

A FAMILY OF METRICS WITH STRICTLY POSITIVE SECTIONAL CURVATURE ON A RIEMANNIAN MANIFOLD

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Abstract

The aim of present paper is to find a family of metrics with strictly positive sectional curvature on the Riemannian manifold $N_1^\kappa \times N_2^\lambda$, where N_1, N_2 are special subsets of \mathbb{R}^2 .

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1 Introduction

The paper includes four sections which are organised as follows. The first section is the introduction. The second section includes a study of some special Riemannian metrics. In the third section we solve some special partial differential equations. The last section contains a study of the sectional curvature.

2 Study of special metrics

Let \mathbb{R}_1^2 be a Euclidian plane with an orthogonal coordinate system (u_1, u_2) which as we already know is covered with an atlas with only one chart. Let us also consider a subset $N_1 \subset \mathbb{R}_1^2$ which is defined as follows:

$$N_1 = \{(u_1, u_2) \in \mathbb{R}_1^2, 0 < u_1 < \infty, -\infty < u_2 < \infty\}. \quad (1)$$

We consider a Riemannian metric on N_1 which is defined as below

$$\omega_1 = \{\omega_{11} = 1, \omega_{12} = \omega_{21} = 0, \omega_{22} = u_1\}. \quad (2)$$

The sectional curvature is given by the formula:

$$K_P(\lambda) = \frac{-R_{ijkl}X^iY^jX^kY^l}{(\omega_{ik}\omega_{jl} - \omega_{il}\omega_{jk})X^iY^jX^kY^l}, i, j, k, l = 1, 2, \tag{3}$$

where $P \in N_1$ and λ is a plane of $T_P(N_1)$.

The components of curvature tensor are given by:

$$R_{ijkl} = \frac{1}{2} \left\{ \frac{\partial^2 \omega_{ik}}{\partial u_j \partial u_l} + \frac{\partial^2 \omega_{jl}}{\partial u_i \partial u_k} - \frac{\partial^2 \omega_{jk}}{\partial u_i \partial u_l} - \frac{\partial^2 \omega_{il}}{\partial u_j \partial u_k} \right\} - \omega_{rs} \{ \Gamma_{jk}^r \Gamma_{il}^s - \Gamma_{jl}^r \Gamma_{ik}^s \}. \tag{4}$$

Using the above formula we have:

$$R_{2121} = R_{1212} = -\frac{1}{4u_1} \text{ and } R_{2112} = R_{1221} = \frac{1}{4u_1} \tag{5}$$

and all the rest components of curvature tensor are zero.

In the sequence we get:

$$K_P(\lambda) = \frac{1}{4u_1^2} > 0, \forall P \in N_1, \forall \lambda \in T_P(N_1). \tag{6}$$

Hence (N_1, ω_1) has strictly positive sectional curvature. We have similar results for the subset:

$$N_1 = \{(u_3, u_4) \in \mathbb{R}_2^2, 0 < u_3 < \infty, -\infty < u_4 < \infty\} \tag{7}$$

of another Euclidian plane \mathbb{R}_2^2 with orthogonal coordinate system (u_3, u_4) and metric:

$$\omega_2 = \{\omega_{33} = 1, \omega_{34} = \omega_{43} = 0, \omega_{44} = u_3\}. \tag{8}$$

Consequently we obtain the following:

Proposition 1 *The Riemannian manifolds (N_1, ω_1) and (N_2, ω_2) have strictly positive sectional curvature.*

We now consider the Riemannian manifolds $\mathbb{R}_1^{2\kappa}$ with orthogonal coordinate system $\{u_1, u_2, \dots, u_{2\kappa}\}$ and $\mathbb{R}_2^{2\lambda}$ with orthogonal coordinate system $\{u_{2\kappa+1}, u_{2\kappa+2}, \dots, u_{2\kappa+2\lambda}\}$, $\kappa, \lambda \in \mathbb{N}^*$. Each of them is covered with an atlas with only one chart. Let us consider their subsets N_1^κ, N_2^λ , which are defined as follows:

$$N_2^\kappa = \{(u_1, u_2, \dots, u_{2\kappa}) \in \mathbb{R}_1^{2\kappa}, 0 < u_1, u_3, \dots, u_{2\kappa-1} < \infty, -\infty < u_2, u_4, \dots, u_{2\kappa} < \infty\}, \tag{9}$$

$$N_1^\lambda = \{(u_{2\kappa+1}, u_{2\kappa+2}, \dots, u_{2\kappa+2\lambda}) \in \mathbb{R}_2^{2\lambda}, 0 < u_{2\kappa+1}, u_{2\kappa+3}, \dots, u_{2\kappa+2\lambda-1} < \infty, -\infty < u_{2\kappa+2}, u_{2\kappa+4}, \dots, u_{2\kappa+2\lambda} < \infty\}. \tag{10}$$

We consider a Riemannian metric on N_1^κ which is defined as below:

$$\begin{aligned} h_1 &= \{h_{ii} = 1, i = 1, 3, \dots, 2\kappa - 1, h_{ii} = u_{i-1}, i = 2, 4, \dots, 2\kappa, \\ h_{ij} &= 0, i \neq j, i, j = 1, 2, \dots, 2\kappa\}. \end{aligned} \quad (11)$$

Similarly on the Riemannian manifold N_2^λ we consider the metric:

$$\begin{aligned} h_2 &= \{h_{ii} = 1, i = 2\kappa + 1, 2\kappa + 3, \dots, 2\kappa + 2\lambda - 1, \\ h_{ii} &= u_{i-1}, i = 2\kappa + 2, 2\kappa + 4, \dots, 2\kappa + 2\lambda, \\ h_{ij} &= 0, i \neq j, i, j = 2\kappa + 1, 2\kappa + 2, \dots, 2\kappa + 2\lambda\}. \end{aligned} \quad (12)$$

We easily find that the Riemannian manifolds (N_1^κ, h_1) and (N_2^λ, h_2) have positive sectional curvature.

