Finslerian convexity on positive orthant

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Abstract. Section 1 reconsiders wellknown facts about Finslerian convexity. Section 2 gives examples of functions which are convex on $\mathbb{R}^n_+$ with respect to the affine Finslerian metric. Section 3 extends the results of Section 2 to others Finsler metrics. Section 4 discusses the Finslerian convexity of separable functions and Section 5 gives a Finsler metric that ensures the convexity of the Rosenbrock banana function.

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1 Convexity on Finslerian manifolds

Let M be an n-dimensional connected $\mathcal{C}^\infty$ manifold and TM its tangent bundle. Denote by (x, y) an arbitrary point in TM and by x the corresponding point in M.

1.1 Definition. A symmetric tensor field $F(x, y)$ of type $(0, 2)$ is called:

(a) *strongly* positive definite if $F(x, y)(v, v) > 0$, $\forall x \in M$, $\forall y \in T_xM$, $\forall v \in T_xM - \{0\}$;

(b) *weakly* positive definite if $F(x, y)(y, y) > 0$, $\forall x \in M$, $\forall y \in T_xM \setminus \{0\}$;

(c) *nearly weakly* positive definite if there exists $v \in T_xM$ such that $F(x, v)(y, y) > 0$, $\forall x \in M$, $\forall y \in T_xM \setminus \{0\}$.

Weakening the conditions in the Definition 1.1 we obtain three types of semidefiniteness.

1.2. Definition. A tensor field $g(x, y)$ of type $(0, 2)$ which is symmetric, strongly positive definite and homogeneous of degree zero, i.e.,

$$g(x, \lambda y) = g(x, y), \quad \forall \lambda \in \mathbb{R} \setminus \{0\}$$

is called a *Finslerian metric* on M. The pair $(M, g(x, y))$ is called a Finslerian manifold. The function $L : TM \to \mathbb{R}$, $L(x, y) = (g_{ij}(x, y)y^iy^j)^{\frac{1}{2}}$ is called the *fundamental Finslerian* function and L^2 is called the *absolute Finslerian* energy.

Denote by $g_{ij}(x, y)$, $i, j = 1, \ldots, n$ the components of $g(x, y)$ and by $g^{ij}(x, y)$ the components of $g^{-1}(x, y)$, i.e. $g^{ij}g_{jk} = \delta^i_k$. Note that $g_{ij}(x, y)$ and $g^{ij}(x, y)$ are homogeneous functions of degree zero with respect to y.

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The fundamental Finslerian function induces a vector field on the manifold $TM \setminus \{0\}$. Namely, for a tangent vector $y = y^i \frac{\partial}{\partial x^i}\Big|_x \in TM \setminus \{0\}$, define a vector $G \in Ty(TM)$ by $G = y^i \left.\frac{\partial}{\partial x^i}\right|_y - 2G^i(y) \left.\frac{\partial}{\partial y^i}\right|_y$, where G^i are local functions defined by

$$G^i(y) = \frac{1}{4} g^{il}(y) \left[\frac{\partial^2 L^2}{\partial y^l \partial x^k}(y) y^k - \frac{\partial L^2}{\partial x^l}(y)\right].$$

G^i can also be expressed by

$$G^i(y) = \frac{1}{4} g^{il}(y) \left[2\frac{\partial g_{jl}}{\partial x^k}(y) - \frac{\partial g_{jk}}{\partial x^l}(y)\right] y^j y^k.$$

Note that G^i is positively homogeneous in $y \in T_xM$ for every $x \in M$, $G^i(ty) = t^2 G^i(y)$, $(\forall) t > 0$. The vector field G is called *Finsler spray.*

Let $(M, g(x,y))$ be a Finslerian manifold with Chern connection $\nabla(x,y)$. The theory of geodesics $(M, g(x,y), \nabla(x,y))$ is similar to the one of Riemannian manifolds.

Geodesics are locally minimal and any two points of a geodesically complete Finslerian manifold can be joined by a minimal geodesic.

Recall that the geodesics are characterized by the following differential system:

$$\frac{d^2 x^i}{dt^2} + 2G^i\left(\frac{dx}{dt}\right) = 0,$$

where $\gamma : I \subset \mathbb{R} \to M$, $\gamma(t) = = (x^1(t), \ldots, x^n(t))$ is a geodesic with constant speed parametrization.

1.3. Definition. Let us consider now a $\mathcal{C}^2$ function $f : M \to \mathbb{R}$. The *Hessian* of f is a map $\text{Hess}\, f : TM \to \mathbb{R}$ defined by

$$\text{Hess}\, f(x,y)(v,v) = \left.\frac{d^2}{ds^2}(f \circ \gamma)\right|_{s=0},$$

where $\gamma : (-\varepsilon, \varepsilon) \to M$ is the geodesic with $\frac{d\gamma}{ds}(0) = v \in \in T_xM$. In local coordinates, we have

$$\begin{aligned}\text{Hess}\, f(x,y)(v,v) \quad &= \frac{\partial^2 f}{\partial x^i \partial x^j}(x) \frac{d\gamma^i}{ds}(0) \frac{d\gamma^j}{ds}(0) + \\ &+ \frac{\partial f}{\partial x^i}(x) \frac{d^2\gamma^i}{ds^2}(0) = \frac{\partial^2 f}{\partial x^i \partial x^j}(x) v^i v^j - 2G^i \frac{\partial f}{\partial x^i}(x).\end{aligned}$$

1.4. Theorem. *Let A be an open totally convex subset of M ([5]), and $f : A \to \mathbb{R}$ a $\mathcal{C}^2$ function. Then f is convex if and only if Hess f is weakly positive semidefinite.*

1.5. Definition. The function f is called *linear* (convex and concave simultaneoulsy) if

$$\text{Hess}\, f(x,y)(y,y) = 0, \quad \forall x \in M, \quad \forall y \in T_xM.$$

In this paper we use the following

Remark. If $(M, g(x))$ is a Riemannian manifold and ϕ is a strictly positive $\mathcal{C}^\infty$ function, homogeneous of degree zero, then $(M, g(x,y) = \phi(y)g(x))$ is a Finslerian manifold having the same geodesics as the Riemannian manifold $(M, g(x))$.

