Killing symmetries of Einstein equations in geometrized fibered framework

V. Balan and P.C. Stavrinos

Abstract. In the framework of geometrized jet bundles of first order, the paper presents the extended Einstein and Maxwell equations with sources, and generalize the classical Killing equations for the case of adapted Cartan connection, emphasizing the non-holonomy and presence of torsion.

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1. Einstein equations with sources in $J^1(T,M)$-geometrized framework

Consider two real $\mathcal{C}^\infty$ differentiable manifolds T and M are ($\dim T = m$, $\dim M = n$). Let $\xi = (E = J^1(T,M), \pi, T \times M)$ be the first order jet bundle of mappings $\varphi : T \to M$, with local coordinates on E

$$(t^\alpha, x^i, y^A)_{(\alpha,i,A)\in I_*} \equiv (y^\mu)_{\mu\in I},$$

with $I_* = I_{h_1} \times I_{h_2} \times I_v$ and

$$I_{h_1} = \overline{1,m},\ I_{h_2} = \overline{m+1, m+n},\ I_v = \overline{m+n+1, m+n+mn},\ I_h = I_{h_1} \cup I_{h_2},\ I = I_h \cup I_v.$$

The indices will implicitly take values as follows:

$$\alpha, \beta, \ldots \in I_{h_1};\ \ i, j, \ldots \in I_{h_2};\ \ A, B, \ldots \in I_v;\ \ \lambda, \mu, \ldots \in I.$$

Throughout the paper we shall identify a given index $A = m + n + n(i - m - 1) + \alpha$ to $A \equiv \binom{i}{\alpha}$ and $y^A \equiv x^{\binom{i}{\alpha}} = \frac{\partial x^i}{\partial t^\alpha}$.

The *non-linear connection* $N = \{N^A_\mu\}_{\mu\in I_h, A\in I_v}$ on E provides the splitting [2]

$$TE = HE \oplus VE, \tag{1.1}$$

and the local adapted basis of $\mathcal{X}(E)$

$$\mathcal{B} = \{\delta_\alpha, \delta_i, \delta_A\}_{(\alpha,i,A)\in I_*} \equiv \{\delta_\mu\}_{\mu\in I}, \tag{1.2}$$

where we denoted

$$\delta_\alpha = \partial_\alpha - N^A_\alpha \delta_A,\ \delta_i = \partial_i - N^A_i \delta_A,\ \delta_A = \dot{\partial}_A = \frac{\partial}{\partial y^A},$$

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and $\partial_\alpha = \frac{\partial}{\partial t^\alpha}$, $\partial_i = \frac{\partial}{\partial x^i}$. We remark that these span the modules of local sections,

$$\mathcal{S}(HE) = Span(\{\delta_\mu\})_{\mu\in I_h}, \ \ \mathcal{S}(VE) = Span(\{\delta_\mu\})_{\mu\in I_v}.$$

The dual basis of $\mathcal{B}$ writes as well

$$\mathcal{B}^* = \{\delta^\alpha, \delta^i, \delta^A\}_{(\alpha,i,A)\in I_*} \equiv \{\delta^\mu\}_{\mu\in I},$$

where $\delta^\alpha = dt^\alpha, \delta^i = dx^i, \delta^A \equiv \delta y^A = dy^A + N^A_\alpha dt^\alpha + N^A_i dx^i$.

Consider on E a fixed non-linear connection and let $\nabla = \{L^\lambda_{\mu\nu}\}_{\lambda,\mu,\nu\in I}$ be a linear connection on E. Then its coefficients relative to the adapted basis (1.2) are given by

$$\delta^\lambda(\nabla_{\delta_\nu}\delta_\mu) = L^\lambda_{\mu\nu}, \quad \forall\lambda,\mu,\nu\in I = I_{h_1}\cup I_{h_2}\cup I_v. \tag{1.3}$$

and generally form $3^3 = 27$ distinct subsets, according to the three subsets of indices. The torsion and curvature