From the above we get:

Proposition 2 *The Riemannian manifolds (N_1^κ, h_1) and (N_2^λ, h_2) have positive sectional curvature.*

On the Cartesian product $N_1^\kappa \times N_2^\lambda$ we consider a special monoparametrical family of Riemannian metrics:

$$d(t) = \begin{cases} d_{ii} = 1 + tf_i, i = 1, 3, \dots, 2\kappa - 1, \\ d_{ii} = u_{i-1}(1 + tf_i), i = 2, 4, \dots, 2\kappa, \\ d_{jj} = 1 + t\varphi_j, j = 2\kappa + 1, 2\kappa + 3, \dots, 2\kappa + 2\lambda - 1, \\ d_{jj} = u_{j-1}(1 + t\varphi_j), j = 2\kappa + 2, 2\kappa + 4, \dots, 2\kappa + 2\lambda, \\ d_{ij} = 0, i \neq j, i, j = 1, 2, \dots, 2\kappa + 2\lambda, \end{cases} \quad (13)$$

where

$$\begin{aligned} f_i &= f_i(u_{2\kappa+1}, u_{2\kappa+2}, \dots, u_{2\kappa+2\lambda}), i = 1, 2, \dots, 2\kappa + 2\lambda, \\ \varphi_j &= \varphi_j(u_1, u_2, \dots, u_{2\kappa}), j = 2\kappa + 1, 2\kappa + 2, \dots, 2\kappa + 2\lambda \end{aligned} \quad (14)$$

are arbitrary functions and $-\varepsilon < t < \varepsilon$, ε is a small positive number.

Our aim is to define $f_i, i = 1, 2, \dots, 2\kappa$ and $\varphi_j, j = 2\kappa + 1, 2\kappa + 2, \dots, 2\kappa + 2\lambda$ in order that the Riemannian manifold $(N_1^\kappa \times N_2^\lambda, d(t))$ had strictly positive sectional curvature.

It is obvious that:

$$d(0) = h_1 \times h_2. \quad (15)$$

Let now P be an arbitrary point of $N_1^\kappa \times N_2^\lambda$. We know that the sectional curvature of a plane which is spanned by the vectors X and Y of the tangent space $T_P(N_1^\kappa \times N_2^\lambda)$ is given by the formula:

$$\sigma(X, Y)(t) = -\frac{\langle R(X, Y)X, Y \rangle}{\|X\|^2 \|Y\|^2 - \langle X, Y \rangle}. \quad (16)$$

According to the Mac-Lauren theorem for the function of one variable $\sigma(X, Y)(t)$ we can claim that:

$$\sigma(X, Y)(t) = \sigma(X, Y)(0) + \sigma'_t(X, Y)(0) \frac{t}{1!} + \sigma''_t(X, Y)(0) \frac{t^2}{2!} + \dots \quad (17)$$

From the formula (17) it is clear that the sign of $\sigma(X, Y)(t)$ depends on the sign of $\sigma(X, Y)(0)$, if $\sigma(X, Y)(0) \neq 0$ and t is a small positive number. In case where $\sigma(X, Y)(0) = 0$, the sign of $\sigma(X, Y)(t)$ depends on the sign of $\sigma'_t(X, Y)(0)$, since $t > 0$. We also know [1, p. 287] that:

$$\sigma(X, Y)(0), \text{ if } X \in T_P(N_1^\kappa) \text{ and } Y \in T_P(N_2^\lambda). \quad (18)$$

We have the following relation for $\sigma(X, Y)(t)$:

$$\sigma(X, Y)(t) = -\frac{A(t)}{B(t)}, \quad (19)$$

where

$$A(t) = R_{ij\rho\sigma} X^i Y^j X^\rho Y^\sigma, \quad i, \rho = 1, 2, \dots, 2\kappa \\ j, \sigma = 2\kappa + 1, 2\kappa + 2, \dots, 2\kappa + 2\lambda \quad (20)$$

and

$$B(t) = 4 \left\{ \sum_{i=1}^{2\kappa} d_{ii} (X^i)^2 \right\} \left\{ \sum_{j=2\kappa+1}^{2\kappa+2\lambda} d_{jj} (Y^j)^2 \right\} > 0, \quad (21)$$

because in this case $\langle X, Y \rangle = 0$.

From the relation (19) we can obtain that:

$$\sigma(X, Y)(0) = 0 \Leftrightarrow -\frac{A(0)}{B(0)} = 0 \Leftrightarrow A(0) = 0. \quad (22)$$

By derivation of (19) with respect to t it holds:

$$\sigma'_t(X, Y)(0) = -\frac{A'(0)B(0) - A(0)B'(0)}{B^2(0)}, \quad (23)$$

which due to (22) becomes:

$$\sigma'_t(X, Y)(0) = -\frac{A'(0)}{B(0)}. \quad (24)$$

From the formula (20) we get:

$$A'(0) = R_{ij\rho\sigma}(0) X^i Y^j X^\rho Y^\sigma, \quad i, \rho = 1, 2, \dots, 2\kappa, \\ j, \sigma = 2\kappa + 1, 2\kappa + 2, \dots, 2\kappa + 2\lambda. \quad (25)$$

As known, the components of curvature tensor $R_{ij\rho\sigma}$ are given by:

$$R_{ij\rho\sigma} = \frac{1}{2} \left\{ \frac{\partial^2 d_{i\rho}}{\partial u_j \partial u_\sigma} + \frac{\partial^2 d_{j\sigma}}{\partial u_i \partial u_\rho} - \frac{\partial^2 d_{j\rho}}{\partial u_i \partial u_\sigma} - \frac{\partial^2 d_{i\sigma}}{\partial u_j \partial u_\rho} \right\} - \{ \Gamma_{j\rho}^r \Gamma_{i\sigma}^s - \Gamma_{j\sigma}^r \Gamma_{i\rho}^s \}, \quad (26)$$

$i, \rho = 1, 2, \dots, 2\kappa, j, \sigma = 2\kappa + 1, 2\kappa + 2, \dots, 2\kappa + 2\lambda,$

where $\Gamma_{j\rho}^r, \Gamma_{i\sigma}^s, \Gamma_{j\sigma}^r, \Gamma_{i\rho}^s$ are the Christoffel symbols of second kind.

