Area conditions associated to thermodynamic and economic systems

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Abstract. Our paper underlines some similarities between thermodynamic and economic systems, via two area conditions.
§1 gives criteria for the separation of variables in a function of two variables, based on the first area condition. §2 describes a thermodynamic system via the Gibbs-Pfaff equation in a space with 5 dimensions. §3 formulates an optimization problem with a nonholonomic constraint and rises an open problem to treat an economic system like a thermodynamic system. Both paragraphs 2 and 3 contain the second area conditon.

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1 Introduction

In this paper we provide two area conditions which are naturally associated to the behavior of thermodynamic and economic systems.

The first area condition appears when we suppose that a plane is foliated by two families of smooth curves like in Fig. 1. Two such families of curves satisfy the first area condition if any three curves in the first family, and any three curves in the second family determine 4 domains whose areas A, B, C, D are in proportion:

$$\frac{A}{B} = \frac{C}{D}.$$

If we think the first family like the family of isotherms and the second as adiabatics, then we are in thermodynamics. If the first family consists of graph of labor, as function of its own price indexed after the quantity of capital, and the second family consists of graph of labor, as function of its own price indexed after the price of the capital, then we have a monopolist problem. Both these problems have a "saddle" behavior in a suitable space.

The second area condition appears when we look for the existence of bidimensional integral manifolds of a 4-dimensional contact distribution which implies the equality of two elements of area. This is strongly connected to the possibility of describing

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a closed thermodynamic (economic) system via a Gibbs-Pfaff equation, or to the possibility of having extrema problems with nonholonomic constraints.

This paper is an answer to some open problems and similarities suggested by the references [6]-[9]. Some of this topics were brought into a first discussion by M. Ferrara and A. Niglia, but all the members of Prof.Dr. Maria Calapso teams have provided valuable feedback on model test reports introduced in the Lectures of Prof. Dr. Constantin Udrişte at Faculty of Economics, University of Messina, May 12-19, 2002.

2 Multiplicative splitting of a function and first area condition

Let us consider a function $f : S\times T \to R$, where $S, T \subset R$. We introduce the following lemmas useful for the sequel:

2.1. Lemma. *The nonconstant function* $f : S \times T \to R$ *can be expressed as a product* gh, $g : S \to R$, $h : T \to R$ *iff, for any pair of points* $s_1, s_2 \in S$, *and* $t_1, t_2 \in T$, *we have*

$$f(s_1,t_1)f(s_2,t_2) = f(s_1,t_2)f(s_2,t_1).$$

Proof. In $f(s_1,t_1)f(s_2,t_2) = f(s_2,t_1)f(s_1,t_2)$ we put $s_1 = s$, $t_1 = t$, and we preserve s_2, t_2 as fixed points. Denoting $f(s_2,t_2) = a \neq 0$, $f(s_2,t) = h(t)$, $\frac{1}{a}f(s,t_2) = g(s)$, we find

$$f = g \cdot h.$$

The converse is obvious.

2.2. Lemma. *Let* I, J *be two intervals and* $f : I \times J \to (0,\infty)$ *a continuous nonconstant function. The function* f *can be expressed as a product* gh, $g : I \to (0,\infty)$, $h : J \to (0,\infty)$ *iff for any pairs of intervals* $I_1, I_2 \subset I$, $J_1, J_2 \subset J$, *we have*

$$\frac{\iint_{I_1\times J_1} f(s,t)dsdt}{\iint_{I_1\times J_2} f(s,t)dsdt} = \frac{\iint_{I_2\times J_1} f(s,t)dsdt}{\iint_{I_2\times J_2} f(s,t)dsdt}.$$

Proof. Suppose the equality is true, and we use the mean value theorem for double integrals. We fix the points (s_i, t_j), $i, j = 1, 2$. We denote by I_{1n}, I_{2n} the intervals centered at s_1 respectively s_2, of length $\frac{1}{n}$, and by J_{1n}, J_{2n} the intervals centered at t_1 respectively t_2, of length $\frac{1}{n}$. The hypotheses and the mean value theorem shows that $f(s_1,t_1)$ is the limit of the mean value of f on $I_{1n} \times J_{1n}$ when $n \to \infty$.