$$\begin{aligned} &\mathcal{T}(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y], \\ &\mathcal{R}(X,Y)Z = \nabla_{[X}\nabla_{Y]}Z - \nabla_{[X,Y]}Z, \ \ \forall\, X,Y,Z\in\mathcal{S}(TE) \end{aligned}$$

have the *adapted coefficients* defined by

$$\delta^\lambda(\mathcal{T}(\delta_\nu,\delta_\mu)) = T^\lambda_{\mu\nu}, \ \ \delta^\lambda(\mathcal{R}(\delta_\nu,\delta_\mu)\delta_\rho) = R_\rho{}^\lambda{}_{\mu\nu}, \quad \forall\,\lambda,\mu,\nu,\rho\in I.$$

Within the set of linear connections on E we evidentiate the ones (called N-connections) which preserve the distributions related to the adapted basis and have hence just 9 nontrivial sets of coefficients, since

$$L^\lambda_{\mu\nu} = 0, \ \ \forall\,(\lambda,\mu)\in(I_h\times I_v)\cup(I_v\times I_h)\cup(I_{h_1}\times I_{h_2})\cup(I_{h_2}\times I_{h_1}), \forall\nu\in I. \tag{1.4}$$

Among them, a central role is played by the so-called "Γ-linear h-normal connections" [4], which depend on the four essential components

$$\nabla \equiv (L^\alpha_{\beta\gamma}, L^i_{j\gamma}, L^i_{jk}, L^i_{jA}), \tag{1.5}$$

and whose the other 5 nontrivial components given by

$$\begin{aligned} &L^A_{B\gamma} \equiv L^{\binom{i}{\alpha}}_{\binom{j}{\beta}\gamma} = \delta^\beta_\alpha L^i_{j\gamma} - \delta^i_j \left|{}^{\ \beta}_{\alpha\gamma}\right|, \ \ L^A_{Bk} \equiv L^{\binom{i}{\alpha}}_{\binom{j}{\beta}k} = \delta^\beta_\alpha \left|{}^{\ i}_{jk}\right|, \\ &L^A_{BC} \equiv L^{\binom{i}{\alpha}}_{\binom{j}{\beta}C} = \delta^\beta_\alpha L^i_{jC}, \ \ L^\alpha_{\beta j} = 0, \quad L^\alpha_{\beta C} = 0. \end{aligned}$$

Further, if E is endowed with a semi-Riemannian metric

$$G = \underbrace{h_{\alpha\beta}(t,x)dt^\alpha\otimes dt^\beta}_{h} + \underbrace{g_{ij}(t,x,y)dx^i\otimes dx^j}_{g} + \underbrace{g_{AB}(t,x,y)\delta y^A\otimes\delta y^B}_{\tilde{g}}, \tag{1.6}$$

with the vertical part given by the Kronecker product

$$g_{AB} \equiv g_{\binom{i}{\alpha}\binom{j}{\beta}} = g_{ij}(t,x,y)h^{\alpha\beta}(t),$$

then E naturally admits *the Cartan linear connection* which is metrical and has, for $m \geq 2$, just 8 nontrivial sets of coefficients for torsion and 7 curvature sets for curvature. Its essential coefficients (1.5) are

$$\begin{aligned} &L^{\alpha}_{\beta\gamma} = \left|{\alpha \atop \beta\gamma}\right| = \tfrac{1}{2}h^{\alpha\varepsilon}(\delta_{\{\beta}h_{\varepsilon\}\gamma} - \delta_{\varepsilon}h_{\beta\gamma}),\\ &L^{i}_{j\gamma} = \tfrac{1}{2}g^{ik}\delta_{\gamma}g_{kj}, \ \ L^{i}_{jk} = \tfrac{1}{2}g^{il}(\delta_{\{k}g_{j\}l} - \delta_{l}g_{jk}),\\ &L^{i}_{jA} \equiv L^{i}_{j\binom{k}{\gamma}} = \tfrac{1}{2}g^{il}(\delta_{\binom{\{k}{\gamma}}g_{j\}l} - \delta_{\binom{l}{\gamma}}g_{jk}), \end{aligned} \tag{1.7}$$

where we use the notations $\tau_{[i\ldots j]} = \tau_{i\ldots j} - \tau_{j\ldots i}$, $\tau_{\{i\ldots j\}} = \tau_{i\ldots j} + \tau_{j\ldots i}$.