From the form of the Riemannian metric given by the relation (13) we obtain:

$$d_{j\rho} = 0, d_{i\sigma} = 0. \quad (27)$$

Hence the formula (26) becomes:

$$R_{ij\rho\sigma} = \frac{1}{2} \left\{ \frac{\partial^2 d_{i\rho}}{\partial u_j \partial u_\sigma} + \frac{\partial^2 d_{j\sigma}}{\partial u_i \partial u_\rho} \right\} - d_{rs} \{ \Gamma_{j\rho}^r \Gamma_{i\sigma}^s - \Gamma_{j\sigma}^r \Gamma_{i\rho}^s \}, \quad (28)$$

$i, \rho = 1, 2, \dots, 2\kappa, j, \sigma = 2\kappa + 1, 2\kappa + 2, \dots, 2\kappa + 2\lambda.$

It is known that the Christoffel symbols are given by the below formula:

$$\Gamma_{\alpha\gamma}^\beta = \frac{1}{2} d^{\beta s} \left(\frac{\partial d_{\alpha s}}{\partial u_\gamma} + \frac{\partial d_{\gamma s}}{\partial u_\alpha} - \frac{\partial d_{\alpha\gamma}}{\partial u_s} \right), \quad (29)$$

where $(d^{\beta s})$ is the inverse matrix of $(d_{\beta j})$.

After some calculations the Christoffel symbols of the Riemannian manifold $N_1^\kappa \times N_2^\lambda$ are given by the following formulas:

$$\Gamma_{j\rho}^r = \begin{cases} \frac{t}{2} \frac{1}{1 + t\varphi_j} \frac{\partial \varphi_j}{\partial u_\rho}, & \text{for } r = j, \\ \frac{t}{2} \frac{1}{1 + t f_\rho} \frac{\partial f_\rho}{\partial u_j}, & \text{for } r = \rho, \\ 0 & \text{in any other case,} \end{cases} \quad (30)$$

$$\Gamma_{i\sigma}^\tau = \begin{cases} \frac{t}{2} \frac{1}{1 + t f_i} \frac{\partial f_i}{\partial u_\sigma}, & \text{for } \tau = i, \\ \frac{t}{2} \frac{1}{1 + t \varphi_\sigma} \frac{\partial \varphi_\sigma}{\partial u_i}, & \text{for } \tau = \sigma, \\ 0 & \text{in any other case,} \end{cases} \quad (31)$$

$$\Gamma_{j\sigma}^r = \begin{cases} \frac{1}{2u_{j-1}}, r \neq \sigma = j-1, r = j \in \{2\kappa+2, 2\kappa+4, \dots, 2\kappa+2\lambda\}, \\ \frac{1}{2u_{\sigma-1}}, r \neq j = \sigma-1, r = \sigma \in \{2\kappa+2, 2\kappa+4, \dots, 2\kappa+2\lambda\}, \\ -\frac{t}{2} \frac{1}{1+tf_r} \frac{\partial \varphi_j}{\partial u_r}, r \in \{1, 3, \dots, 2\kappa-1\}, \\ r \neq \sigma = j \in \{2\kappa+1, 2\kappa+3, \dots, 2\kappa+2\lambda-1\}, \\ -\frac{t}{2} \frac{1}{u_{r-1}(1+tf_r)} \frac{\partial \varphi_j}{\partial u_r}, r \in \{2, 4, \dots, 2\kappa\}, \\ r \neq \sigma = j \in \{2\kappa+1, 2\kappa+3, \dots, 2\kappa+2\lambda-1\}, \\ -\frac{t}{2} \frac{u_{j-1}}{1+tf_r} \frac{\partial \varphi_j}{\partial u_r}, r \in \{1, 3, \dots, 2\kappa-1\}, \\ r \neq \sigma = j \in \{2\kappa+2, 2\kappa+4, \dots, 2\kappa+2\lambda\}, \\ -\frac{t}{2} \frac{u_{j-1}}{u_{r-1}(1+tf_r)} \frac{\partial \varphi_j}{\partial u_r}, r \in \{2, 4, \dots, 2\kappa\}, \\ r \neq \sigma = j \in \{2\kappa+2, 2\kappa+4, \dots, 2\kappa+2\lambda\}, \\ -\frac{1}{2} \frac{1+tf_j}{1+tf_r} r = j-1, r \neq \sigma = j \in \{2\kappa+2, 2\kappa+4, \dots, 2\kappa+2\lambda\}, \\ 0 \text{ in any other case.} \end{cases} \quad (32)$$

