Bayes estimators for a multivariate survival model

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Abstract. In a system with three components which are independently on test with exponential life distributions,the failure of one reduces the mean life of the remaining. This system functions only as long as least one of three identical components functions. We established clasical estimators of maximum likelihood, Bayes with prior informative and also with noninformative prior.
Also, confidence intervals and credible intervals are established for both situations with informative prior and noninformative prior and a comparison between them is made.

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§1. Introduction

Let X, Y, Z denote the lifetimes of three components. Initially these three components have as their life distribution an exponential one with parameter λ.

Failure of the first changes the life distribution of the second in $Exp(\lambda\theta_1)$ and this changes the life distribution of the last component in $Exp(\lambda\theta_1\theta_2)$.

We assume that $X \sim Exp(\lambda)$, $Y \sim Exp(\theta_1\lambda)$, $Z \sim Exp(\theta_1\theta_2\lambda)$.

We define: $u = min(X,Y,Z)$, $v = \max\{\min(X,Y), \min(X,Z), \min(Y,Z)\} - \min(X,Y,Z)$, and $w = \max(X,Y,Z) - \max\{\min(X,Y), \min(X,Z), \min(Y,Z)\}$.

The joint density of u, v, w is:

$$f_{(u,v,w)}(u,v,w) = 6\lambda^3\theta_1^2\theta_2 \exp\{-3\lambda u - 2\lambda\theta_1 v - \lambda\theta_1\theta_2 w\} \quad , u, v, w > 0.$$

For a random sample $(x_1, y_1, z_1), \ldots, (x_n, y_n, z_n)$ size n the likelihood function is:

$$L(\lambda, \theta_1, \theta_2) = 3^n 2^n \lambda^{3n} \theta_1^{2n} \theta_2^n \exp(-3\lambda U - 2\lambda\theta_1 V - \lambda\theta_1\theta_2 W),$$

where $U = \sum\limits_1^n u_j$, $V = \sum\limits_1^n v_j$, $W = \sum\limits_1^n w_j$.

The estimators of maximum likelihood are:

$$\hat{\lambda} = n/(3U), \hat{\theta}_1 = (3U)/(2V), \hat{\theta}_2^{=}(2V)/W.$$

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Because U, V and W are sums of exponential variables, U is distributed Gamma$(3\lambda, n)$, $V \sim$ Gamma $(2\theta_1\lambda, n)$ and $W \sim$ Gamma$(\lambda\theta_1\theta_2, n)$. Then

$$M(\hat{\lambda}) = \lambda\frac{n}{n-1},\ M(\hat{\theta}_1) = \theta_1\frac{n}{n-1},\ M(\hat{\theta}_2) = \theta_2\frac{n}{n-1},$$

$$D(\hat{\lambda}) = \frac{\lambda^2 n^2}{(n-1)^2(n-2)};\ D(\hat{\theta}_1) = \frac{n(2n-1)\theta_1^2}{(n-1)^2(n-2)};\ D(\hat{\theta}_2) = \frac{n(2n-1)\theta_2^2}{(n-1)^2(n-2)}.$$

§2. Bayes estimators

One suppsoses that the priors are: for $\lambda \sim \mathrm{Exp}(\alpha_1)$, for $\theta_1|\lambda \sim \mathrm{Exp}(\lambda\alpha_2)$ and for $\theta_2|(\lambda, \theta_1) \sim \mathrm{Exp}(\lambda\theta_1\alpha_3)$.

In this case the posterior of $\lambda, \theta_1, \theta_2$ given U, V, W is given by:

$$\begin{aligned} g(\lambda, \theta_1, \theta_2|U, V, W) = \; & \frac{1}{(n!)^3}\lambda^{3n+2}\theta_1^{2n+1}\theta_2^{n}(3U+\alpha_1)^{n+1}(2V+\alpha_2)^{n+1}(W+\alpha_3)^{n+1}\cdot \\ & \cdot\exp\{-\lambda[3U + 2\theta_1 V + \theta_1\theta_2 W + \alpha_1 + \alpha_2\theta_1 + \alpha_3\theta_1\theta_2]\}. \end{aligned}$$

As well, in this case, the Bayes estimators of $\lambda, \theta_1, \theta_2$ for squared error loss are:

$$\tilde{\lambda} = \frac{n+1}{3U+\alpha_1};\quad \tilde{\theta}_1 = \frac{n+1}{n}\frac{3U+\alpha_1}{2V+\alpha_2} \quad \text{and} \quad \tilde{\theta}_2 = \frac{n+1}{n}\frac{2V+\alpha_2}{W+\alpha_3}.$$