Its essential torsion coefficients are given by [4]

$$\begin{cases} T_{\gamma}{}^{\binom{i}{\alpha}}{}_{\binom{j}{\beta}} = \partial_{\binom{j}{\beta}}N_{\gamma}^{\binom{i}{\alpha}} - \delta^{\beta}_{\alpha}L^{i}_{j\gamma} + \delta^{i}_{j}L^{\beta}_{\alpha\gamma}\\ T_{k}{}^{\binom{i}{\alpha}}{}_{\binom{j}{\beta}} = \partial_{\binom{j}{\beta}}N_{k}^{\binom{i}{\alpha}} - \delta^{\beta}_{\alpha}L^{i}_{jk}\\ T_{\binom{j}{\beta}}{}^{\binom{i}{\alpha}}{}_{\binom{k}{\gamma}} = \delta^{\beta}_{i}L^{i}_{j\binom{k}{\gamma}} - \delta^{\gamma}_{i}L^{i}_{k\binom{j}{\beta}}\\ T_{\beta}{}^{i}{}_{j} = -L^{i}_{\beta j}, \ \ T^{i}_{jA} = L^{i}_{jA}\\ T_{\beta}{}^{A}{}_{\gamma} = \delta_{[\gamma}N^{A}_{\beta]}, \ T_{\beta}{}^{A}{}_{j} = \delta_{[j}N^{A}_{\beta]}, \ T_{i}{}^{A}{}_{j} = \delta_{[j}N^{A}_{i]}, \end{cases} \tag{1.8}$$

and the nontrivial *non-holonomy coefficients* $\omega^{\lambda}_{\mu\nu}$ are provided by the relations

$$\begin{aligned} &[\delta_{\mu}, \delta_{\nu}] = \omega^{A}_{\mu\nu}\delta_{A} \equiv T^{A}_{\mu\nu}\delta_{A}, \ \forall \mu,\nu \in I_h,\\ &[\delta_{\mu}, \delta_{B}] = \omega^{A}_{\mu B}\delta_{A} \equiv \partial_{B}N^{A}_{\mu}\delta_{A}, \ \forall \mu \in I_h. \end{aligned}$$

As well, the nontrivial essential curvature N-tensor fields are

$$\begin{cases} R_{\beta}{}^{\alpha}{}_{\gamma\delta} = \partial_{[\delta}L^{\alpha}_{\beta\gamma]} + L^{\varepsilon}_{\beta[\gamma}L^{\alpha}_{\varepsilon\delta]}\\ R_{j}{}^{i}{}_{km} = \partial_{[k}L^{i}_{jm]} + L^{\beta}_{j[m}L^{i}_{\beta k]} + L^{i}_{jA}T^{A}_{mk}\\ R_{j}{}^{i}{}_{\gamma\mu} = \partial_{[\mu}L^{i}_{j\gamma]} + L^{\varepsilon}_{j[\gamma}L^{i}_{\varepsilon\mu]} + L^{i}_{jA}T^{A}_{\gamma\mu}, \ \forall \mu \in I_h\\ R_{j}{}^{i}{}_{\mu A} = \partial_{A}L^{i}_{j\mu} - L^{i}_{jA|\mu} + L^{i}_{jB}T^{B}_{\mu A}\\ R_{j}{}^{i}{}_{CD} = \partial_{[D}L^{i}_{jC]} + L^{k}_{j[C}L^{i}_{kD]}, \ \ \forall \mu \in I_h \end{cases} \tag{1.9}$$