$$\Gamma_{i\rho}^\tau = \begin{cases} \frac{1}{2u_{i-1}}, \tau \neq \rho = i-1, \tau = i \in \{2, 4, \dots, 2\kappa\}, \\ \frac{1}{2u_{\rho-1}}, \tau \neq i = \rho-1, \tau = \rho \in \{2, 4, \dots, 2\kappa\}, \\ -\frac{t}{2} \frac{1}{1+t\varphi_\tau} \frac{\partial f_i}{\partial u_\tau}, \tau \in \{2\kappa+1, 2\kappa+3, \dots, 2\kappa+2\lambda-1\}, \\ \tau \neq i = \rho \in \{1, 3, \dots, 2\kappa-1\}, \\ -\frac{t}{2} \frac{1}{u_{\tau-1}(1+t\varphi_\tau)} \frac{\partial f_i}{\partial u_\tau}, \tau \in \{2\kappa+2, 2\kappa+4, \dots, 2\kappa+2\lambda\}, \\ \tau \neq i = \rho \in \{1, 3, \dots, 2\kappa-1\}, \\ -\frac{t}{2} \frac{u_{i-1}}{1+t\varphi_\tau} \frac{\partial f_i}{\partial u_\tau}, \tau \in \{2\kappa+1, 2\kappa+3, \dots, 2\kappa+2\lambda-1\}, \\ \tau \neq i = \rho \in \{2, 4, \dots, 2\kappa\}, \\ -\frac{t}{2} \frac{u_{i-1}}{u_{\tau-1}(1+t\varphi_\tau)} \frac{\partial f_i}{\partial u_\tau}, \tau \in \{2\kappa+2, 2\kappa+4, \dots, 2\kappa+2\lambda\}, \\ \tau \neq i = \rho \in \{2, 4, \dots, 2\kappa\}, \\ -\frac{1}{2} \frac{1+tf_i}{1+tf_\tau}, \tau = i-1, \tau \neq i = \rho \in \{2, 4, \dots, 2\kappa\}, \\ 0, \text{ in any other case} \end{cases} \quad (33)$$

$$R_{ijij} = \begin{cases} \frac{t}{2} \left\{ \frac{\partial^2 f_i}{\partial u_j^2} + \frac{\partial^2 \varphi_j}{\partial u_i^2} \right\} - \frac{t^4}{4} \left\{ \frac{1}{1+t\varphi_j} \left(\frac{\partial \varphi_j}{\partial u_i} \right)^2 + \frac{1}{1+tf_i} \left(\frac{\partial f_i}{\partial u_j} \right)^2 \right\}, \\ i \in \{1, 3, \dots, 2\kappa - 1\}, j \in \{2\kappa + 1, 2\kappa + 3, \dots, 2\kappa + 2\lambda - 1\}, \\ \frac{t}{2} \left\{ \frac{\partial^2 f_i}{\partial u_j^2} + u_{j-1} \frac{\partial^2 \varphi_j}{\partial u_i^2} \right\} - \frac{t^4}{4} \left\{ u_{j-1} \frac{1}{1+t\varphi_j} \left(\frac{\partial \varphi_j}{\partial u_i} \right)^2 + \right. \\ \left. + \frac{1}{1+tf_i} \left(\frac{\partial f_i}{\partial u_j} \right)^2 \right\} + \frac{t}{4} \left\{ \frac{1+t\varphi_j}{1+t\varphi_{j-1}} \frac{\partial f_i}{\partial u_j} \right\}, \\ i \in \{1, 3, \dots, 2\kappa - 1\}, j \in \{2\kappa + 2, 2\kappa + 4, \dots, 2\kappa + 2\lambda\}, \\ \frac{t}{2} \left\{ u_{i-1} \frac{\partial^2 f_i}{\partial u_j^2} + \frac{\partial^2 \varphi_j}{\partial u_i^2} \right\} - \frac{t^4}{4} \left\{ \frac{1}{1+t\varphi_j} \left(\frac{\partial \varphi_j}{\partial u_i} \right)^2 + u_{i-1} \frac{1}{1+tf_i} \left(\frac{\partial f_i}{\partial u_j} \right)^2 \right\} + \\ \frac{t}{4} \left\{ \frac{1+tf_i}{1+tf_{i-1}} \frac{\partial \varphi_j}{\partial u_{i-1}} \right\}, \\ i \in \{2, 4, \dots, 2\kappa\}, j \in \{2\kappa + 1, 2\kappa + 3, \dots, 2\kappa + 2\lambda - 1\}, \\ \frac{t}{2} \left\{ u_{i-1} \frac{\partial^2 f_i}{\partial u_j^2} + u_{j-1} \frac{\partial^2 \varphi_i}{\partial u_i^2} \right\} - \frac{t^4}{4} \left\{ u_{j-1} \frac{1}{1+t\varphi_j} \left(\frac{\partial \varphi_j}{\partial u_i} \right)^2 + \right. \\ \left. u_{i-1} \frac{1}{1+tf_i} \left(\frac{\partial f_i}{\partial u_j} \right)^2 + \frac{t}{4} \left\{ u_{j-1} \frac{1+tf_i}{1+tf_{i-1}} \frac{\partial \varphi_j}{\partial u_{i-1}} + u_{i-1} \frac{1+t\varphi_j}{1+t\varphi_{j-1}} \frac{\partial f_i}{\partial u_{j-1}} \right\} \right\}, \\ i \in \{2, 4, \dots, 2\kappa\}, j \in \{2\kappa + 2, 2\kappa + 4, \dots, 2\kappa + 2\lambda\}, \end{cases} \quad (34)$$

$$R_{ij\rho j} = \begin{cases} \frac{t}{2} \frac{\partial^2 f_i}{\partial u_j \partial u_\sigma} - \left\{ \frac{t}{u_{j-1}} \frac{\partial f_i}{\partial u_j} + \frac{t^4}{4} \frac{1}{1+tf_i} \frac{\partial f_i}{\partial u_j} \frac{\partial f_i}{\partial u_\sigma} + \frac{t}{4u_{\sigma-1}} \frac{\partial f_i}{\partial u_\sigma} \right\} \\ i = \{1, 3, \dots, 2\kappa - 1\}, j \neq \sigma, j, \sigma \in \{2\kappa + 1, 2\kappa + 2, \dots, 2\kappa + 2\lambda\}, \\ \frac{t}{2} u_{i-1} \frac{\partial^2 f_i}{\partial u_j \partial u_\sigma} - \left\{ \frac{t}{u_{j-1}} u_{i-1} \frac{\partial f_i}{\partial u_j} + \frac{t^4}{4} u_{i-1} \frac{1}{1+tf_i} \frac{\partial f_i}{\partial u_j} \frac{\partial f_i}{\partial u_\sigma} + \right. \\ \left. \frac{t}{4u_{\sigma-1}} u_{i-1} \frac{\partial f_i}{\partial u_\sigma} \right\} \\ i = 2, 4, \dots, 2\kappa, j \neq \sigma, j, \sigma \in \{2\kappa + 1, 2\kappa + 3, \dots, 2\kappa + 2\lambda\} \end{cases} \quad (35)$$

Remark 2.1 In relation (35) the components written with bold characters appear only in the case $\sigma = j - 1$ and the components written with bigger characters appear only in the case $j = \sigma - 1$.