2 Finslerian convexity on $\mathbf{R}^n_+$ with respect to the affine Finslerian metric

Let $\mathbb{R}^n_+ = \{x = (x^1, \dots, x^n) \in \mathbb{R}^n \mid x^i > 0,\ i = \overline{1,n}\}$ denote *the positive orthant*, which is considered as a subset of $\mathbb{R}^n \setminus \{0\}$.

Let $\mathbb{R}^n \setminus \{0\}$ endowed with the affine Finslerian metric $g(x,y)$ of components

$$\begin{cases} g_{ii}(x,y) = \phi(y) \cdot \dfrac{1}{(x^i)^2},\ i = \overline{1,n} \\ g_{ij}(x,y) = 0, \quad i \neq j, \end{cases}$$

where $\phi(y)$ is a strictly positive $\mathcal{C}^\infty$ homogeneous function of degree zero.

Theorem 2.1. *$(\mathbb{R}^n_+, g(x,y))$ is totally convex in $(\mathbb{R}^n \setminus \{0\}, g(x,y))$.*

Proof. We use the formula

$$G^i(y) = \frac{1}{4} g^{il}(y) \left[\frac{2g_{jl}}{\partial x^k}(y) - \frac{\partial g_{jk}}{\partial x^l}(y) \right] y^j y^k.$$

$$g^{ij}(x,y) = \begin{cases} \dfrac{(x^i)^2}{\phi(y)}, & i = j \\ 0, & i \neq j. \end{cases}$$

Hence $g^{il} \neq 0$ only for $l = i$. Similary, $\dfrac{\partial g_{jl}}{\partial x^k} \neq 0$ only for $j = l = k$ and $\dfrac{\partial g_{jk}}{\partial x^l} \neq 0$ only for $j = l = k$. Similarly

$$\begin{aligned} G^i(y) &= \frac{1}{4} g^{ii}(y) \left[\frac{2g_{ii}}{\partial x^i}(y) - \frac{\partial g_{ii}}{\partial x^i}(y) \right] (y^i)^2 = \\ &= \frac{1}{4} \frac{(x^i)^2}{\phi(y)} \cdot \phi(y) \frac{-2}{(x^i)^3} (y^i)^2 = -\frac{(y^i)^2}{2x^i}, \quad \forall i = \overline{1,n}. \end{aligned}$$

The geodesics are described by

$$\begin{aligned}&\frac{d^2x^i}{dt^2}+2G^i\left(\frac{dx}{dt}\right)=0\Leftrightarrow x''^i(t)=\frac{[x'^i(t)]^2}{x^i(t)},\ t\in[t_1,t_2],\ i=\overline{1,n}\Leftrightarrow\\&\Leftrightarrow\left[\frac{x^i(t)}{x'^i(t)}\right]'=0\Leftrightarrow\frac{x^i(t)}{x'^i(t)}=\frac{1}{a_i}\Leftrightarrow[x^i(t)\cdot e^{-a_it}]'=0\Leftrightarrow x^i(t)e^{-a_it}=e^{b_i}\Leftrightarrow\\&\Leftrightarrow x^i(t)=e^{a_it+b_i},\quad\forall i=\overline{1,n}.\end{aligned}$$

It follows $t_1=-\infty$, $t_2=\infty$, i.e., each geodesic is defined on the whole $\mathbb{R}$. If we consider two arbitrary points $x,z\in\mathbb{R}^n_+$, then a geodesic joining them, i.e. $\gamma(0)=x$, $\gamma(1)=z$, is

$$\begin{aligned}\gamma(t)&=(x^1e^{(\ln z^1-\ln x^1)t},\ldots,x^ne^{(\ln z^n-\ln x^n)t})\\&=(x_1^{1-t}\cdot z_1^t,\ldots,x_n^{1-t}z_n^t),\quad t\in[0;1]\end{aligned}$$

and the coordinate functions are positive, so their values are completely included in $\mathbb{R}^n_+$.

Theorem 2.2. *The domain* $S=\{x\in\mathbb{R}^n_+\mid x^1+\ldots+x^n\le 1\}$ *is totally convex in* $(\mathbb{R}^n_+,g(x,y))$.

Proof. We consider $f(x)=x^1+\ldots+x^n$; $\dfrac{\partial f}{\partial x^i}(x)=1$, $\dfrac{\partial^2 f}{\partial x^i\partial x^j}(x)=0$, $\forall i,j=\overline{1,n}$, we easily infer

$$\begin{aligned}&\operatorname{Hess}f(x,y)(y,y)=\frac{\partial^2 f}{\partial x^i\partial x^j}(x)y^iy^j-2G^i(y)\frac{\partial f}{\partial x^i}(x)=\\&=0-\sum_{i=1}^n 2\cdot\frac{-(y^i)^2}{2x^i}\cdot 1=\frac{(y^1)^2}{x^1}+\ldots+\frac{(y^n)^2}{x^n}\ge 0,\quad\forall x\in\mathbb{R}^n_+,\ \forall y\in\mathbb{R}^n.\end{aligned}$$

Hess f is weakly positive semidefinite and hence f is convex. But the sublevel sets of a convex function are totally convex.

Theorem 2.3. *The function* $f:\mathbb{R}^n_+\to\mathbb{R}$, $f(x)=\displaystyle\sum_{i=1}^n\ln x^i$ *is linear on* $(\mathbb{R}^n_+,g(x,y))$.

