Fiducial inference for discrete and continuous distributions

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Abstract. This paper describes the general principles and methods of the fiducial inference. A brief survey of its competing inferencial theories as well as a comparison with them are also provided. Arguments in favour of the application of the fiducial method to the parameters of discrete random variables are given, and, as an application, the fiducial distribution associated to the binomial proportions is shown to be of the beta family.

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1 Introduction

The concept of fiducial probability was introduced by Fisher in 1930, in his paper "Inverse probability". The idea behind fiducial inference is the following: Suppose there is a population characterised by a density function $f(x;\theta)$, and, as usual in the inferencial theory, the form of f is known, but there is no prior information available about the true value of the parameter θ. Given a set of observations, one wants to assign epistemic probabilities to subsets of the set of admissible values of the parameter θ, indicating the belief that the true value of the parameter is contained in the respective subsets. If no particular subset is of special interest, then this problem transforms itself in the idea of specifying a probability distribution on the respective set, provided that these epistemic probabilites are assigned coherently to all subsets. The "classical" method of deriving such distributional inferences is by applying the Bayesian theory. The drawback of this method, is, however, that it requires the specification of a prior distribution. Fisher regarded the specification of a prior distribution as being in conflict with the assumption that no prior information is available. He especially rejected the use of uniform priors, pointing out the inconsistencies resulting from different parameterisations of the same inference problem. He also criticized the use of subjective priors because of the subjective element that would inflict upon the posterior distribution. He thus conceived the fiducial inference as an alternative to the Bayes approach, aiming to obtain a distribution for the unknown parameter without

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the help of priors. Another difference to the Bayesian approach is that the fiducial distribution is not the actual distribution of an unknown parameter, but it describes the uncertainty about the value of the fixed unknown parameter. Fisher himself derived a number of fiducial inferences for various problems, without being sufficiently clear about the underlying principles and without reaching undeniable conclusions.

Briefly speaking, the fiducial argument provides an alternative method to generate distributional inferences in an entirely objective way and exclusively on the basis of the given sample, so that it can be applied without specifying a prior.

In other words, a fiducial distribution for the parameter θ can be found without any prior distribution - only on the basis of a sample $x_1, \ldots, x_K$ for a probability/density function $f(x;\theta)$.

Nevertheless, there is a lack of agreement among statisticians about fiducial probabilities and a strong controversy about this idea which seems at the first glance to be so very clear and simple. This is due -among other things- to the fact that, at its beginnings, the fiducial method has been put forward as a general logical principle, but yet been illustrated mainly by means of particular examples rather than broad requirements.

The paper is organised as follows: first, the foundations of the fiducial inference are described; the pivotal and the non-pivotal approach are presented and then discussed. In the second part an extension for parameters of discrete random variables is provided, and a fiducial distribution for the binomial proportions is derived.

2 On fiducial inference

2.1 Fiducial inference on the basis of pivotal variables

The ingredients of the fiducial approach are, according to Fisher,

- a sufficient statistic for the parameter in question, derived on the basis of the sample,
- a pivot, function of both sufficient statistic and true value of the parameter, whose distribution though does not depend on either the sampled value of the sufficient statistic or the parameter, and
- the fiducial argument, which states that, from the distribution of the pivot, a distribution for the parameter can be derived on the basis of the sampled sufficient statistic: the fiducial distribution associated with the parameter.

Because of the misgivings and misunderstandings which have arisen concerning the fiducial argument, it would be better to illustrate it through some examples taken from literature.