We also have:

$$R_{ij\rho j} = \begin{cases} \frac{t}{2} \frac{\partial^2 \varphi_j}{\partial u_i \partial u_\rho} - \left\{ \frac{t}{u_{\rho-1}} \frac{\partial \varphi_j}{\partial u_\rho} + \frac{t^4}{4} \frac{1}{1+t\varphi_j} \frac{\partial \varphi_j}{\partial u_i} \frac{\partial \varphi_j}{\partial u_\rho} + \frac{t}{4u_{i-1}} \frac{\partial \varphi_j}{\partial u_i} \right\}, \\ j = 2\kappa + 1, 2\kappa + 3, \dots, 2\kappa + 2\lambda - 1, i \neq \rho, i, \rho \in \{1, 2, \dots, 2\kappa\}, \\ \frac{t}{2} u_{j-1} \frac{\partial^2 \varphi_j}{\partial u_i \partial u_\rho} - \left\{ \frac{t}{u_{\rho-1}} u_{j-1} \frac{\partial \varphi_j}{\partial u_\rho} + \frac{t^4}{4} \frac{1}{1+t\varphi_j} u_{j-1} \frac{\partial \varphi_j}{\partial u_i} \frac{\partial \varphi_j}{\partial u_\rho} + \right. \\ \left. \frac{t}{4u_{i-1}} u_{j-1} \frac{\partial \varphi_j}{\partial u_i} \right\}, \\ j = \{2\kappa + 2, 2\kappa + 4, \dots, 2\kappa + 2\lambda\}, i \neq \rho, i, \rho \in \{1, 2, \dots, 2\kappa\}. \end{cases} \quad (36)$$

Remark 2.2 In relation (36) the components written with bold characters appear only in the case $i = \rho - 1$ and the components written with bigger characters appear only in the case $\rho = i - 1$.

From the form of the Riemannian metric given by the formula (13) we get:

$$R_{ij\rho\sigma} = 0, i \neq \rho, j \neq \sigma, i, \rho \in \{1, 2, \dots, 2\kappa\}, j, \sigma \in \{2\kappa + 1, 2\kappa + 2, \dots, 2\kappa + 2\lambda\} \quad (37)$$

3 Special partial differential equations

We consider the functions:

$$f_i, i = 1, 2, \dots, 2\kappa \text{ and } \varphi_j, j = 2\kappa + 1, 2\kappa + 2, \dots, 2\kappa + 2\lambda, \quad (38)$$

so that they satisfy the following partial differential equations:

$$\frac{\partial^2 f_i}{\partial u_j \partial u_\sigma} = 0, i = 1, 3, \dots, 2\kappa - 1, j \neq \sigma, j, \sigma \in \{2\kappa + 1, 2\kappa + 2, \dots, 2\kappa + 2\lambda\}, \quad (39)$$

$$\frac{\partial^2 f_i}{\partial u_j \partial u_{j-1}} - \frac{1}{u_{j-1}} \frac{\partial f_i}{\partial u_j} = 0, i = 1, 3, \dots, 2\kappa - 1, \sigma = j - 1, \quad (40)$$

$$j, \sigma \in \{2\kappa + 1, 2\kappa + 2, \dots, 2\kappa + 2\lambda\},$$

$$\frac{\partial^2 f_i}{\partial u_\sigma \partial u_{\sigma-1}} - \frac{1}{2u_{\sigma-1}} \frac{\partial f_i}{\partial u_\sigma} = 0, i = 1, 3, \dots, 2\kappa - 1, j = \sigma - 1, \quad (41)$$

$$j, \sigma \in \{2\kappa + 1, 2\kappa + 2, \dots, 2\kappa + 2\lambda\},$$

$$u_{i-1} \frac{\partial^2 f_i}{\partial u_j \partial u_\sigma} = 0, i = 2, 4, \dots, 2\kappa, j \neq \sigma, j, \sigma \in \{2\kappa + 1, 2\kappa + 2, \dots, 2\kappa + 2\lambda\}, \quad (42)$$

$$u_{i-1} \frac{\partial^2 f_i}{\partial u_j \partial u_{j-1}} - \frac{u_{i-1}}{u_{j-1}} \frac{\partial f_i}{\partial u_j} = 0, i = 2, 4, \dots, 2\kappa, \sigma = j - 1; \quad (43)$$

$$j, \sigma \in \{2\kappa + 1, 2\kappa + 2, \dots, 2\kappa + 2\lambda\},$$

$$u_{i-1} \frac{\partial^2 f_i}{\partial u_\sigma \partial u_{\sigma-1}} - \frac{u_{i-1}}{u_{\sigma-1}} \frac{\partial f_i}{\partial u_\sigma} = 0, i = 2, 4, \dots, 2\kappa, j = \sigma - 1, \quad (44)$$

$$j, \sigma \in \{2\kappa + 1, 2\kappa + 2, \dots, 2\kappa + 2\lambda\},$$

$$\frac{\partial^2 \varphi_j}{\partial u_i \partial u_\rho}, j = 2\kappa + 1, 2\kappa + 3, \dots, 2\kappa + 2\lambda - 1, i \neq \rho, i, \rho \in \{1, 2, \dots, 2\lambda\}, \quad (45)$$