Proof. For this function, $\dfrac{\partial f}{\partial x^i}=\dfrac{1}{x^i}$, $\dfrac{\partial^2 f}{\partial (x^i)^2}=\dfrac{-1}{(x^i)^2}$, $\dfrac{\partial^2 f}{\partial x^i\partial x^j}=0$, $\forall i\ne j$, hence

$$\begin{aligned}\operatorname{Hess}f(x,y)(y,y)&=\frac{\partial^2 f}{\partial x^i\partial x^j}(x)y^iy^j-2G^i(y)\frac{\partial f}{\partial x^i}(x)=\\&=\frac{-1}{(x^1)^2}(y^1)^2+\frac{-1}{(x^2)^2}(y^1)^2+\ldots+\frac{-1}{(x^n)^2}(y^n)^2+\\&\quad+\frac{(y^1)^2}{x^1}\cdot\frac{1}{x^1}+\frac{(y^2)^2}{x^2}\cdot\frac{1}{x^2}+\ldots+\frac{(y^n)^2}{x^n}\cdot\frac{1}{x^n}=0.\end{aligned}$$

Theorem 2.4. *The function* $f : \mathbb{R}^n_+ \to (0;\infty),\ f(x) = \prod_{i=1}^{n-1} x^i$ *is convex on* $(\mathbb{R}^n_+, g(x,y))$.

Proof. For this function $\frac{\partial f}{\partial x^i} = x_1 x_2 \cdot \ldots \cdot \hat{x}_i \cdot \ldots \cdot x_{n-1}$, $\forall i = \overline{1, n-1}$, $\frac{\partial f}{\partial x^n} = 0$, $\frac{\partial^2 f}{\partial (x^i)^2} = 0$, $\forall i = \overline{1, n-1}$,

$$\frac{\partial^2 f}{\partial x^i \partial x^j} = x_1 x_2 \cdot \ldots \cdot \hat{x}^i \cdot \ldots \cdot \hat{x}^j \cdot \ldots \cdot x^{n-1},\ \forall i \neq j,\ i,j \in \{1, \ldots, n-1\}.$$

$$\operatorname{Hess} f(x,y)(y,y) = 2 \sum_{1 \le i < j \le n-1} x^1 x^2 \cdot \ldots \cdot \hat{x}^i \cdot \ldots \cdot \hat{x}^j \cdot \ldots \cdot x^{n-1} y^i y^j +$$
$$+ \frac{(y^1)^2}{x^1} \cdot x^2 \cdot \ldots \cdot x^{n-1} + \frac{(y^2)^2}{x^2} \cdot x^1 x^3 \cdot \ldots \cdot x^{n-1} + \ldots + \frac{(y^{n-1})^2}{x^{n-1}} \cdot$$
$$\cdot x^1 x^2 \cdot \ldots \cdot x^{n-2} = \frac{\left(\sum_{1 \le i < j \le n-1} x^1 x^2 \cdot \ldots \cdot \hat{x}^i \cdot \ldots \cdot \hat{x}^j \cdot \ldots \cdot x^{n-1} y^i y^j \right)^2}{x^1 x^2 \cdot \ldots \cdot x^{n-1}} \ge 0,$$

$\forall x \in \mathbb{R}^n_+,\ \forall y \in \mathbb{R}^n_+$.

Theorem 2.5. *Let* $\mu > 0$. *The logbarrier function*

$$f : \mathbb{R}^n_+ \to R,\ f(x) = \sum_{i=1}^n c_i x^i - \mu \sum_{i=1}^n \ln x^i, \quad c_1, \ldots, c_k \ge 0,\ c_{k+1}, \ldots, c_n < 0$$

is convex with respect to the Finslerian metric $\tilde{g}_{ii}(x,y) = \frac{\phi(y)}{(x^i)^2}$, $\forall i = \overline{1,k}$, $\tilde{g}_{ii}(x,y) = \phi(y)$, $\forall i = \overline{k+1,n}$, $\tilde{g}_{ij}(x,y) = 0$, $\forall i \neq j$.

Proof. $\tilde{g}^{ii}(x,y) = \frac{(x^i)^2}{\phi(y)}$, $\forall i = \overline{1,k}$, $\tilde{g}^{ii}(x,y) = \frac{1}{\phi(y)}$, $\forall i = \overline{k+1,n}$ and $\tilde{g}^{ij}(x,y) = 0$, $\forall i \neq j$. Hence $\tilde{g}^{il} \neq 0$ only for $l = i$, $\frac{\partial \tilde{g}_{jl}}{\partial x^p} \neq 0$ only for $j = l = p \in \{1, \ldots, k\}$. Thus, $\forall i = \overline{1,k}$,

$$G^i(y) = \frac{1}{4} \tilde{g}^{ii}(y) \cdot \cdot \frac{\partial \tilde{g}_{ii}}{\partial x^i}(y) \cdot (y^i)^2 = \frac{1}{4} \cdot \frac{(x^i)^2}{\phi(y)} \cdot \phi(y) \cdot \cdot \frac{-2}{(x^i)^3} \cdot (y^i)^2 = -\frac{(y^i)^2}{2x^i}$$

and $\forall i \in \{k+1, \ldots, n\}$, $G^i(y) = 0$. For the logbarrier function we obtain $\frac{\partial f}{\partial x^i} =$

$c_i - \dfrac{\mu}{x^i}$, $\dfrac{\partial^2 f}{\partial (x^i)^2} = \dfrac{\mu}{(x^i)^2}$, $\forall i = \overline{1,n}$ and $\dfrac{\partial^2 f}{\partial x^i \partial x^j} = 0$, $\forall i \neq j$. Hence

$$\begin{aligned}
&\mathrm{Hess} f(x,y)(y,y) = \frac{\mu}{(x^1)^2}(y^1)^2 + \frac{\mu}{(x^2)^2}(y^2)^2 + \ldots + \frac{\mu}{(x^n)^2}(y^n)^2 + \\
&+ \frac{(y^1)^2}{x^1}\left(c_1 - \frac{\mu}{x^1}\right) + \frac{(y^2)^2}{x^2}\left(c_2 - \frac{\mu}{x^2}\right) + \ldots + \frac{(y^k)^2}{x^k}\left(c_k - \frac{\mu}{x^k}\right) = \\
&= \frac{\mu}{(x^{k+1})^2}(y^{k+1})^2 + \ldots + \frac{\mu}{(x^n)^2}(y^n)^2 + c_1\frac{(y^1)^2}{x^1} + \ldots + c_k\frac{(y^k)^2}{x^k} \geq 0, \\
&\quad \forall x \in \mathbb{R}^n_+, \quad \forall y \in \mathbb{R}^n.
\end{aligned}$$

