

Market competition via geometric dynamics

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Abstract. In this paper we use the Geometric Dynamics (see [3], [4]) to analyse some economical problems concerning market competition. We look at the qualitative differences across two sectors and at the peculiarities of each market as at the determinants of the economic dynamics.

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1 Introduction

Dynamics is the study of how systems change in time. Geometric dynamics [3] was created to show that any dynamical system of order one can be extended to a dynamical systems of order two coming from a first order Lagrangian of least square type. According to this theory, any flow is included into a geodesic motion in a gyroscopic field. Also the Hamiltonian describing such a problem is a difference of two squares, having negative, null and positive values. By this note we are going to introduce new economic growth models to study market competition.

2 Market competition model

Consider a generic market where two different type of firms produce and trade their own homogeneous good. We are interested to study the dynamical evolution of the two sectors in which the two classes of firms follow a certain growth. For studying the dynamical competition between these two sectors we have to elaborate a model to capture this evolution in an interval of time. In the sequel let us denote by E_1 and E_2 two populations of new firms born in the above sectors; in particular, we consider the dynamical growth of two sectors of the new economy. To study this realistic economical phenomena we adopt the first order differential system used by Udrişte and Postolache [5]

$$\frac{dE_1}{dt} = g_1 E_1 \left(1 - \frac{E_1}{K_1} - \beta_1 \frac{E_2}{K_1} \right)$$

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$$\frac{dE_2}{dt} = g_2 E_2 \left(1 - \frac{E_2}{K_2} - \beta_2 \frac{E_1}{K_2} \right)$$

to analyse the population competition.

To be in the context of economics, we interpret the constants g_1 and g_2 as the growth rates of the two sectors respectively; the constants K_1 and K_2 as investments of capitals; the constants β_1 and β_2 as competitive interaction coefficients. All these quantities are chosen strictly positive.

The evolution of the competition between E_1 and E_2 is like a flow described by the previous first order differential system. The geometric dynamics of the competition between E_1 and E_2 is described by the second order differential system

$$\begin{aligned} \frac{d^2 E_1}{dt^2} &= \frac{\partial f}{\partial E_1} - \left(\beta_1 g_1 \frac{E_1}{K_1} - \beta_2 g_2 \frac{E_2}{K_2} \right) \frac{dE_2}{dt} \\ \frac{d^2 E_2}{dt^2} &= \frac{\partial f}{\partial E_2} - \left(\beta_2 g_2 \frac{E_2}{K_2} - \beta_1 g_1 \frac{E_1}{K_1} \right) \frac{dE_1}{dt}, \end{aligned}$$

where

$$f = \frac{1}{2} \left[g_1^2 E_1^2 \left(1 - \frac{E_1}{K_1} - \beta_2 \frac{E_2}{K_1} \right)^2 + g_2^2 E_2^2 \left(1 - \frac{E_2}{K_2} - \beta_2 \frac{E_1}{K_2} \right)^2 \right]$$

is the density of economical growth process. These second order differential equations describe the extremals of the following Lagrangian

$$\begin{aligned} L &= \frac{1}{2} \left[\left(\frac{dE_1}{dt} \right)^2 + \left(\frac{dE_2}{dt} \right)^2 - g_1 E_2 \left(1 - \frac{E_1}{K_1} - \beta_1 \frac{E_2}{K_1} \right) \frac{dE_1}{dt} - \right. \\ &\quad \left. - g_2 E_2 \left(1 - \frac{E_1}{K_2} - \beta_2 \frac{E_1}{K_2} \right) \frac{dE_2}{dt} + f \right], \end{aligned}$$

and the associated Hamiltonian is

$$H = \frac{1}{2} \left[\left(\frac{dE_1}{dt} \right)^2 + \left(\frac{dE_2}{dt} \right)^2 \right] - f.$$

By using

$$\begin{aligned} \bar{g}_{ij} &= (H + f) \delta_{ij}, \quad N_j^i = \Gamma_{jk}^i y^k - F_j^i \\ F_j^i &= \delta^{ih} F_{jh}, \quad F_{ij} = \frac{\partial X_j}{\partial E_i} - \frac{\partial X_i}{\partial E_j}, \quad i, j, h = 1, 2, \\ X_1 &= g_1 E_1 \left(1 - \frac{E_1}{K_1} - \beta_1 \frac{E_2}{K_1} \right), \quad X_2 = g_2 E_2 \left(1 - \frac{E_2}{K_2} - \beta_2 \frac{E_1}{K_2} \right), \end{aligned}$$

the solutions of the differential system are horizontal pregeodesics of the Riemann-Jacobi-Lagrange manifold

$$(R^2 \setminus \mathcal{E}, \bar{g}_{ij}, N_j^i),$$

where

$$\mathcal{E} = \left\{ (0, 0), (0, K_2), (K_1, 0), \left(\frac{K_1 - \beta_1 K_2}{1 - \beta_1 \beta_2}, \frac{K_2 - \beta_2 K_1}{1 - \beta_1 \beta_2} \right) \right\}$$

is the set of equilibrium points of the previous first order differential system.