$$\frac{\partial^2 \varphi_j}{\partial u_\rho \partial u_{\rho-1}} - \frac{1}{u_{\rho-1}} \frac{\partial \varphi_j}{\partial u_\rho} = 0, j = 2\kappa + 1, 2\kappa + 3, \dots, 2\kappa + 2\lambda - 1, \quad (46)$$

$$i = \rho - 1, i, \rho \in \{1, 2, \dots, 2\kappa\},$$

$$\frac{\partial^2 \varphi_j}{\partial u_i \partial u_{i-1}} - \frac{1}{2u_{i-1}} \frac{\partial \varphi_j}{\partial u_\rho} = 0, j = 2\kappa + 1, 2\kappa + 3, \dots, 2\kappa + 2\lambda - 1, \quad (47)$$

$$\rho = i - 1, i, \rho \in \{1, 2, \dots, 2\kappa\},$$

$$u_{j-1} \frac{\partial^2 \varphi_j}{\partial u_i \partial u_\rho} = 0, j = 2\kappa + 2, 2\kappa + 4, \dots, 2\kappa + 2\lambda, \rho \neq i, \quad (48)$$

$$i, \rho \in \{1, 2, \dots, 2\kappa\},$$

$$u_{j-1} \frac{\partial^2 \varphi_j}{\partial u_\rho \partial u_{\rho-1}} - \frac{u_{j-1}}{u_{\rho-1}} \frac{\partial \varphi_j}{\partial u_\rho} = 0, j = 2\kappa + 2, 2\kappa + 4, \dots, 2\kappa + 2\lambda, i = \rho - 1, \quad (49)$$

$$i, \rho \in \{1, 2, \dots, 2\kappa\},$$

$$u_{j-1} \frac{\partial^2 \varphi_j}{\partial u_i \partial u_{i-1}} - \frac{u_{j-1}}{u_{i-1}} \frac{\partial \varphi_j}{\partial u_i} = 0, j = 2\kappa + 2, 2\kappa + 4, \dots, 2\kappa + 2\lambda, \rho = i - 1, \quad (50)$$

$$i, \rho \in \{1, 2, \dots, 2\kappa\}.$$

A solution of the system of partial differential equations (39)-(50) is given below:

$$f_i = V_{i, 2\kappa+1}(u_{2\kappa+1}), i = 1, 2, \dots, 2\kappa \quad (51)$$

and

$$\varphi_j = V_{j,1}(u_1), j = 2\kappa + 1, 2\kappa + 2, \dots, 2\kappa + 2\lambda. \quad (52)$$

Remark 3.1 We note here that if the system of partial differential equations (39)-(50) is satisfied, then the components $R_{ij\sigma}$, $R_{ij\rho j}$ of the curvature tensor will have the form:

$$R_{ij\sigma} = \begin{cases} -\frac{t^2}{4} \frac{1}{1 + t f_i} \frac{\partial f_i}{\partial u_j} \frac{\partial f_i}{\partial u_\sigma}, i = 1, 3, \dots, 2\kappa - 1, \\ j \neq \sigma, j, \sigma \in \{2\kappa + 1, 2\kappa + 2, \dots, 2\kappa + 2\lambda\} \\ -\frac{t^2}{4} u_{i-1} \frac{1}{1 + t f_i} \frac{\partial f_i}{\partial u_j} \frac{\partial f_i}{\partial u_\sigma}, i = 2, 4, \dots, 2\kappa, \\ j \neq \sigma, j, \sigma \in \{2\kappa + 1, 2\kappa + 2, \dots, 2\kappa + 2\lambda\} \end{cases} \quad (53)$$

and

$$R_{ij\rho j} = \begin{cases} -\frac{t^2}{4} \frac{1}{1 + t \varphi_j} \frac{\partial \varphi_j}{\partial u_i} \frac{\partial \varphi_j}{\partial u_\rho}, j = 2\kappa + 1, 2\kappa + 3, \dots, 2\kappa + 2\lambda - 1, \\ i \neq \rho, i, \rho \in \{1, 2, \dots, 2\kappa\}, \\ -\frac{t^2}{4} \frac{1}{1 + t \varphi_j} u_{j-1} \frac{\partial \varphi_j}{\partial u_i} \frac{\partial \varphi_j}{\partial u_\rho}, j = 2\kappa + 2, 2\kappa + 4, \dots, 2\kappa + 2\lambda, \\ i \neq \rho, i, \rho \in \{1, 2, \dots, 2\kappa\}. \end{cases} \quad (54)$$

From the formulas (53)-(54) we obtain:

$$R_{ij\sigma}(0) = R_{ij\rho j}(0) = 0, i, \rho \in \{1, 2, \dots, 2\kappa\},$$

$$j, \sigma \in \{2\kappa + 1, 2\kappa + 2, \dots, 2\kappa + 2\lambda\}. \tag{55}$$

From relation (34) we get:

