written $\widehat{A}$.

4 Isometric Space

On the isoset $\widehat{\mathfrak{R}}^n$we consider isofunction $\widehat{d}$:

$$\widehat{d}:\widehat{\mathfrak{R}}^n x \widehat{\mathfrak{R}}^n \to \mathfrak{R}_+ \tag{18}$$

$$\widehat{d}:(X=(\widehat{x}_1,\ldots,\widehat{x}_n)),\ Y=(\widehat{y}_1,\ldots,\widehat{y}_n))\ \mapsto \widehat{d}(X,Y) \stackrel{def}{=} \tag{19}$$
$$\stackrel{def}{=} T\sqrt{(\widehat{x}_1-\widehat{y}_1)^2+\ldots+(\widehat{x}_n-\widehat{y}_n)^2}=\sqrt{(x_1-y_1)^2+\ldots+(x_n-y_n)^2}$$

Isofunction $\widehat{d}$, following properties:

$$i)\ \widehat{d}(X,Y)>0 \Leftrightarrow X\neq Y \tag{20}$$

$$ii)\ \widehat{d}(X,Y)=0 \Leftrightarrow X=Y \tag{21}$$

$$iii)\ \widehat{d}(x,Y)=\widehat{d}(Y,X) \tag{22}$$

$$iv)\ \widehat{d}(X,Y)\preceq \widehat{d}(X,Z)+\widehat{d}(Z,Y),\ \forall X,Y,Z\in\widehat{\mathfrak{R}}^n \tag{23}$$

is called isodistance and $\widehat{\mathfrak{R}}^n$ with isodistance $\widehat{d}$ is called Cartesian real metric isospace.

The isosubset $\widehat{S}(P_0,a)$ of $\widehat{\mathfrak{R}}^n$:

$$\widehat{S}(P_0,a)=\{P=(\widehat{x}_1,\ldots,\widehat{x}_n)\in\widehat{\mathfrak{R}}^n/\widehat{d}(P,P_0)<a,a\in\mathfrak{R}_+\} \tag{24}$$

is called open isosphere with center $P_0=(p_1^0,\ldots,p_n^0)$ and radius a.

5 Isorthogonal Space

On $\widehat{\mathfrak{R}}^n$, equiped with isovector structure, we consider an isoinner product defined as follows:

$$<\hat{,}>:\widehat{\mathfrak{R}}^n x \widehat{\mathfrak{R}}^n \to \mathfrak{R} \tag{25}$$

$$<\hat{,}>:(X=(\widehat{x}_1,\ldots,\widehat{x}_n),Y=(\widehat{y}_1,\ldots,\widehat{y}_n))\mapsto<X,Y>\stackrel{def}{=} \tag{26}$$
$$\stackrel{def}{=} T(\widehat{x}_1\widehat{y}_1+\ldots+\widehat{x}_n\widehat{y}_n)=x_1y_1+\ldots+x_ny_n$$

In this case, the attached real Cartesian Affine space $A^n(\widehat{\mathfrak{R}})$ is called isoeuclidean real orthogonal isospace of dimension n and it is symbolized $\widehat{O}^n$.

If $X=Y$, then we have:

$$<X\ \hat{,}\ Y>=x_1^2+\ldots+x_n^2 \tag{27}$$

Therefore, isoinner product on $\widehat{\mathfrak{R}}^n$ defines an isonorm $\|\hat{}\|$ on $\widehat{\mathfrak{R}}^n$, that is:

$$\|\hat{}\|:\widehat{\mathfrak{R}}^n\to\mathfrak{R}_+ \tag{28}$$

$$\|\hat{}\| : X = (\widehat{x}_1, \ldots, \widehat{x}_n) \longmapsto \|\hat{}\| \overset{def}{=} \sqrt{< X \hat{;} X >} = \sqrt{x_1^2 + \ldots + x_n^2} = |X| \tag{29}$$

We accept that $\widehat{\mathfrak{R}}^n$ has the isovector structure, the isoaffine structure and the isoinner product $<\hat{,}>$, from which it is clear the isometric structure on $\widehat{\mathfrak{R}}^n$. On this $\widehat{\mathfrak{R}}^n$ we will study isofunctions of several variables.

The points $Q_1,\ Q_2,\ \ldots,\ Q_n$ of isoeuclidean orthogonal real isospace $\widehat{O}^n = A^n(\widehat{\mathfrak{R}})$ constitute an orthogonal base of $\widehat{O}^n$, if and only if, the vectors $Q_0Q_1,\ Q_0Q_2,\ \ldots,\ Q_0Q_n$ form an orthogonal base of $V^n(\widehat{\mathfrak{R}})$.

Fundamental affine base $\{P_0, P_1, \ldots, P_n\}$ of $A^n(\widehat{\mathfrak{R}}^n)$ is an orthogonal base of $\widehat{O}^n$.