3 Finslerian convexity on $\mathbf{R}^n_+$ with respect to other Finsler metrics

Let $\mathbb{R}^n \setminus \{0\}$ endowed with the Finslerian metric components of

$$\begin{cases} g_{ii}(x,y) = \dfrac{\phi(y)}{(x^i)^4}, & i = \overline{1,n} \\ g_{ij}(x,y) = 0, & \forall i \neq j, \end{cases}$$

where $\phi(y)$ is also a strictly positive C^∞ homogeneous function of degree zero.

Theorem 3.1. *$(\mathbb{R}^n_+, g(x,y))$ is totally convex in $(\mathbb{R}^n \setminus \{0\}, g(x,y))$.*

Proof. $g^{ii}(x,y) = \dfrac{(x^i)^4}{\phi(y)}$, $i = \overline{1,n}$, $g^{ij}(x,y) = 0$, $\forall i \neq j$. Thus, $\forall i = \overline{1,n}$,

$$G^i(y) = \frac{1}{4} g^{ii}(x,y) \cdot \frac{\partial g_{ii}}{\partial x^i}(y)(y^i)^2 = \frac{1}{4}\frac{(x^i)^4}{\phi(y)}\phi(y)\frac{-4}{(x^i)^5}(y^i)^2 = \frac{-(y^i)^2}{x^i}.$$

The geodesic arcs are given by

$$\frac{d^2 x^i}{dt^2} + 2G^i\left(\frac{dx}{dt}\right) = 0 \Leftrightarrow x''^i(t) - \frac{2[x'^i(t)]^2}{x^i(t)} = 0 \quad t \in [t_1, t_2] \Leftrightarrow$$

$$\begin{aligned}
\frac{x''^i(t)}{x'^i(t)} &= 2\frac{x'^i(t)}{x^i(t)} \Leftrightarrow \ln x'^i(t) = 2\ln x^i(t) + \ln c_i^2 \Leftrightarrow \\
&\Leftrightarrow \quad x'^i(t) = [c_i x^i(t)]^2 \Leftrightarrow \frac{x'^i(t)}{[x^i(t)]^2} = c_i^2 \Leftrightarrow \left(\frac{1}{x^i(t)}\right)' = c_i^2 \Leftrightarrow \\
&\Leftrightarrow \quad \frac{1}{x^i(t)} = c_i^2 t + b_i,\ c_i^2 = a_i \to x^i(t) = \frac{1}{a_i t + b_i},\ \forall i = \overline{1,n},\ t \in [t_1, t_2].
\end{aligned}$$

It follows that the solution $\gamma(t) = (x^1(t), \ldots, x^n(t))$ is defined on the whole $\mathbb{R}$.

If we consider two arbitrary points $x, z \in \mathbb{R}^n_+$, then a geodesic joining them, i.e. $\gamma(0) = x$, $\gamma(1) = z$, is

$$\gamma(t) = \left(\frac{1}{\left(\frac{1}{z_1} - \frac{1}{x_1}\right) t + \frac{1}{x_1}}, \ldots, \frac{1}{\left(\frac{1}{z_n} - \frac{1}{x_n}\right) t + \frac{1}{x_n}} \right) =$$

$$= \left(\frac{x_1 z_1}{t x_1 + (1-t) z_1}, \ldots, \frac{x_n z_n}{t x_n + (1-t) z_n} \right), \quad t \in [0;1]$$

and the coordinate functions are positive, so their values are completely included in $\mathbb{R}^n_+$.

Theorem 3.2. *The domain* $S : x^1 + \ldots + x^n \leq 1$ *is totally convex in* $(\mathbb{R}^n_+, g(x,y))$.

Proof. We consider $f(x) = x^1 + \ldots + x^n$; $\dfrac{\partial f}{\partial x^i}(x) = 1$, $\dfrac{\partial^2 f}{\partial x^i \partial x^j}(x) = 0$, $\forall i, j = \overline{1, n}$ so

$$\text{Hess} f(x,y)(y,y) = \frac{\partial^2 f}{\partial x^i \partial x^j}(x) y^i y^j - 2G^i(y) \frac{\partial f}{\partial x^i}(x) =$$

$$= 0 - \sum_{i=1}^{n} 2 \cdot \frac{-(y^i)^2}{x^i} \cdot 1 = 2 \left[\frac{(y^1)^2}{x^1} + \ldots + \frac{(y^n)^2}{x^n} \right] \geq 0, \forall x \in \mathbb{R}^n_+, \forall y \in \mathbb{R}^n .$$

Hence f is convex and the sublevel sets of a convex function are totally convex.

Theorem 3.3. *The function* $f : \mathbb{R}^n_+ \to \mathbb{R}$, $f(x) = \sum_{i=1}^{n} \ln x^i$ *is convex on* $(\mathbb{R}^n_+, g(x,y))$.