$$R'_{ijij}(0) = \begin{cases} \frac{1}{2} \left\{ \frac{\partial^2 f_i}{\partial u_j^2} + \frac{\partial^2 \varphi_j}{\partial u_i^2} \right\}, i \in \{1, 3, \dots, 2\kappa - 1\}, \\ j \in \{2\kappa + 1, 2\kappa + 3, \dots, 2\kappa + 2\lambda - 1\}, \\ \frac{1}{2} \left\{ \frac{\partial^2 f_i}{\partial u_j^2} + u_{j-1} \frac{\partial^2 \varphi_j}{\partial u_i^2} \right\} + \frac{1}{4} \frac{\partial f_i}{\partial u_{j-1}}, i \in \{1, 3, \dots, 2\kappa - 1\}, \\ j \in \{2\kappa + 2, 2\kappa + 4, \dots, 2\kappa + 2\lambda\}, \\ \frac{1}{2} \left\{ u_{i-1} \frac{\partial^2 f_i}{\partial u_j^2} + \frac{\partial^2 \varphi_j}{\partial u_i^2} \right\} + \frac{1}{4} \frac{\partial \varphi_j}{\partial u_{i-1}}, i \in \{2, 4, \dots, 2\kappa\}, \\ j \in \{2\kappa + 1, 2\kappa + 3, \dots, 2\kappa + 2\lambda - 1\}, \\ \frac{1}{2} \left\{ u_{i-1} \frac{\partial^2 f_i}{\partial u_j^2} + u_{j-1} \frac{\partial^2 \varphi_j}{\partial u_i^2} \right\} + \frac{1}{4} u_{i-1} \frac{\partial f_i}{\partial u_{j-1}} + \frac{1}{4} u_{i-1} \frac{\partial \varphi_j}{\partial u_{i-1}} \\ i \in \{2, 4, \dots, 2\kappa\}, j \in \{2\kappa + 2, 2\kappa + 4, \dots, 2\kappa + 2\lambda\}. \end{cases} \tag{56}$$

From relation (37) it is also obvious that:

$$R_{ij\rho\sigma}(0) = 0, i \neq \rho, j \neq \sigma, i, \rho \in \{1, 2, \dots, 2\kappa\},$$

$$j, \sigma \in \{2\kappa + 1, 2\kappa + 2, \dots, 2\kappa + 2\lambda\}. \tag{57}$$

Remark 3.2 Using the above formulas we conclude that the only non-zero components $R_{ijij}(0)$ are those given below:

$$R'_{1j1j}(0) = \frac{1}{2} \left\{ \frac{\partial^2 f_1}{\partial u_j^2} + \frac{\partial^2 \varphi_j}{\partial u_1^2} \right\} = \begin{cases} \frac{1}{2} \{V'_{1,2\kappa+1}(u_{2\kappa+1}) + V'_{1,2\kappa+1}(u_1)\}, \\ j = 2\kappa + 1, \\ \frac{1}{2} V'_{j,1}(u_1), \\ j = 2\kappa + 3, 2\kappa + 5, \dots, 2\kappa + 2\lambda - 1, \end{cases} \tag{I}$$

$$R'_{i2\kappa+1i2\kappa+1}(0) = \frac{1}{2} \left\{ \frac{\partial^2 f_i}{\partial u_{2\kappa+1}^2} + \frac{\partial^2 \varphi_{2\kappa+1}}{\partial u_i^2} \right\} = \begin{cases} \frac{1}{2} \{V'_{1,2\kappa+1}(u_{2\kappa+1}) + \\ + V'_{2\kappa+1,1}(u_1)\}, i = 1, \\ \frac{1}{2} V'_{i,2\kappa+1}(u_{2\kappa+1}), \\ i = 3, 5, \dots, 2\kappa - 1, \end{cases} \tag{II}$$

$$R'_{ijij}(0) = \begin{cases} \frac{1}{2}u_{j,1}V'_{j,1}(u_1), \\ i = 1, j = 2\kappa + 4, 2\kappa + 6, \dots, 2\kappa + 2\lambda, \\ \frac{1}{2}u_{2\kappa+1}V'_{2\kappa+2,1}(u_1) + \frac{1}{4}V'_{1,2\kappa+1}(u_{2\kappa+1}), \\ i = 1, j = 2\kappa + 2, \\ \frac{1}{4}V'_{1,2\kappa+1}(u_{2\kappa+1}), \\ i = 3, 5, \dots, 2\kappa - 1, j = 2\kappa + 2, \\ 0 \text{ in any other case.} \end{cases} \quad (\text{III})$$

$$R'_{ijij}(0) = \begin{cases} \frac{1}{2}u_{i-1}V'_{i,2\kappa+1}(u_{2\kappa+1}), \\ j = 2\kappa + 1, i = 4, 6, \dots, 2\kappa, \\ \frac{1}{2}u_1V'_{2,2\kappa+1}(u_{2\kappa+1}) + \frac{1}{4}V'_{2\kappa+1,1}(u_1), \\ i = 2, j = 2\kappa + 1, \\ \frac{1}{4}V'_{j,1}(u_1), \\ i = 2, j = 2\kappa + 3, 2\kappa + 5, \dots, 2\kappa + 2\lambda - 1, \\ 0 \text{ in any other case.} \end{cases} \quad (\text{IV})$$

$$R'_{ijij}(0) = \begin{cases} \frac{1}{4}u_{j-1}V'_{j,1}(u_1), \\ j = 2\kappa + 4, 2\kappa + 6, \dots, 2\kappa + 2\lambda, i = 2, \\ \frac{1}{4}u_{i-1}V'_{i,2\kappa+1}(u_{2\kappa+1}), \\ i = 4, 6, \dots, 2\kappa; j = 2\kappa + 2, \\ \frac{1}{4}u_{2\kappa+1}V'_{2\kappa+1,1}(u_1) + \frac{1}{4}u_1V'_{2,2\kappa+1}(u_{2\kappa+1}), \\ i = 2, j = 2\kappa + 2, \\ 0 \text{ in any other case.} \end{cases} \quad (\text{V})$$

4 Study of the sectional curvature

Using the formula (20) we have:

$$A'(0) = R'_{ijij}(0)(X^i)^2(Y^j)^2 < 0, \\ i = 1, 2, \dots, 2\kappa, j = 2\kappa + 1, 2\kappa + 2, \dots, 2\kappa + 2\lambda, \quad (\text{58})$$

which is equivalent to $\sigma(X, Y)(0) > 0$, because from (21) it holds $B(0) > 0$.