3 Model test and simulations

In this paragraph we are going to present the results of some tests using MAPLE 6.

Let us consider a market described by the following data:

$$g_1 = 0, 3; \quad K_1 = 0, 005; \quad \beta_1 = 0, 001$$

$$g_2 = 0, 2; \quad K_2 = 0, 007; \quad \beta_2 = 0, 003.$$

Starting from these data we obtain:

1. The first order differential system

$$\begin{cases} \frac{dE_1}{dt} = 0, 3E_1(1 - 0, 005E_1 - 0, 001E_2) \\ \frac{dE_2}{dt} = 0, 2E_2(1 - 0, 007E_2) - 0, 003E_1. \end{cases}$$

2. Equilibrium points

$$\begin{cases} x = 0 \\ y = 0 \end{cases} ; \begin{cases} x = 0 \\ y = 142, 85 \end{cases} ; \begin{cases} x = 200 \\ y = 0 \end{cases} ; \begin{cases} x = 187, 5 \\ y = 62, 5 \end{cases} .$$

Around these four equilibrium points appear interesting phenomena concerning the dynamical evolution of the populations of the firms.

3. The *economical growth process function*

$$f = \frac{1}{2}[(0, 3E_1(1 - 0, 005E_1 - 0, 001E_2))^2 + (0, 2E_2(1 - 0, 007E_2 - 0, 003E_1))^2].$$

4. The second order differential

$$\begin{cases} \frac{d^2 E_1}{dt^2} = \frac{\partial f}{\partial E_1} - (0, 06E_1 - 0, 08E_2) \frac{\partial E_2}{dt} \\ \frac{d^2 E_2}{dt^2} = \frac{\partial f}{\partial E_2} - (0, 08E_2 - 0, 08E_1) \frac{\partial E_1}{dt}. \end{cases}$$

Conclusion. The market competition has heteroclinic connections.