and
$$\begin{cases} R_{\binom{j}{\beta}\ \gamma\delta}^{\binom{i}{\alpha}} = \delta^\beta_\alpha R_j{}^i{}_{\gamma\delta} + \delta^i_j R_\alpha{}^\beta{}_{\gamma\delta} \\ R_{\binom{j}{\beta}\ \mu k}^{\binom{i}{\alpha}} = \delta^\beta_\alpha R_j{}^i{}_{\mu k}, \ \ \forall \mu \in I_h \\ R_{\binom{j}{\beta}\ \mu A}^{\binom{i}{\alpha}} = \delta^\beta_\alpha R_j{}^i{}_{\mu A}, \forall \mu \in I, \end{cases}$$
where we denote by $|\alpha$, $|i$ and $|A$ the covariant derivations given by ∇_{δ_μ}, for $\mu \in I_{h_1}, I_{h_2}$ and I_v respectively.

Then the associated *Ricci* N-tensor fields are
$$\begin{array}{llll} R_{\alpha\beta} = R^\gamma_{\alpha\beta\gamma}, & R_{i\alpha} = R^k_{i\alpha k}, & R_{ij} = R^k_{ijk}, & R_{iA} = -R^j_{ijA}, \\ R_{\binom{i}{\alpha}\beta} = R^k_{i\beta\binom{k}{\alpha}}, & R_{\binom{i}{\alpha}j} = R^k_{ij\binom{k}{\alpha}}, & R_{\binom{i}{\alpha}\binom{j}{\beta}} = R^k_{i\binom{j}{\beta}\binom{k}{\alpha}}, & \end{array} \tag{1.10}$$
and the scalar of curvature $R = R_h + R_g + R_v$, where
$$R_h = h^{\alpha\beta} R_{\alpha\beta}, \quad R_g = g^{ij} R_{ij}, \quad R_v = \tilde{g}^{AB} R_{AB}.$$
Then we have the following result:

Theorem 1. a) *Within the framework of first-order geometrized jet spaces endowed with arbitrary nonlinear connection, vertical Kronecker-type metric and Cartan linear N-connection* the Einstein equations with sources *are*
$$\begin{cases} R_{\alpha\beta} - \frac{1}{2} R h_{\alpha\beta} = \kappa \mathcal{T}_{\alpha\beta} \\ R_{ij} - \frac{1}{2} R g_{ij} = \kappa \mathcal{T}_{ij} \\ R_{AB} - \frac{1}{2} R g_{AB} = \kappa \mathcal{T}_{AB}, \end{cases} \quad \begin{cases} 0 = \mathcal{T}_{\alpha i}, & 0 = \mathcal{T}_{\alpha A}, \\ R_{i\alpha} = \kappa \mathcal{T}_{i\alpha}, & R_{A\alpha} = \kappa \mathcal{T}_{A\alpha} \\ R_{iA} = \kappa \mathcal{T}_{iA}, & R_{Ai} = \kappa \mathcal{T}_{Ai}, \end{cases} \tag{1.11}$$
where $\mathcal{T} = \mathcal{T}_{\mu\nu} \delta^\mu \otimes \delta^\nu \in \mathcal{T}^0_2(E)$ *is the* energy-momentum tensor field *and κ is the cosmological constant.*

b) *The equations (1.11) satisfy the* conservation laws
$$E^\mu{}_{\nu|\mu} = \kappa \mathcal{T}^\mu{}_{\nu|\mu}, \ \ \forall \mu \in I = I_{h_1} \cup I_{h_2} \cup I_v,$$
where $E_{\mu\nu} = R_{\mu\nu} + \frac{1}{2} R G_{\mu\nu}$ *is the Einstein N-tensor field and the indices are raised by means of the metric G on E.*

We note that the explicit expressions of (1.11) were for the first time provided in [4].