6 Isoeuclidean Manifold

On the isoset $\widehat{\mathfrak{R}}^n$, equipped with all the above isostructures, we consider isomapping f:

$$f : \widehat{\mathfrak{R}}^n \to \widehat{\mathfrak{R}}^n \tag{30}$$

$$f : P = (\widehat{p}_1, \ldots, \widehat{p}_n) \longmapsto f(P) = P, \forall P \in \widehat{\mathfrak{R}}^n \tag{31}$$

The pair $\left(\widehat{\mathfrak{R}}^n, f\right)$is called linear isomanifold of dimension n or Euclidean real isomanifold of dimension n. In short the $\left(\widehat{\mathfrak{R}}^n, f\right)$ we note $\widehat{\mathfrak{R}}^n$.

Let $\widehat{\mathfrak{R}}^n$ be the Euclidean real isomanifold its point $P = (\widehat{p}_1, \ldots, \widehat{p}_n)$ we consider the following special functions $\widehat{x}_1, \ldots, \widehat{x}_n$ on $\widehat{\mathfrak{R}}^n$, which are defined as follows:

$$\begin{aligned} \widehat{x}_1 &: \widehat{\mathfrak{R}}^n \to \mathfrak{R}, \\ \widehat{x}_1 &: P = (\widehat{p}_1, \ldots, \widehat{p}_n) \longmapsto \widehat{x}_1(P) = \widehat{x}_1(\widehat{p}_1, \ldots, \widehat{p}_n) \overset{def}{=} x_1(p_1, \ldots, p_n) = p_1 \end{aligned} \tag{32}$$

$$\begin{aligned} \widehat{x}_n &: \widehat{\mathfrak{R}}^n \to \mathfrak{R}, \\ \widehat{x}_n &: P = (\widehat{p}_1, \ldots, \widehat{p}_n) \longmapsto \widehat{x}_n(P) = \widehat{x}_n(\widehat{p}_1, \ldots, \widehat{p}_n) \overset{def}{=} x_n(p_1, \ldots, p_n) = p_n \end{aligned} \tag{33}$$

The above functions $\widehat{x}_1, \ldots, \widehat{x}_n$ are called natural cordinates isofunctions on $\widehat{\mathfrak{R}}^n$, or simply, cordinates isofunctions on $\widehat{\mathfrak{R}}^n$.

7 Some Basic Sets of $\widehat{\mathfrak{R}}^n$

Let $\widehat{\mathfrak{R}}^n$ be the Euclidean real isomanifold and its point $P_0 = (p_1^0, \ldots, p_n^0)$.

Its open isosubset $\widehat{\Delta}\ (P_0, a)$, or $\widehat{S}_d$, or $\widehat{S}(P_0, a)$ defined as follows:

$$\widehat{\Delta}(P_0, a) = \{(\widehat{x}_1, \ldots, \widehat{x}_n) \in \widehat{\mathfrak{R}}^n / (\widehat{x}_1 - \widehat{p}_1^0)^2 + \ldots + (\widehat{x}_n - \widehat{p}_n^0)^2 < \widehat{a}^2, a \in \mathfrak{R}_+\} \tag{34}$$

is called open isosphere with center $P_0 = (\widehat{p}_1^0, \ldots, \widehat{p}_n^0)$ and radius a.

The isosubset $\bar{\widehat{\Delta}}\ (P_0, a)$ of $\widehat{\mathfrak{R}}^n$ defined as follows:

$$\bar{\widehat{\Delta}}\ (P_0, a) = \{(\widehat{x}_1, \ldots, \widehat{x}_n) \in \mathfrak{R}^n / (\widehat{x}_1 - \widehat{p}_1^0)^2 + \ldots + (\widehat{x}_n - \widehat{p}_n^0)^2 < \widehat{a}^2, a \in \mathfrak{R}_+\} \quad (35)$$

is called closed isosphere with center $P_0 = (\widehat{p}_1^0, \ldots, \widehat{p}_n^0)$ and radius a.

The isosubset $\widehat{S}^{n-1}$ of $\widehat{\mathfrak{R}}^n$ defined as follows:

$$\begin{aligned} \widehat{S}^{n-1} &= \bar{\widehat{\Delta}}\ (P_0, a) - \widehat{\Delta}(P_0, a) = \\ &= \{(\widehat{x}_1, \ldots, \widehat{x}_n) \in \widehat{\mathfrak{R}}^n / (\widehat{x}_1 - \widehat{p}_1^0)^2 + \ldots + (\widehat{x}_n - \widehat{p}_n^0)^2 = \widehat{a}^2, a \in \mathfrak{R}_+\} \end{aligned} \quad (36)$$

is called isosphere of dimension $n - 1$ with center $P_0 = (\widehat{p}_1^0, \ldots, \widehat{p}_n^0)$ and radius a.

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