Let Δ and Σ be two domains in R^2, and $F : \Delta \to \Sigma$, $F(u,v) = (x,y)$, $x = x(u,v)$, $y = y(u,v)$ a diffeomorphism having the Jacobian

$$J(u,v) = \frac{D(x,y)}{D(u,v)} = x_u y_v - x_v y_u > 0.$$

If we accept that (x, y) are Cartesian coordinates, then the area of the domain Σ is

$$\iint_{\Sigma} dxdy = \iint_{\Delta} J(u,v)dudv.$$

The family of coordinate curves $u = c_1$, and the family of coordinate curves $v = c_2$ split the domain Δ into subdomains (see Fig. 1). We select four arbitrary subdomains as in the Fig.1. According Lemma 2, the multiplicative splittig of the Jacobian J is equivalent to the *first area condition*

$$AD = BC.$$

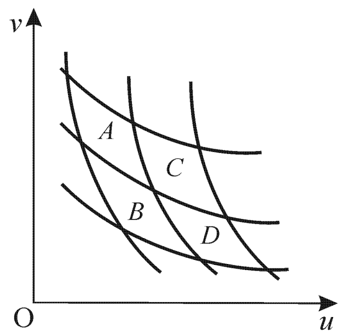

Fig. 1

2.3. Proposition. The following statements are equivalent:
1) The first area condition $AD = BC$ is satisfied.
2) The Jacobian J can be written as a product,

$$J(u,v) = p(u)q(v).$$

3) The Jacobian J is the general solution of the PDE

$$\frac{\partial^2}{\partial u \partial v}(\ln J) = 0.$$

4) There are recalibrations of u and v such that $J = 1$.

Economic interpretation. First version. Let p be the price and q be the quantity of the input. The dynamics of the price p and quantity q is described by the change of variables

$$p = p(u,v), \quad q = q(u,v).$$

Consequently the coordinate curves $u = c_1$, and $v = c_2$, will divide the domain in the plane uOv as in Fig.1. The first area condition shows to the monopolist that the demand function $d(p,q)$ is decomposed like a product between a function of price $g(p)$

and a function of quantity $h(q)$. Equivalently, the market has "saddle" behavior in the space $Oghd$ (Fig.2):

$$\text{the saddle } S : d = gh, \quad \text{with the rulings}$$

$$D_\lambda : d = \lambda g,\ h = \lambda \quad \text{(straight lines, } \lambda = \text{parameter)}$$

$$D_\mu : d = \mu h,\ g = \mu \quad \text{(straight lines, } \mu = \text{parameter)}$$

$$S = \bigcup_{\lambda \in \bar{R}} D_\lambda = \bigcup_{\mu \in \bar{R}} D_\mu .$$

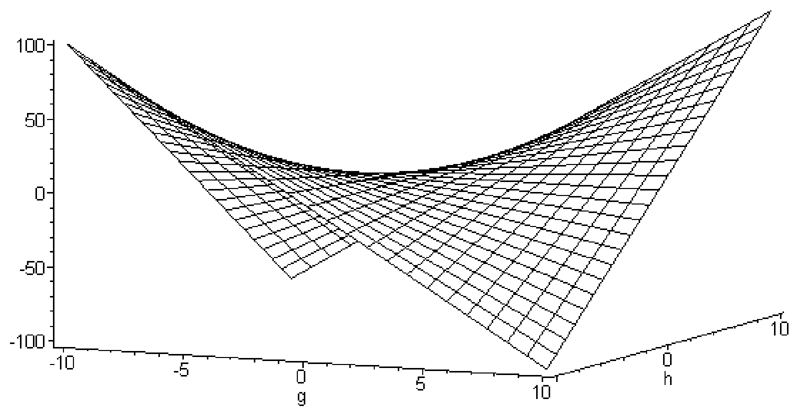

Fig. 2. Saddle: $d = g * h$

Any two straight lines in the same family are not coplanar. A straight line from the family D_λ and a straight line from the family D_μ are concurrent. Any three straight lines in the same family are parallel with the same plane.

In other words, each decision of the monopolist to optimize the demand d can be attend by using partial politics corresponding to rectilinear trajectories in the space $Oghd$, since for the function $d = gh$ there exist directions producing minimum value or maximum value, with jump from minimum to maximum or conversely.