Proof. $\dfrac{\partial f}{\partial x^i} = \dfrac{1}{x^i}$, $\dfrac{\partial^2 f}{\partial (x^i)^2} = \dfrac{-1}{(x^i)^2}$, $\dfrac{\partial^2 f}{\partial x^i \partial x^j} = 0$, $\forall i \neq j$, hence

$$\begin{aligned} \text{Hess} f(x,y)(y,y) &= \frac{-1}{(x^1)^2}(y^1)^2 - \frac{1}{(x^2)^2}(y^2)^2 - \ldots - \frac{1}{(x^n)^2}(y^n)^2 + \\ &+ 2\frac{(y^1)^2}{x^1} \cdot \frac{1}{x^1} + 2\frac{(y^2)^2}{x^2} \cdot \frac{1}{x^2} + \ldots + 2\frac{(y^n)^2}{x^n} \cdot \frac{1}{x^n} = \\ &= \frac{(y^1)^2}{(x^1)^2} + \ldots + \frac{(y^n)^2}{(x^n)^2} \geq 0, \quad \forall x \in \mathbb{R}^n_+, \forall y \in \mathbb{R}^n . \end{aligned}$$

Theorem 3.4. *The function* $f : \mathbb{R}^n_+ \to (0, \infty)$, $f(x) = \prod_{i=1}^{n-1} x^i$ *is convex on* $(\mathbb{R}^n_+, g(x,y))$.

Proof. Using the result from Theorem 2.4, we obtain

$$\begin{aligned}
&\operatorname{Hess} f(x,y)(y,y) = 2 \sum_{1\le i<j\le n-1} (x^1 x^2 \cdot \ldots \cdot \hat{x}_i \cdot \ldots \cdot \hat{x}_j \cdot \ldots \cdot x_{n-1} y^i y^j) + \\
&+2 \cdot \frac{(y^1)^2}{x^1} \cdot x^2 \cdot \ldots \cdot x^{n-1} + 2\frac{(y^2)^2}{x^2} \cdot x^1 x^3 \cdot \ldots \cdot x^{n-1} + \ldots + 2\frac{(y^{n-1})^2}{x^{n-1}} \cdot \\
&\cdot x^1 x^2 \cdot \ldots \cdot x^{n-2} = \frac{\left(\displaystyle\sum_{1\le i<j\le n-1} (x^1 x^2 \cdot \ldots \cdot \hat{x}^i \cdot \ldots \cdot \hat{x}^j \cdot \ldots \cdot x^{n-1} y^i y^j \right)^2}{x^1 x^2 \cdot \ldots \cdot x^{n-1}} + \\
&+\frac{(y^1)^2}{x^1} \cdot x^2 \cdot \ldots \cdot x^{n-1} + \frac{(y^{n-1})^2}{x^{n-1}} x^1 x^2 \cdot \ldots \cdot x^{n-2} \ge 0, \\
&\forall x \in \mathbb{R}^n_+,\ \forall y \in \mathbb{R}^n.
\end{aligned}$$

Now we consider the Finslerian metric of components

$$\begin{cases} g_{ii}(x,y) = \dfrac{\phi(y)}{(x^i)^{\frac{2(p+1)}{p}}}, & i = \overline{1,n} \\[2ex] g_{ij}(x,y) = 0, & \forall i \ne j, \end{cases}$$

where $\phi(y)$ is also a strictly positive C^∞ homogeneous function of degree zero and $p \in \mathbb{R}$, $p > 0$.

Theorem 3.5. $(\mathbb{R}^n_+, g(x,y))$ *is totally convex in* $(\mathbb{R}^n \setminus \{0\}, g(x,y))$.

Proof. $g^{ii}(x,y) = \dfrac{(x^i)^{\frac{2(p+1)}{p}}}{\phi(y)}$, $\forall i = \overline{1,n}$, $g^{ij}(x,y) = 0$, $\forall i \ne j$. Thus, $\forall i = \overline{1,n}$,

$$\begin{aligned}
G^i(y) &= \frac{1}{4} g^{ii}(x,y) \frac{\partial g_{ii}}{\partial x^i}(y)(y^i)^2 = \\
&= \frac{1}{4} \cdot \frac{(x^i)^{\frac{2(p+1)}{p}}}{\phi(y)} \cdot \phi(y) \cdot \frac{-2(p+1)}{p} \cdot \frac{1}{(x^i)^{\frac{2(p+1)}{p}+1}} \cdot (y^i)^2 = \frac{-(p+1)}{2p} \cdot \frac{(y^i)^2}{x^i}.
\end{aligned}$$

The geodesic arc is given by

$$\frac{d^2x^i}{dt^2} + 2G^i\left(\frac{dx}{dt}\right) = 0 \Leftrightarrow x''^i(t) - 2\cdot\frac{p+1}{2p}\cdot\frac{[x'^i(t)]^2}{x^i(t)} = 0 \Leftrightarrow$$

$$\Leftrightarrow \frac{x''^i(t)}{x'^i(t)} = \frac{p+1}{p}\cdot\frac{x'^i(t)}{x^i(t)} \Leftrightarrow \ln x'^i(t) = \frac{p+1}{p}\ln x^i(t) + \ln a_i \Leftrightarrow$$

$$\Leftrightarrow x'^i(t) = a_i[x^i(t)]^{\frac{p+1}{p}} \Leftrightarrow \frac{x'^i(t)}{[x^i(t)]^{\frac{p+1}{p}}} = a_i \Leftrightarrow$$

$$\Leftrightarrow \frac{(x^i(t))^{-\frac{p+1}{p}+1}}{-\dfrac{p+1}{p}+1} = a_it + b_i \Leftrightarrow [x^i(t)]^{-\frac{1}{p}} = -\frac{1}{p}(a_it + b_i) \Leftrightarrow$$

$$\Leftrightarrow x^i(t) = \frac{(-p)^p}{(a_it+b_i)^p} = \frac{1}{\left(\dfrac{a_i}{-p}t + \dfrac{b_i}{-p}\right)^p} =$$

$$= \frac{1}{(m_it+n_i)^p}, \quad \forall i = \overline{1,n}, \quad t \in [t_1, t_2].$$

It follows that the solution $\gamma(t) = (x^1(t), \ldots, x^n(t))$ is defined on the whole $\mathbb{R}$.