Using the following MAPLE routines we find Figs. 1-6.

```

with(plottools):
with(plots):
with(DEtools):
E:=DEplot({D(x)(t)=0.3*x(t)*(1-0.005*x(t)-0.001*y(t)),
D(y)(t)=0.2*y(t)*(1-0.003*x(t)-0.007*y(t))},{x(t),y(t)},t=0..10,
[[x(0)=5,y(0)=10]],x=-10..205,y=-10..150,scene=[x,y],stepsize=.1,linestyle=4,
linecolor=t, method=classical[rk4],startinit=true,maxfun=5000);
N:=implicitplot({0.3*x*(1-0.005*x-0.001*y),0.2*y*(1-0.003*x-0.007*y) },
x=-10..205,y=-10..150,color=green);
plots[display](E,N,axes=none,title='FIG.1. COMPETITION PHASE PORTRAIT,
NULLCLINES, (E1,E2)-PLANE ');

DE1 := {diff(x(t), t^2) = 0.09 * x(t) * (1 - 0.005 * x(t) - 0.001 * y(t)) * (1 - 0.001 *
y(t) - 0.00012 * y^2(t) * (1 - 0.007 * y(t) - 0.003 * x(t)) + (0.06 * x(t) - 0.08 * y(t)) *
diff(y(t), t), diff(y(t), t^2) = -0.00009 * x^2(t) * (1 - 0.005 * x(t) - 0.001 * y(t)) + 0.04 *
y(t) * (1 - 0.007 * y(t) - 0.003 * x(t)) * (1 - 0.003 * x(t)) + (0.08 * y(t) - 0.06 * x(t)) *
diff(x(t), t)};
DEplot(DE1,[x(t),y(t)],t=-15..15,[[x(0)=0.1,y(0)=0.3,D(x)(0)=1,D(y)(0)=-0.1],
[x(0)=1,y(0)=3,D(x)(0)=-1,D(y)(0)=-0.1]],x=-10..20,y=-10..15,
scene=[x(t),y(t)],obsrange=true,title='FIG.2. COMPETITION GEOMETRIC
DYNAMICS',linecolor=red,stepsize=.001,iterations=50);
with(plots):
implicitplot((0.3*x*(1-0.005*x-0.001*y))^2+(0.2*y*(1-0.003*x-0.007*y))^2 =
1, x = -10..205, y = -10..150, color = blue,
title='FIG.3. 1-LEVEL SET OF ENERGY DENSITY');
contourplot((0.3*x*(1-0.005*x-0.001*y))^2+(0.2*y*(1-0.003*x-0.007*y))^2, x =
-10..205, y = -10..150, contours = [0.5, 0.7, 1],
title='FIG.4. LEVEL SETS OF ENERGY DENSITY');
contourplot((0.3*x*(1-0.005*x-0.001*y))^2+(0.2*y*(1-0.003*x-0.007*y))^2, x =
-20..20, y = -20..20, contours = [0.5, 0.7, 1],
title='FIG.5. LEVEL SETS OF ENERGY DENSITY');
with(plots) : listcontplot([seq([seq((0.3 * x * (1 - 0.005 * x - 0.001 * y))^2 + (0.2 *
y * (1 - 0.003 * x - 0.007 * y))^2, x = -10..205)], y = -10..150)],
title='FIG.6. LEVEL SETS OF ENERGY DENSITY');

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FIG. 1. COMPETITION PHASE PORTRAIT, NULLCLINES, (E1,E2)-PLANE

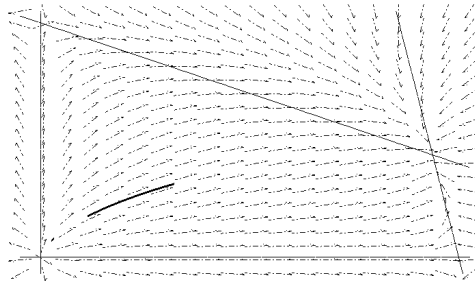


FIG. 2. COMPETITION GEOMETRIC DYNAMICS

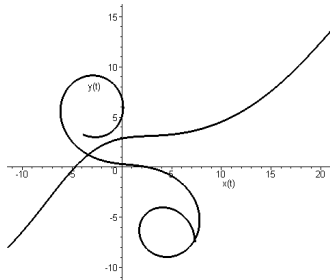


FIG. 3. 1-LEVEL SET OF ENERGY DENSITY

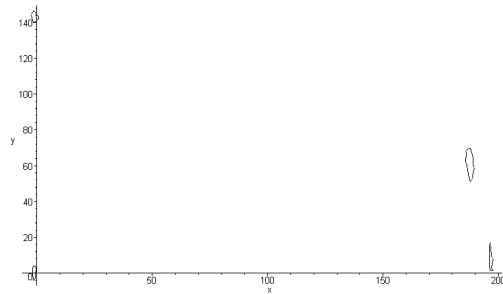


FIG. 4. LEVEL SETS OF ENERGY DENSITY

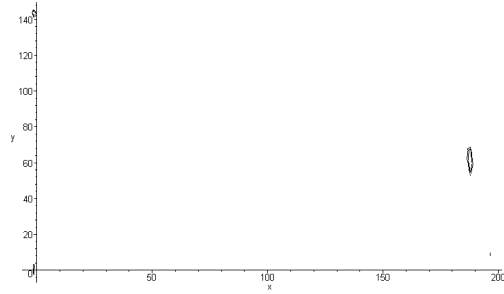


FIG. 5. LEVEL SETS OF ENERGY DENSITY

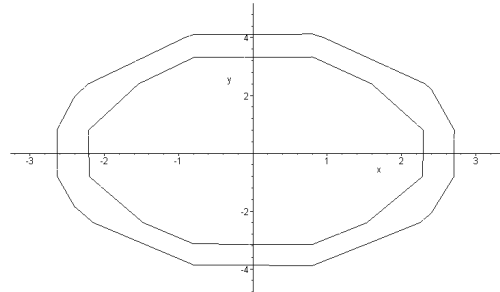
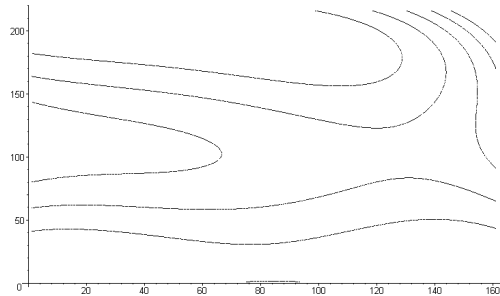


FIG. 6. LEVEL SETS OF ENERGY DENSITY



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