Remark. From microeconomic point of view the product between the functions $g(p)$ and $h(q)$ represents either the demand function or the value of the total demand of the market. In other words, we can have at least too situations:

1) $TR = g(p)h(q)$, where TR denotes the total revenue which also represent the value of the total demand in an equilibrium state of the market. Starting from this model, the monopolist can attend some economical decisions firstly by the analysis of the potential demand of the market and secondly by setting the price policies.

2) $V(d(p,q)) = g(p)h(q)$, where $V : \mathbb{R} \to \mathbb{R}$ denotes a value function of demand. In this frame $V \circ d$ represent the value produced by the total demand in an equilibrium state of the market.

Example. Let us accept $g(p), h(q), V(d)$ are polynomials of different orders in their variables. Of course, generically, we can have $V(d) = g(p)h(q)$ only for suitable polynomials.

MAPLE 6 SIMULATIONS:

PRICE (p)	QUANTITY (d)	SUPPLY (s=q)
30	850	450
35	825	525
40	800	600
45	775	675
50	750	750
55	725	825
60	700	900
65	675	975
70	650	1050

$solve(\{c0+c1*850+c2*850^2 = (a0+a1*30)*(b0+b1*450), c0+c1*825+c2*825^2 = (a0+a1*35)*(b0+b1*525), c0+c1*800+c2*800^2 = (a0+a1*40)*(b0+b1*600), c0+c1*775+c2*775^2 = (a0+a1*45)*(b0+b1*675), c0+c1*750+c2*750^2 = (a0+a1*50)*(b0+b1*750), c0+c1*725+c2*725^2 = (a0+a1*55)*(b0+b1*825), c0+c1*700+c2*700^2 = (a0+a1*60)*(b0+b1*900)\}, \{c0, c1, c2, a0, a1, b0, b1\});$

$\{c2 = 3/5*a1*b1, c1 = -1200*a1*b1 - 3*a0*b1 - 1/5*a1*b0, c0 = 600000*a1*b1+3000*a0*b1+200*a1*b0+a0*b0, a0 = a0, a1 = a1, b0 = b0, b1 = b1\};$

$solve(\{c0+c1*850 = (a0+a1*30)*(b0+b1*450), c0+c1*825 = (a0+a1*35)*(b0+b1*525), c0+c1*800 = (a0+a1*40)*(b0+b1*600), c0+c1*775 = (a0+a1*45)*(b0+b1*675), c0+c1*750 = (a0+a1*50)*(b0+b1*750)\}, \{c0, c1, a0, a1, b0, b1\});$

$\{a0 = a0, a1 = a1, b0 = b0, b1 = 0, c1 = -1/5*a1*b0, c0 = 200*a1*b0+a0*b0\}, \{a0 = a0, b0 = b0, b1 = b1, a1 = 0, c1 = -3*a0*b1, c0 = 3000*a0*b1+a0*b0\};$

LEAST SQUARES METHOD

$minimize((c0+c1*850+c2*850^2-(a0+a1*30)*(b0+b1*450))^2+(c0+c1*825+c2*825^2-(a0+a1*35)*(b0+b1*525))^2+(c0+c1*800+c2*800^2-(a0+a1*40)*(b0+b1*600))^2+(c0+c1*775+c2*775^2-(a0+a1*45)*(b0+b1*675))^2+(c0+c1*750+c2*750^2-(a0+a1*50)*(b0+b1*750))^2+(c0+c1*725+c2*725^2-(a0+a1*55)*(b0+b1*825))^2+(c0+c1*700+c2*700^2-(a0+a1*60)*(b0+b1*900))^2+(c0+c1*675+c2*675^2-(a0+a1*65)*(b0+b1*975))^2+(c0+c1*650+c2*650^2-(a0+a1*70)*(b0+b1*1050))^2, location);$

$0, \{[[a0 = a0, a1 = a1, b0 = b0, b1 = b1, c0 = 600000*a1*b1+3000*a0*b1+200*a1*b0+a0*b0, c1 = -1200*a1*b1-3*a0*b1-1/5*a1*b0, c2 = 3/5*a1*b1], 0]\};$

RULES:

1) If (pn), (dn), (qn) are arithmetic progressions, then pn = p0 + n*a, dn = d0 + n*b,qn = q0 + n*c. Consequently (pn - p0)/a =(dn - d0)/b=(qn - q0)/c and ((pn - p0)/a)*((qn - q0)/c)=$((dn-d0)/b)^2$.