2. Special cases. Maxwell equations with sources.

Regarding the energy-momentum tensor field, we distinguish several special extended cases:

I. The case of *electromagnetic field source*, when

$$T_{\mu\nu} = F_{\mu\rho}F^{\rho}_{\mu} - \frac{1}{4}G_{\mu\nu}F^{\rho\pi}F_{\rho\pi}.$$

Here the *deflection tensor fields* provided by the covariant derivatives of the Liouville field $\mathcal{C} = y^A\delta_A$ via

$$d^A_\mu = \delta^A\nabla_{\delta_\mu}\mathcal{C}, \quad \mu \in I, \ A \in I_v,$$

give raise to the electromagnetic 2-form

$$F = F_{A\mu}\delta y^A \wedge \delta y^\mu, \tag{2.12}$$

explicitely given by its nontrivial components

$$\left\{\begin{array}{lll} F_{A\beta} & \equiv F_{\binom{i}{\alpha}\beta} = \frac{1}{2}\left(h^{\alpha\gamma}g_{ik}y^{\binom{k}{[\gamma}}\right)_{|\beta]} & \\ F_{Aj} & \equiv F_{\binom{i}{\alpha}j} = \frac{1}{2}\, d_{\left(\substack{[i\\ \alpha}\right)j]} = & \\ & = \frac{1}{2}\, y_{\left(\substack{[i\\ \alpha}\right)|j]} = \frac{1}{2}\left(y^{\binom{k}{\gamma}}h^{\alpha\gamma}g_{k[i}\right)_{|j]}, & \\ F_{AB} & = \frac{1}{2}\tilde{g}_{[AC}d^C_{B]} = \frac{1}{2}\tilde{g}_{[AC}y^C_{|B]}. & \end{array}\right. \tag{2.13}$$

where the raising/lowering of the indices was performed using the metric G, as follows

$$\tilde{F} = F_A{}^\mu\delta_\mu \otimes \delta^A, \qquad F^\alpha_A = h^{\alpha\beta}F_{A\beta},\ F^i_A = g^{ij}F_{Aj},\ F^C_A = g^{CD}F_{AD}.$$

The energy-momentum tensor fields have the essential coefficients given by

$$\left\{\begin{array}{l} \mathcal{T}_{\alpha\beta} = F_{A\alpha}F_{B\beta}\tilde{g}^{AB} - \frac{1}{4}h_{\alpha\beta}F_* \\ \mathcal{T}_{ij} = F_{Ai}F_{Bj}\tilde{g}^{AB} - \frac{1}{4}g_{ij}F_* \\ \mathcal{T}_{AB} = F_{AC}F_{BD}\tilde{g}^{CD} - \frac{1}{4}g_{AB}F_* \end{array}\right. \qquad \left\{\begin{array}{l} \mathcal{T}_{\alpha i} = F_{C\alpha}F_{Di}\tilde{g}^{CD} \\ \mathcal{T}_{\alpha A} = F_{C\alpha}F_{DA}\tilde{g}^{CD} \\ \mathcal{T}_{iA} = F_{Ci}F_{DA}\tilde{g}^{CD} \end{array}\right.$$

where $F_* = F_{AC}F_{BD}\tilde{g}^{AB}\tilde{g}^{CD}$.

The 2-form F is subject to the two sets of *the Maxwell extended equations*

$$\left\{\begin{array}{l} F_{\binom{i}{\alpha}k|\gamma} = \frac{1}{2}[d_{\binom{i}{\alpha}\beta|k} + d_{\binom{i}{\alpha}m}T^m_{\beta k} + d_{\binom{i}{\alpha}C}T^C_{\beta k} - (T^j_{\gamma i|k} + L^j_{kC}T^C_{\gamma i})y_{\binom{j}{\alpha}}] \\ F_{\binom{i}{\alpha}\binom{k}{\beta}|\gamma} = \frac{1}{2}[d_{\binom{i}{\alpha}\beta|\binom{k}{\gamma}} + d_{\binom{i}{\alpha}m}T^m_{\beta\binom{k}{\gamma}} + d_{\binom{i}{\alpha}C}T^C_{\beta k} - (\partial_{\binom{k}{\gamma}}T^j_{\gamma i} + L^j_{kC}T^C_{\gamma\binom{i}{\beta}})y_{\binom{j}{\alpha}}] \\ \\ \underset{ijk}{S}\,F_{\binom{i}{\alpha}j|k} = -\frac{1}{2}\underset{ijk}{S}\,(L^m_{iC}y_{\binom{m}{\alpha}} + d_{\binom{i}{\alpha}C})T^C_{jk} \\ \underset{ijk}{S}\,F_{\binom{i}{\alpha}\{j|\binom{k\}}{\gamma}} = 0, \quad \underset{ABC}{S}\,F_{AB|C} = 0 \end{array}\right.$$