§3. Bayes estimators with a Jeffreys noninformative prior given by the Fisher information matrix

We have

$$I = \sqrt{|\det M|},$$

where M is

$$M = \begin{pmatrix} \dfrac{\partial^2 \ln L}{\partial\lambda^2} & \dfrac{\partial^2 \ln L}{\partial\lambda\partial\theta_1} & \dfrac{\partial^2 \ln L}{\partial\lambda\partial\theta_2} \\ \dfrac{\partial^2 \ln L}{\partial\lambda\partial\theta_1} & \dfrac{\partial^2 \ln L}{\partial\theta_1^2} & \dfrac{\partial^2 \ln L}{\partial\theta_1\partial\theta_2} \\ \dfrac{\partial^2 \ln L}{\partial\lambda\partial\theta_2} & \dfrac{\partial^2 \ln L}{\partial\theta_1\partial\theta_2} & \dfrac{\partial^2 \ln L}{\partial\theta_2^2} \end{pmatrix}.$$

Then the noninformative prior distribution is:

$$p(\lambda, \theta_1, \theta_2) = \frac{n^{3/2}}{\lambda\theta_1\theta_2}.$$

For this case the posterior joint distribution of $\lambda, \theta_1, \theta_2$ is:

$\Gamma^3(n)\cdot P(\lambda, \theta_1, \theta_2|u, v, w) = U^n V^n W^n 6^n \lambda^{3n+1}\theta_1^{2n-1}\theta_2^{n-1}\exp\{-\lambda[3U+2\theta_1 V+\theta_1\theta_2 W]\}$,

and the Bayes estimators of squared error loss are:

$$\lambda^* = \frac{n}{3U};\quad \theta_1^* = \frac{3}{2}\frac{n}{n-1}\frac{U}{V} \quad \text{and} \quad \theta_2^* = \frac{2n}{n-1}\frac{V}{W},$$

which are very close to estimators of maximum likelihood.

§4. The confidence intervals and credible Bayes intervals

The confidence interval for λ is

$$\left(\frac{3}{2U}h_{\varepsilon/2}; \frac{3}{2U}h_{1-\varepsilon/2}\right),$$

where $F_{\chi^2(2n)}(h_{\varepsilon/2}) = \varepsilon/2$, and

$$\left(\frac{h_{\varepsilon/2}}{2(3U+\alpha_1)}; \frac{h_{1-\varepsilon/2}}{2(3U+\alpha_1)}\right) \ni \tilde{\lambda}$$

is a Bayesian credible interval in the case of informative prior.

Other better interval is given by: $(2n-2-a; 2n-2+b)$ with

$$\int_{2n-2-a}^{2n-2+b} g(\lambda|U,V,W)d\lambda \geq 1-\varepsilon;$$

$\lambda^* \in (\frac{h_{\varepsilon/2}}{6U}; \frac{h_{1-\varepsilon/2}}{6U})$ is a Bayesian credible interval in the case of noninformative prior.

The lengths of these intervals are respectively:

$$\frac{3}{2U}(h_{1-\varepsilon/2} - h_{\varepsilon/2}), \quad \frac{1}{2(3U+\alpha_1)}(h_{1-\varepsilon/2} - h_{\varepsilon/2}) \quad \text{and} \quad \frac{1}{6U}(h_{1-\varepsilon/2} - h_{\varepsilon/2}).$$

For θ_1, we have

$$\left(\frac{3U}{2Vb_{1-\varepsilon/2}}; \frac{3U}{2Vb_{\varepsilon/2}}\right) \quad , F_{Be(n+1,n+1)}\left(b_{\varepsilon/2}\right) = \varepsilon/2,$$

$$\tilde{\theta}_1 \in \left(\frac{1}{A}\left(\frac{1}{b_{1-\varepsilon/2}} - 1\right); \frac{1}{A}\left(\frac{1}{b_{\varepsilon/2}} - 1\right)\right),$$

where $A = \frac{2V+\alpha_2}{3U+\alpha_1}$. In the noninformative prior case, the credible interval is

$$\left(\frac{3U}{2V}\left(\frac{1}{b_{1-\varepsilon/2}} - 1\right); \frac{3U}{2V}\left(\frac{1}{b_{\varepsilon/2}} - 1\right)\right) \ni \theta_1^*.$$