2) For the continuous case, if p =p0 + l*t, d = d0 + m*t , q = q0 + n*t , t in R (a straight line), then((p-p0)/l)*((q-q0)/n) =$((d-d0)/m)^2$.

Second version. We consider two inputs (input 1 = Capital, input 2 = Labor). We denote by p_1, q_1 the price and the quantity of the first input, and by p_2, q_2 the price and the quantity of the second one. Sometimes is suitable to introduce a demand of the form $d_1(p_1, q_2)$. The first area condition leads to decomposition of d_1 and consequently to "saddle" behavior of the monopolist in a suitable 3-dimensional space.

3 Thermodynamic systems and second area condition

Let R^5 be the real Euclidean space with 5 dimensions. For identification with the space of thermodynamic states label the Cartesian coordinates on R^5 with U, T, S, P, V and adopt for them the following names:

U = internal energy, T = temperature, S = entropy, P = pressure, V = volume.

We consider the Gibbs-Pfaff equation

$$\theta = dU - TdS + PdV = 0.$$

Since $\theta \wedge d\theta \neq 0$, the Gibbs-Pfaff equation is not completely integrable (it does not admit an integrant factor). On the other hand,

$$\theta \wedge (d\theta)^2 \neq 0$$

shows that the Gibbs-Pfaff form is a contact form. Therefore the integral submanifolds of the Gibbs-Pfaff equation are either curves or surfaces. Obviously, the integral curves are included in the integral surfaces.

An integral surface of the Gibbs-Pfaff equation is the image of a C^2 regular function

$$r : D \subset R^2 \to R^5,\ r(x,y) = (U(x,y), T(x,y), S(x,y), P(x,y), V(x,y))$$

whose components verify the PDEs system

$$\frac{\partial U}{\partial x} - T\frac{\partial S}{\partial x} + P\frac{\partial V}{\partial x} = 0, \quad \frac{\partial U}{\partial y} - T\frac{\partial S}{\partial y} + P\frac{\partial V}{\partial y} = 0.$$

The integrability conditions

$$\frac{\partial^2 U}{\partial x \partial y} = \frac{\partial^2 U}{\partial y \partial x}, \ \frac{\partial^2 S}{\partial x \partial y} = \frac{\partial^2 S}{\partial y \partial x}, \ \frac{\partial^2 V}{\partial x \partial y} = \frac{\partial^2 V}{\partial y \partial x}$$

have as consequence the equality of two Jacobians,

$$\frac{\partial T}{\partial x}\frac{\partial S}{\partial y} - \frac{\partial S}{\partial x}\frac{\partial T}{\partial y} = \frac{\partial P}{\partial x}\frac{\partial V}{\partial y} - \frac{\partial V}{\partial x}\frac{\partial P}{\partial y}. \tag{1}$$

3.1. Proposition (*second area condition*). The existence of an integral surface of the Gibbs-Pfaff equation implies the equality of area elements in the planes $\{T, S\}$ and $\{P, V\}$.

For a given point $M_0(U_0, T_0, S_0, P_0, V_0)$ and two noncollinear vectors $a = (a_1, a_2, a_3, a_4, a_5)$, $b = (b_1, b_2, b_3, b_4, b_5)$ fixed by the conditions

$$a_1 - T_0 a_3 + P_0 a_5 = 0,\ b_1 - T_0 b_3 + P_0 b_5 = 0,\ a_2 b_3 - a_3 b_2 = a_4 b_5 - a_5 b_4,$$

there exists an infinity of integral surfaces r which satisfy

$$r(x_0, y_0) = M_0,\ \frac{\partial r}{\partial x}(x_0, y_0) = a,\ \frac{\partial r}{\partial y}(x_0, y_0) = b.$$

The set of all integral surfaces of the Gibbs-Pfaff equation is a *nonholonomic hypersurface* in R^5 which is denoted by $(R^5, \theta = 0)$. All these surfaces are orthogonal to the vector field $(1, 0, -T, 0, P)$ whose field lines are

$$U = t,\ T = c_1,\ S = -c_1 t + d_1,\ P = c_2,\ V = c_2 t + d_2,\ t \in R,$$

c_1, c_2, d_1, d_2 = arbitrary constants (family of straight lines).