If we consider two arbitrary points $x, z \in \mathbb{R}^n_+$, then a geodesic joining them, i.e. $\gamma(0) = x$, $\gamma(1) = z$, is

$$\begin{aligned}\gamma(t) &= \left(\frac{1}{\left[\left(\frac{1}{z^1}\right)^{\frac{1}{p}} - \left(\frac{1}{x^1}\right)^{\frac{1}{p}}\right]t + \left(\frac{1}{x^1}\right)^{\frac{1}{p}}}, \ldots, \frac{1}{\left[\left(\frac{1}{z^n}\right)^{\frac{1}{p}} - \left(\frac{1}{x^n}\right)^{\frac{1}{p}}\right]t + \left(\frac{1}{x^n}\right)^{\frac{1}{p}}}\right) = \\ &= \left(\frac{(x^1)^{\frac{1}{p}}(z^1)^{\frac{1}{p}}}{t(x^1)^{\frac{1}{p}} + (1-t)(z^1)^{\frac{1}{p}}}, \ldots, \frac{(x^n)^{\frac{1}{p}}(z^n)^{\frac{1}{p}}}{t(x^n)^{\frac{1}{p}} + (1-t)(z^n)^{\frac{1}{p}}}\right), \ t \in [0;1]\end{aligned}$$

and the coordinate functions are positive, so their values are completely included in $\mathbb{R}^n_+$.

Theorem 3.6. *The domain* $S : x^1 + \ldots + x^n \leq 1$ *is totally convex in* $(\mathbb{R}^n_+, g(x,y))$.

Proof. By direct computation, for $f(x) = x^1 + \ldots + x^n$, we have

$$\begin{aligned}\mathrm{Hess} f(x,y)(y,y) &= \frac{\partial^2 f}{\partial x^i \partial x^j}(x)y^iy^j - 2G^i(y)\frac{\partial f}{\partial x^i}(x) = \\ &= 0 - 2\cdot -\frac{-(p+1)}{2p}\cdot\frac{(y^i)^2}{x^i}\cdot 1 - \ldots - 2\cdot\frac{-(p+1)}{2p}\cdot\frac{(y^n)^2}{x^n} = \\ &= \frac{p+1}{p}\left[\frac{(y^1)^2}{x^1} + \ldots + \frac{(y^n)^2}{x^n}\right] \geq 0, \quad \forall x \in \mathbb{R}^n_+, \forall y \in \mathbb{R}^n .\end{aligned}$$

Hence f is convex and the sublevel sets of a convex function are totally convex.

Theorem 3.7. *The function* $f : \mathbb{R}^n_+ \to \mathbb{R}$, $f(x) = \sum_{i=1}^{n} \ln x^i$ *is convex on* $(\mathbb{R}^n_+, g(x,y))$.

Proof.

$$\mathrm{Hess} f(x,y)(y,y) = \frac{-1}{(x^1)^2}(y^1)^2 + \frac{-1}{(x^2)^2}(y^2)^2 + \ldots + \frac{-1}{(x^n)^2}(y^n)^2 +$$

$$+2 \cdot \frac{p+1}{2p}\frac{(y^1)^2}{x^1} \cdot \frac{1}{x^1} + 2 \cdot \frac{p+1}{2p}\frac{(y^2)^2}{x^2} \cdot \frac{1}{x^2} + \ldots + 2 \cdot \frac{p+1}{2p}\frac{(y^n)^2}{x^n} \cdot \frac{1}{x^n} =$$

$$= \frac{1}{p}\left[\frac{(y^1)^2}{(x^1)^2} + \ldots + \frac{(y^n)^2}{(x^n)^2}\right] \geq 0, \quad \forall x \in \mathbb{R}^n_+, y \in \mathbb{R}^n.$$

Theorem 3.8. *The function* $f : \mathbb{R}^n_+ \to (0;\infty)$, $f(x) = \prod_{i=1}^{n-1} x^i$ *is convex on* $(\mathbb{R}^n_+, g(x,y))$.

Proof.

$$\mathrm{Hess}\, f(x,y)(y,y) = 2 \sum_{1 \leq i < j \leq n-1} (x^1 x^2 \cdot \ldots \cdot \hat{x}^i \cdot \ldots \cdot \hat{x}^j \cdot \ldots \cdot x^{n-1} y^i y^j) +$$

$$+2 \cdot \frac{p+1}{2p}\frac{(y^1)^2}{x^1} \cdot x^2 \cdot \ldots \cdot x^{n-1} + 2\frac{p+1}{2p}\frac{(y^2)^2}{x^2} \cdot x^1 x^3 \cdot \ldots \cdot x^{n-1} + \ldots$$

$$\ldots + 2\frac{p+1}{2p}\frac{(y^{n-1})^2}{x^{n-1}} \cdot x^1 x^2 \cdot \ldots \cdot x^{n-2} =$$

$$= \frac{\left(\sum_{1 \leq i < j \leq n-1} (x^1 x^2 \cdot \ldots \cdot \hat{x}^i \cdot \ldots \cdot \hat{x}^j \cdot \ldots \cdot x^{n-1} y^i y^j\right)^2}{x^1 x^2 \cdot \ldots \cdot x^{n-1}} +$$

$$+\frac{1}{p}\left[\frac{(y^1)^2}{x^1} \cdot x^2 \cdot \ldots \cdot x^{n-1} + \ldots + \frac{(y^{n-1})^2}{x^{n-1}} x^1 x^2 \cdot \ldots \cdot x^{n-2}\right] \geq 0,$$

$\forall x \in \mathbb{R}^n_+, y \in \mathbb{R}^n,\ p > 0.$

4 Finslerian convexity of separable functions

Let us consider an open totally convex set $A \subseteq \mathbb{R}^n$ and two *separable* functions

$$f(x) = \sum_{i=1}^{n} f_i(x^i),\ g(x) = \sum_{i=1}^{n} g_i(x^i),$$

where $x \in A \subseteq \mathbb{R}^n$, $f_i, g_i \in C^2$, $\forall i = \overline{1,n}$.