From relation (58) we obtain:

$$\begin{cases} R'_{ijij}(0) \leq 0, \forall i = 1, 2, \dots, 2\kappa, \forall j = 2\kappa + 1, 2\kappa + 2, \dots, 2\kappa + 2\lambda \\ \text{and} \\ R'_{ijij}(0) < 0, \text{ for at least pair } (i, j) \\ \text{with } i = 1, 2, \dots, 2\kappa, j = 2\kappa + 1, 2\kappa + 2, \dots, 2\kappa + 2\lambda. \end{cases} \quad (\text{59})$$

By considering cases I-V the system (59) becomes:

$$\left(\begin{array}{l} \frac{1}{2} \{V'_{1,2\kappa+1}(u_{2\kappa+1}) + V'_{2\kappa+1,1}(u_1)\} \leq 0, \\ \frac{1}{2} V'_{j,1}(u_1) \leq 0, j = 2\kappa + 3, 2\kappa + 5, \dots, 2\kappa + 2\lambda - 1, \\ \frac{1}{2} V'_{i,2\kappa+1}(u_{2\kappa+1}) \leq 0, i = 1, 3, \dots, 2\kappa - 1, \\ \frac{1}{2} u_{j-1} V'_{j,1}(u_1) \leq 0, j = 2\kappa + 4, 2\kappa + 6, \dots, 2\kappa + 2\lambda, \\ \frac{1}{2} u_{2\kappa+1} V'_{2\kappa+2,1}(u_1) + \frac{1}{4} V'_{1,2\kappa+1}(u_{2\kappa+1}) \leq 0, \\ \frac{1}{4} V'_{i,2\kappa+1}(u_{2\kappa+1}) \leq 0, i = 3, 5, \dots, 2\kappa - 1, \\ \frac{1}{2} u_{i-1} V'_{i,2\kappa+1}(u_{2\kappa+1}) \leq 0, i = 4, 6, \dots, 2\kappa, \\ \frac{1}{2} u_1 V'_{2,2\kappa+1}(u_{2\kappa+1}) + \frac{1}{4} V'_{2\kappa+1}(u_1) \leq 0, \\ \frac{1}{4} V'_{j,1}(u_1) \leq 0, j = 2\kappa + 3, 2\kappa + 5, \dots, 2\kappa + 2\lambda - 1, \\ \frac{1}{4} u_{j-1} V'_{j,1}(u_1) \leq 0, j = 2\kappa + 4, 2\kappa + 6, \dots, 2\kappa + 2\lambda, \\ \frac{1}{4} u_{i-1} V'_{i,2\kappa+1}(u_{2\kappa+1}) \leq 0, i = 4, 6, \dots, 2\kappa, \\ \frac{1}{4} u_{2\kappa+1} V'_{2\kappa+2,1}(u_1) + \frac{1}{4} u_1 V'_{2,2\kappa+1}(u_{2\kappa+1}) \leq 0, \end{array} \right. \quad (60)$$

where in one at least case the strict inequality holds.

The functions:

$$V'_{i,2\kappa+1}(u_{2\kappa+1}) = 0, i = 2, 4, \dots, 2\kappa, \quad (61)$$

$$\begin{aligned} V'_{i,2\kappa+1}(u_{2\kappa+1}) &= \alpha_i u_{2\kappa+1} + \beta_i, \alpha_i, \beta_i \in \mathbb{R}, \alpha_i < 0, \\ i &= 1, 3, \dots, 2\kappa - 1, \end{aligned} \quad (62)$$

$$V'_{j,1}(u_1) = 0, j = 2\kappa + 2, 2\kappa + 4, \dots, 2\kappa + 2\lambda, \quad (63)$$

$$\begin{aligned} V'_{j,1}(u_1) &= \alpha_j u_1 + \beta_j, \alpha_j, \beta_j \in \mathbb{R}, \alpha_j < 0, \\ j &= 2\kappa + 1, 2\kappa + 3, \dots, 2\kappa + 2\lambda - 1 \end{aligned} \quad (64)$$

are a solution of the system (60).

Therefore the functions

$$f_i, i = 1, 2, \dots, 2\kappa \text{ and } \varphi_j, j = 2\kappa + 1, 2\kappa + 2, \dots, 2\kappa + 2\lambda,$$

can be the following:

$$f_i = f_i(u_{2\kappa+1}) = \begin{cases} \alpha_i u_{2\kappa+1} + \beta_i, \alpha_i, \beta_i \in \mathbb{R}, \alpha_i < 0, \\ i = 1, 3, \dots, 2\kappa - 1 \\ 0, i = 2, 4, \dots, 2\kappa \end{cases}, \quad (65)$$

$$\varphi_j = \varphi_j(u_1) = \begin{cases} \alpha_j u_1 + \beta_j, \alpha_j, \beta_j \in \mathbb{R}, \alpha_j < 0 \\ j = 2\kappa + 1, 2\kappa + 3, \dots, 2\kappa + 2\lambda - 1 \\ 0, j = 2\kappa + 2, 2\kappa + 4, \dots, 2\kappa + 2\lambda \end{cases} \quad (66)$$

Consequently the family of metrics can be:

$$d(t) = \begin{cases} d_{ii} = 1 + t(\alpha_i u_{2\kappa+1} + \beta_i), \alpha_i, \beta_i \in \mathbb{R}, \alpha_i < 0, i = 1, 3, \dots, 2\kappa - 1, \\ d_{ii} = u_{i-1}, i = 1, 3, \dots, 2\kappa - 1, \\ d_{jj} = 1 + t(\alpha_j u_1 + \beta_j), \alpha_j, \beta_j \in \mathbb{R}, \alpha_j < 0, \\ j = 2\kappa + 1, 2\kappa + 3, \dots, 2\kappa + 2\lambda - 1, \\ d_{jj} = u_{j-1}, j = 2\kappa + 2, 2\kappa + 4, \dots, 2\kappa + 2\lambda, \\ d_{ij} = 0, i \neq j, i, j \in \{1, 2, \dots, 2\kappa + 2\lambda\}, \end{cases} \quad (67)$$

which is an answer to our problem.

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