3.2. Definition. 1) An integral surface of the Gibbs-Pfaff equation $\theta = 0$ is called a *simple thermodynamic system*. The variables x, y of the domain space R^2 are called *states* of the system.

2) The nonholonomic hypersurface $(R^5, \theta = 0)$ is called a *closed thermodynamic system*.

Obviously a closed thermodynamic system is a collection of simple thermodynamic systems. In this context, the area condition (1) is called *Maxwell equation* attached to the thermodynamic system.

From local point of view the states x, y of a simple thermodynamic system can be chosen as two of five coordinates U, T, S, P, V. In this sense we have 10 types of simple thermodynamic systems [12].

4 Economic systems and second area condition

The standard framework of economics requires extrema problems on a feasible set described by constraints. On the other hand, in Mechanics, a completely integrable Pfaff equation is called a *holonomic constraint*, and a Pfaff equation which is not completely integrable is called a *nonholonomic constraint*. This remark suggests to introduce "extrema with nonholonomic constraints" in the economic theory.

To exemplify, let $R^5 = \{y = (y_1, y_2, y_3, y_4, y_5) | y_i \in R,\ i = \overline{1,5}\}$. The non-integrable Pfaff equation

$$\omega = dy_1 + y_2 dy_3 + y_4 dy_5 = 0$$

defines a nonholonomic hypersurface $(R^5, \omega = 0)$.

Let $f : R^5 \to R$ be an economic objective function. As in [11], [12] we propose to find the minima of f with the nonholonomic constraint $\omega = 0$. The constrained critical points are solutions of the system

$$\frac{\partial f}{\partial y_1} + \lambda = 0,\ \frac{\partial f}{\partial y_2} = 0,\ \frac{\partial f}{\partial y_3} + \lambda y_2 = 0,\ \frac{\partial f}{\partial y_4} = 0,\ \frac{\partial f}{\partial y_5} + \lambda y_4 = 0,$$

which is obtained from the condition $df + \lambda\omega = 0$. Suppose we have a solution $y_0(\lambda) = (y_{i_0} = y_{i_0}(\lambda),\ i = \overline{1,5})$. If the quadratic form

$$d^2 f + \lambda(dy_2 dy_3 + dy_4 dy_5)|_{y_0(\lambda)}$$

is positive definite for $\lambda \in I$, then $y_0(\lambda)$, $\lambda \in I$ is a family of minimum points, and $f(y_0(\lambda)$, $\lambda \in I$, is a family of minimum values (depending of λ).

As model we can use the objective function $f(y) = \frac{1}{2}\sum_{i=1}^{5} y_i^2$. Then the constrained critical point is $y_0(\lambda) = (y_1 = -\lambda,\ y_2 = 0,\ y_3 = 0,\ y_4 = 0,\ y_5 = 0)$, and the associated quadratic form is positive definite for $\lambda \in (-2, 2)$. Consequently, all the points $y_0(\lambda)$, $\lambda \in (-2, 2)$ are minimum points and $\min f = \frac{\lambda^2}{2}$.

4.1. Proposition (*second area condition*). The existence of an integral surface of the Pfaff equation $\omega = 0$ implies the equality of area elements in the planes $y_2 O y_3$ and $y_4 O y_5$.

We can realise a theoretical identification of a closed economic system with a closed thermodynamic system. For example we can use the following correspondence for coordinates:

U = internal energy $\to$ G = potential growth
T = temperature $\to$ I = internal stability
S = entropy $\to$ E = entropy
P = pressure $\to$ P = price
V = volume $\to$ Q = quantity of production.

Open problem. The Gibbs-Pfaff equation is replaced by a suitable Pfaff equation and each economic system consists of 10 types of simple economic systems. What is the meaning of Thermodynamic Laws in Economy ?

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