Theorem 4.1. *A separable function* $f \in C^2$ *defined on an open totally convex set* $A \subseteq \mathbb{R}^n$ *is convex with respect to the Finsler metric* $g(x,y)$ *of components*

$$\begin{aligned} &g_{ii}(x,y) = \phi(y) \cdot e^{-2cg_i(x^i)}, \quad \forall i = \overline{1,n}, \\ &g_{ij}(x,y) = 0,\ \forall i \neq j,\ x \in A,\ c \geq 0 \text{ constant} \end{aligned}$$

if and only if the inequalities $f_i'' + cf_i'g_i' \geq 0$, $\forall i = \overline{1,n}$ *hold on* A.

Proof.

$$\begin{aligned} G^i(y) &= \frac{1}{4} g^{ii}(x,y) \frac{\partial g_{ii}}{\partial x^i}(y)(y^i)^2 = \\ &= \frac{1}{4} \cdot \frac{e^{2cg_i(x^i)}}{\phi(y)} \cdot \phi(y) \cdot e^{-2cg_i(x^i)} \cdot [-2cg_i'(x^i)](y^i)^2 \\ &= -\frac{1}{2} cg_i'(x^i)(y^i)^2, \quad \forall i = \overline{1,n}. \end{aligned}$$

But $f(x^1, \ldots, x^n) = f_1(x^1) + f_2(x^2) + \ldots + f_n(x^n)$,

$$\frac{\partial f}{\partial x^i} = \frac{\partial f_i}{\partial x^i} = f_i', \quad \frac{\partial^2 f}{\partial (x^i)^2} = \frac{\partial^2 f_i}{\partial (x^i)^2} = f_i'', \quad \frac{\partial^2 f}{\partial x^j \partial x^i} = 0, \quad \forall j \neq i.$$

Hence

$$\begin{aligned} &\text{Hess} f(x,y)(y,y) = \frac{\partial^2 f}{\partial x^i \partial x^j}(x) y^i y^j - 2G^i(y) \frac{\partial f}{\partial x^i}(x) = \\ &= \frac{\partial^2 f}{\partial (x^1)^2}(y^1)^2 + \ldots + \frac{\partial^2 f}{\partial (x^n)^2}(y^n)^2 - 2 \cdot \frac{-1}{2} cg_1'(x^1)(y^1)^2 \cdot \frac{\partial f}{\partial x^1} - \\ &\quad - \ldots - 2 \cdot \frac{-1}{2} cg_n'(x^n)(y^n)^2 \cdot \frac{\partial f}{\partial x^n} = f_1''(y^1)^2 + \ldots + f_n''(y^n)^2 + \\ &\quad + cg_1'(y^1)^2 f_1' + \ldots + cg_n'(y^n)^2 f_n' = \\ &= (f_1'' + cg_1'f_1')(y^1)^2 + \ldots + (f_n'' + cg_n'f_n')(y^n)^2 \geq 0 \Leftrightarrow \\ &\Leftrightarrow f_i'' + cf_i'g_i' \geq 0, \quad \forall i = \overline{1,n}. \end{aligned}$$

Corollary 4.2. *A separable function* $f \in C^2$ *defined on an open geodesic convex set* $A \subseteq \mathbb{R}^n$ *is convex with respect to the Finsler metric* $g(x,y)$ *of components*

$$\begin{aligned} &g_{ii}(x,y) = \phi(y) \cdot e^{-2cf_i(x^i)}, \quad \forall i = \overline{1,n} \\ &g_{ij}(x,y) = 0,\ \forall i \neq j, \end{aligned}$$

$x \in A$, $c \geq 0$ *(constant) iff the inequalities* $f_i'' + c(f_i')^2 \geq 0$, $i = \overline{1,n}$ *hold on* A.

Proof. We consider $g_i = f_i$, $\forall i = \overline{1,n}$ in the above theorem.

Remark. 1) In order to show, e.g., that the Finslerian metric from the Theorem 3.5 belong to the metric class from Theorem 4.1, set $c = \dfrac{p+1}{p}$ and $g_i(x_i) = \ln(x_i)$, $\forall i = \overline{1,n}$.

2) It is worthwhile being careful with the application of Theorem 4.1, because the Finsler metric from these theorems does not always lead to a complete Finsler connection, i.e., the geodesics are not geodesically complete in every case. For example, if we take $g_i(x^i) = x^i$, $\forall i = \overline{1,n}$ in the Theorem 4.1, then

$$G^i(y) = -\frac{1}{2}c(y^i)^2, \ \ \forall i = \overline{1,n}$$

and the differential equations of geodesic arcs are

$$\frac{d^2x^i}{dt^2} + 2G^i\left(\frac{dx}{dt}\right) = 0, \ \forall i = \overline{1,n}$$

or, equivalently,

$$x''^i(t) - c[x'^i(t)]^2 = 0, \ t \in [t_1; t_2], \ \forall i = \overline{1,n}.$$

It follows that the solution $\gamma(t) = (x^2(t), \ldots, x^n(t))$, is given by $x^i(t) = \dfrac{1}{c}\ln(a_i t + b_i)$, $\forall i = \overline{1,n}$, $t \in \mathbb{R}$, which means that the definition domain of the geodesics must satisfy the positivity of the values $a_i t + b_i$, $\forall i = \overline{1,n}$.

5 Finslerian convexity of Rosenbrock banana function

Let us consider $\mathbb{R}^2$ endowed with the Finsler metric

$$g(x,y) = \begin{pmatrix} \phi(y)[4(x^1)^2 + 1] & -2\phi(y)x^2 \\ -2\phi(y)x^1 & \phi(y) \end{pmatrix},$$

where $\phi(y)$ is also a strictly positive C^∞ homogeneous function of degree zero.