and

$$\begin{cases} g^{BC}F_{B\alpha|C} = -4\pi J_\alpha \\ g^{BC}F_{Bi|C} = -4\pi J_i \\ g^{BC}F_{AB|C} + g^{ij}F_{Ai|j} + h^{\alpha\beta}F_{A\alpha|\beta} = 4\pi J_A, \end{cases}$$

with $J = J_\mu \delta^\mu \in \mathcal{X}^*(E)$ the adapted electric current, and where we denoted by S the cyclic summation of the corresponding indices below.

II. In the case of a *perfect fluid* with the extended velocity vector field $\mathcal{V} = \mathcal{V}^\mu \delta_\mu \in \mathcal{X}(E)$, the energy-momentum N-tensor field is given by

$$\mathcal{T}_{\mu\nu} = (P+\rho)\mathcal{V}_\mu \mathcal{V}_\nu + pG_{\mu\nu},$$

where ρ is the *mass-energy density*, p is the *pressure* and, in the 4-dimensional Lorentz-metric case $\mathcal{V}$ satisfies the condition $\mathcal{V}^i\mathcal{V}_i = -1$. In particular, for $p = 0$, is obtained the case of cosmologic dust (presureless fluid).

III. In the case of source given by *the Klein-Gordon field* Φ, we have

$$\mathcal{T}_{\mu\nu} = \Phi_{|\mu}\Phi_{|\nu} - \frac{1}{2}(G^{\pi\rho}\Phi_{|\pi}\Phi_{|\rho} + m^2\Phi^2)G_{\mu\nu},$$

where m is the mass, and the field Φ satisfies the condition $G^{\mu\nu}\Phi_{|\mu|\nu} = m^2\Phi$ [10].

IV. In the case of *the radiation field*, we get

$$\mathcal{T}_{\mu\nu} = \Phi^2 K_\mu K_\nu,$$

where the N-vector field $K = K^\mu \delta_\mu \in \mathcal{X}(E)$ obeys in the Lorentz case the condition $G_{\mu\nu}K^\mu K^\nu = 0$.

3. Killing equations in $J^1(T,M)$-geometrized framework

In the process of searching solutions for the Einstein equations, the simplifying assumptions usually refer to symmetries of solutions or eigenvectors of the gravitational field. In the first case, roughly speaking, the symmetries provide charts in which the components of the metric G are independent of one or more coordinates. Rigorously, this happens when there exist a (Killing) vector field $\xi = \xi^\mu \delta_\mu$, such that the associated *Lie derivative operator*, given, e.g., by

$$L_\xi U^\mu_\nu = \xi^\rho U^\mu_{\nu|\rho} - U^\rho_\nu \xi^\mu_{|\rho} + U^\mu_\rho \xi^\rho_{|\nu}, \ \forall \mu,\nu \in I, \forall \xi \in \mathcal{X}(E).$$

vanishes on the metric tensor filed G, i.e., $L_\xi G = 0$. The Killing fields form a Lie algebra, as consequence of the property $L_{[\xi,\zeta]} = [L_\xi, L_\zeta]$. The Killing equations rewrite

$$(L_\xi G)_{\mu\nu} \equiv \xi(G_{\mu\nu}) + G_{\sigma\{\mu}(\delta_{\nu\}}\xi^\sigma - \omega^\sigma_{\rho\nu\}}\xi^\rho) = 0, \ \ \forall \mu,\nu \in I,$$