The lengths of these intervals are:

$$\frac{3U}{2V}\left(\frac{1}{b_{\varepsilon/2}} - \frac{1}{b_{1-\varepsilon/2}}\right); \quad \frac{3U+\alpha_1}{2V+\alpha_2}\left(\frac{1}{b_{\varepsilon/2}} - \frac{1}{b_{1-\varepsilon/2}}\right) \quad \text{and} \quad \frac{3U}{2V}\left(\frac{1}{b_{\varepsilon/2}} - \frac{1}{b_{1-\varepsilon/2}}\right).$$

The classical confidence interval for θ_2 is

$$\left(\frac{2V}{W}\frac{1}{b_{1-\varepsilon/2}}; \frac{2V}{W}\frac{1}{b_{\varepsilon/2}}\right).$$

In the informative prior case, the interval is

$$\left(\frac{2V+\alpha_2}{W+\alpha_3}\left(\frac{1}{b_{1-\varepsilon/2}}-1\right);\frac{2V+\alpha_2}{W+\alpha_3}\left(\frac{1}{b_{\varepsilon/2}}-1\right)\right)\ni\tilde{\theta}_2.$$

A better interval in this case is one as $(1/2-a;1/2+l)$ such that

$$\int_{1/2-a}^{1/2+b} g(\theta_2|U,V,W)d\theta_2\geq 1-\varepsilon,$$

where the modul of Beta is in this case 1/2. For the noninformative prior, the credible interval is

$$\left(\frac{2V}{W}\left(\frac{1}{b_{1-\varepsilon/2}}-1\right);\frac{2V}{W}\left(\frac{1}{b_{\varepsilon/2}}-1\right)\right)\ni\theta_2^*.$$

The lengths of these intervals are:

$$\frac{2V}{W}\left(\frac{1}{b_{\varepsilon/2}}-\frac{1}{b_{1-\varepsilon/2}}\right);\qquad\frac{2V+\alpha_2}{W+\alpha_3}\left(\frac{1}{b_{\varepsilon/2}}-\frac{1}{b_{1-\varepsilon/2}}\right);\quad\text{and}\quad\frac{2V}{W}\left(\frac{1}{b_{\varepsilon/2}}-\frac{1}{b_{\varepsilon/2}}\right).$$

Theorem. *The distribution of the variable* $\frac{2(n+1)\lambda}{\tilde{\lambda}}=2n\lambda(3U+\alpha_1)$ *is a* $\chi^2_{(2n+2)}$.

The clasical probability of coverage for $\tilde{\lambda}$ is:

$$P\left(\lambda\in\left(\tfrac{h_{\varepsilon/2}}{2(3U+\alpha_1)},\tfrac{h_{1-\varepsilon/2}}{2(3U+\alpha_1)}\right)\right)=P(2n\lambda(3U+\alpha_1)\in(nh_{\varepsilon/2};nh_{1-\varepsilon/2}))=$$
$$=F_{\chi^2(2n+2)}(nh_{\varepsilon/2})-F_{\chi^2(2n+2)}(nh_{1-\varepsilon/2}).$$

§5. Reliability estimations

By definition, $R(u_0,v_0,w_0)=P(u>u_0,v>v_0,w>w_0)$. For noninformative prior, we have

$$\hat{R}(u_0,v_0,w_0)=\left(\frac{U}{U+u_0}\frac{V}{V+v_0}\frac{W}{W+w_0}\right)^n.$$

The probability that the system will continue to function through at least time t_0 is given by:

$$\begin{aligned}\hat{R}&=P(max(X,Y,Z)>t_0)=P(u+v+w>t_0)=\\&=2\theta_2e^{-3\lambda t_0}+\frac{6(e^{-3\lambda t_0}-e^{-\lambda\theta_1\theta_2t_0})}{(2-\theta_2)(3-\theta_1\theta_2)}+\frac{6(e^{-3\lambda t_0}-e^{-2\lambda\theta_1t_0})}{(2-\theta_2)(3-2\theta_1)},\end{aligned}$$

if $\theta_1\neq 3/2$ and $\theta_2\neq 2$;

$$\hat{R}=e^{-3\lambda t_0}+\frac{3\lambda t_0e^{-3\lambda t_0}}{(1-\theta_2/2)}+\frac{e^{-3\lambda\theta_2t_0/2}-e^{-3\lambda t_0}}{(1-\theta_2/2)^2},$$

if $\theta_1 = 3/2$ and $\theta_2 \neq 2$;

$$\hat{R} = e^{-3\lambda t_0} + \frac{6\lambda\theta_1}{3-2\theta_1}\left(t_0 e^{-2\lambda\theta_1 t_0} + \frac{1-e^{-3\lambda t_0}}{\lambda(3-2\theta_1)}\right),$$

if $\theta_1 \neq 3/2$ and $\theta_2 = 2$.

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