Theorem 5.1. *The Rosenbrock banana function* $f : \mathbb{R}^2 \to \mathbb{R}$,

$$f(x^1, x^2) = 100[x^2 - (x^1)^2] + (1 - x^1)^2$$

is convex with respect to g.

Proof.

$$
\begin{aligned}
&G^1(y) = \frac{1}{4} g^{1l}(y) \left[2 \frac{\partial g_{jl}}{\partial x^k}(y) - \frac{\partial g_{jk}}{\partial x^l} \right] y^j y^k = \\
&= \frac{1}{4} g^{11}(y) \left(2 \frac{\partial g_{11}}{\partial x^1} - \frac{\partial g_{11}}{\partial x^1} \right) (y^1)^2 + \frac{1}{4} g^{11} \left(2 \frac{\partial g_{11}}{\partial x^2} - \frac{\partial g_{12}}{\partial x^1} \right) y^1 y^2 + \\
&\quad + \frac{1}{4} g^{11} \left(2 \frac{\partial g_{21}}{\partial x^1} - \frac{\partial g_{21}}{\partial x^1} \right) y^2 y^1 + \frac{1}{4} g^{11} \left(2 \frac{\partial g_{21}}{\partial x^2} - \frac{\partial g_{22}}{\partial x^1} \right) (y^2)^2 + \\
&\quad + \frac{1}{4} g^{12} \left(2 \frac{\partial g_{12}}{\partial x^1} - \frac{\partial g_{11}}{\partial x^2} \right) (y^1)^2 + \frac{1}{4} g^{12} \left(2 \frac{\partial g_{12}}{\partial x^2} - \frac{\partial g_{12}}{\partial x^2} \right) y^1 y^2 + \\
&\quad + \frac{1}{4} g^{12} \left(2 \frac{\partial g_{22}}{\partial x^1} - \frac{\partial g_{21}}{\partial x^2} \right) y^2 y^1 + \frac{1}{4} g^{12} \left(2 \frac{\partial g_{22}}{\partial x^2} - \frac{\partial g_{22}}{\partial x^2} \right) (y^2)^2,
\end{aligned}
$$

which rewrites

$$
\begin{aligned}
G^1(y) &= \frac{1}{4} \cdot \frac{1}{\phi(y)} \cdot \phi(y) \cdot 8x^1 (y^1)^2 - \frac{1}{4} \frac{1}{\phi(y)} \cdot [-2\phi(y)] y^1 y^2 + \\
&\quad + \frac{1}{4} \cdot \frac{1}{\phi(y)} \cdot [-2\phi(y)] y^1 y^2 + \frac{1}{4} \cdot \frac{2x^1}{\phi(y)} \cdot 2[-2\phi(y)] (y^1)^2 = \\
&= 2x^1 (y^1)^2 + \frac{1}{2} y^1 y^2 - \frac{1}{2} y^1 y^2 - 2x^1 (y^1)^2 = 0.
\end{aligned}
$$

Hence $G^1(y) = 0$. By a similar computation, $G^2(y) = -(y^1)^2$. But

$$
f(x^1, x^2) = 100(x^1)^4 + (x^1)^2 - 200x^2(x^1)^2 - 2x^1 + 100(x^2)^2 + 1
$$

and

$$
\frac{\partial f}{\partial x^1} = 400(x^1)^3 + 2x^1 - 400x^1x^2 - 2, \; \frac{\partial^2 f}{\partial (x^1)^2} = 1200(x^1)^2 + 2 - 400x^2,
$$

$$
\frac{\partial^2 f}{\partial x^1 \partial x^2} = -400x^1, \; \frac{\partial^2 f}{\partial x^2} = -200(x^1)^2 + 200x^2, \; \frac{\partial^2 f}{\partial (x^2)^2} = 200.
$$

Therefore we have

$$
\begin{aligned}
&\mathrm{Hess} f(x,y)(y,y) = \frac{\partial^2 f}{\partial (x^1)^2} (y^1)^2 + 2 \cdot \frac{\partial^2 f}{\partial x^1 \partial x^2} y^1 y^2 + \frac{\partial^2 f}{\partial (x^2)^2} (y^2)^2 - 2G^2(y) \frac{\partial f}{\partial x^2} = \\
&= [1200(x^1)^2 + 2 - 400x^2](y^1)^2 + 2(-400x^1) y^1 y^2 + 200(y^2)^2 + \\
&\quad + 2(y^1)^2 [-200(x^1)^2 + 200x^2] = \\
&= (y^1)^2 [800(x^1)^2 + 2] - 800x^1 y^1 y^2 + 200(y^2)^2 = \\
&= 200[4(x^1)^2 (y^1)^2 - 4x^1 y^1 y^2 + (y^2)^2] + 2(y^1)^2 = \\
&= 200(2x^1 y^1 - y^2)^2 + 2(y^1)^2 \geq 0, \quad \forall x, y \in \mathbb{R}^2.
\end{aligned}
$$

References

[1] D. Bao, S.S. Chern, Z. Shen, *An Introduction to Riemann-Finsler Geometry*, Springer, 2000.

[2] T. Rapcák, *Smooth Nonlinear Optimization in* $\mathbb{R}^n$, Kluwer Academic Publishers, Dordrecht, 1997.

[3] C. Udrişte, *Convex Functions and Optimization Methods on Riemannian Manifolds*, Kluwer Academic Publishers, Dordrecht, 1994.

[4] C. Udrişte, *Convex programs on Finsler manifolds*, in New Developments in Differential Geometry, J. Szenthe, ed., Kluwer Academic Publishers, Dordrecht, 1996, 443-458.

[5] C. Udrişte, *Finslerian convexity and optimization*, in Finslerian Geometries, P.L. Antonelli, ed., Kluwer Academic Publishers, Dordrecht, 2000, 283-196.

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