Theorem 2. The Killing equations expressed with respect to the Cartan canonic connection of the geometrized first-order jet bundle endowed with Kronecker lifted metric have the form

$$\left\{\begin{array}{l} \xi_{\{\alpha|\beta\}} = 0 \\ \xi_{\{\alpha|j\}} = -g_{kj}(T^k_{\alpha m}\xi^m + T^k_{\alpha C}\xi^C) \\ \xi_{\{\alpha|A\}} = -\tilde{g}_{AB}T^B_{\alpha\mu}\xi^\mu \\ \xi_{\{i|j\}} = -g_{k\{j}(T^k_{i\}\alpha}\xi^\alpha + T^k_{i\}C}\xi^C) \\ \xi_{\{i|A\}} = -g_{ij}T^j_{Am}\xi^m - \tilde{g}_{AB}T^B_{i\mu}\xi^\mu \\ \xi_{\{A|B\}} = -\tilde{g}_{C\{A}T^C_{B\}\mu}\xi^\mu. \end{array}\right. \tag{3.14}$$

Proof. Using $T^\rho_{\mu\nu} = L^\rho_{[\mu\nu]} - \omega^\rho_{\mu\nu}$, the Killing equations gets the equivalent form

$$G_{\sigma\{\mu}\xi^\sigma_{|\nu\}} + G_{\sigma\{\mu}T^\sigma_{\nu\}\rho}\xi^\rho = 0, \quad \forall \mu,\nu \in I.$$

The metricity of the Cartan connection, the splitting $I = I_{h_1} \cup I_{h_2} \cup I_v \ni \mu,\nu$ and (1.8) povide the claim. □

Corollary 1. Provided that the nonlinear connection N and the metric G satisfy the relations

$$\left\{\begin{array}{l} \delta^\beta_\alpha g^{ik}\delta_\gamma g_{kj} = 2\delta^i_j \left|{}^{\beta}_{\alpha\gamma}\right| \\ \delta_{\{j}g_{k\}i} = \delta_i g_{jk}, \\ \delta_{\binom{\{k}{\gamma}}g_{j\}l} = \delta_{\binom{l}{\gamma}}g_{jk}, \end{array}\right.$$

the structure (E,N,G) admits v-torsionless Killing equations:

$$\left\{\begin{array}{l} \xi_{\{\alpha|j\}} = -g_{kj}T^k_{\alpha m}\xi^m \\ \xi_{\{i|j\}} = -g_{k\{j}T^k_{i\}\alpha}\xi^\alpha \\ \xi_{\{\alpha|\beta\}} = 0, \;\; \xi_{\{\alpha|A\}} = 0, \;\; \xi_{\{i|A\}} = 0, \;\; \xi_{\{A|B\}} = 0. \end{array}\right.$$

Corollary 2. a) For $s = 1$, (3.14) provide the Killing-Yawata equations for the Sasaki lift, with autonomous field ξ.

b) For $s = 1$ and $\xi = \xi^m\delta_m \in Span(\delta_\mu)_{\mu\in I_{h_2}}$, the third equation of (3.14) is the classical Killing equation on M,

$$g_{i\{j}\xi^i{}_{|k\}} \equiv g_{ij}\xi^i_{|k} + g_{ik}\xi^i_{|j} = 0.$$

Conclusions. In the framework of first-order geometrized jet bundle, were presented the Einstein and Maxwell equations with sources, which extend the classical Riemannian ones, and special particular cases were pointed out. The extended Killing equations were derived, and the presence of torsion and non-holonomy emphasized, for the case of canonic Cartan connection on $J^1(T,M)$. The existence of Killing fields providing symmetries for the Kronecker-type metric structure represents the subject of further concern.

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Vladimir Balan
University Politehnica of Bucharest, Department Mathematics I
Splaiul Independenţei 313, RO-77206 Bucharest, Romania
Email address: vbalan@mathem.pub.ro

Panayiotis C. Stavrinos
Department of Mathematics, University of Athens,
Panepistemiopolis, GR-15784 Athens, Greece
Email address: pstavrin@cc.uoa.gr