Example 1. [Fisher, 1935] Let x be an observed value of a random variable having the normal distribution, with the density

$$f\left(x;\mu,\sigma^2\right)=\frac{1}{\sqrt{2\pi}\sigma}\cdot e^{-\frac{(x-\mu)^2}{2\sigma^2}}.$$

A sample $x_1, x_2, ..., x_n$ is taken from this population,

$$\overline{x}=\frac{1}{n}\sum_{i=1}^{n}x_i \text{ and } s^2=\frac{1}{n-1}\sum_{i=1}^{n}\left(x_i-\overline{x}\right)^2.$$

are the estimates of the mean and of the squared standard deviation, respectively. The pivotal random variable is then:

$$T=\frac{\overline{X}-\mu}{S}\sqrt{n}\sim t_{n-1};$$

Because $\overline{X}$ is a sufficient statistic for the unknown parameter μ and based on the distribution of $\overline{X}$, a proper distribution for μ can be obtained; if we let $\widetilde{t}$ be some constant, then, as Fisher stated:

> "[...] the inequality
>
> $$t>\widetilde{t}$$
>
> is equivalent to the inequality
>
> $$\mu<\overline{x}-\frac{s\cdot\widetilde{t}}{\sqrt{n}},$$
>
> so that this last inequality must be satisfied with the same probability as the first. This probability is known for all values of $\widetilde{t}$, and decreases continuously as $\widetilde{t}$ is increased. Since, therefore, the right-hand side of the inequality takes, by varying $\widetilde{t}$, all real values, we may state the probability that μ is less than any assigned value, or the probability that it lies between any assigned values, or, in short, its probability distribution in the light of the sample observed."

In conclusion, the fiducial distribution is:

$$\mu\sim t_{n-1},$$

where $\overline{x}$ and s are fixed constants on the basis of a given sample. □

Example 2. [Sprott, 1965] The random variable X has the exponential distribution with the parameter θ,

$$X\sim Exp\left(\theta\right).$$

Suppose that $x_1, \ldots, x_n$ are n observations of an i.i.d. sample from X. Because the density of X is

$$f(x;\theta) = \theta \cdot e^{-\theta x},\ x \geq 0,$$

the joint density of the random vector $(X_{1,\ldots,}X_n)$ is:

$$f(x_1, \ldots, x_n; \theta) = \theta^n e^{-\theta \sum_{i=1}^{n} x_i} = \theta^n e^{-\theta s},$$

where $s = \sum_{i=1}^{n} x_i$ is a sufficient statistic for θ. The goal of the example is to derive a fiducial distribution for θ -that is a distribution for θ for a given s.

The density of the random variable $S = \sum_{i=1}^{n} X_i$ is the n times convolution of the density of X, that is:

$$f(s;\theta) = \frac{\theta^n s^{n-1} e^{-\theta s}}{(n-1)!},\ s \geq 0.$$

With the substitution

$$z = \theta s \quad (\theta = const;\ Z = \theta S)$$

the following form is obtained for the density distribution of the pivot:

$$f(z) = \frac{(\theta s)^{n-1} e^{-(\theta s)}}{(n-1)!}\theta = \frac{z^{n-1} e^{-z}}{(n-1)!},\ z \geq 0.$$

The following step consists in deriving the fiducial distribution for θ given the $x_1, \ldots, x_n$ values, that is for a fixed s. With the converse substitution:

$$\theta = \frac{z}{s}\ (s = const;\ \Theta = \frac{1}{s} Z)$$

the fiducial distribution is obtained:

$$(*) \qquad f(\theta\,|_{s=konst}) = \frac{s^{n-1}\theta^{n-1}e^{-s\theta}}{(n-1)!} \cdot s,\ \theta \geq 0.$$

Note that the parameter is initially assumed to be a fixed unknown constant. Although it is a constant, the knowledge concerning this parameter provided by the observations may be expressed as a probability distribution, the fiducial distribution. As Fisher points out (1959), that doesn't mean that θ is a random variable, only that "the exact nature and degree of our incertainty (about θ) is just as if we knew it to have been one chosen at random from such a population" (in our example, the one described by (*)). □

The examples clearly express the fact that no prior distribution for the parameter is involved and that the Bayesian methods are neither used nor usable in the above inversion.

As simple and compelling as the fiducial argument is when applied to a simple case, it soon becomes lost in complexity. Fisher has provided a list of caveats and situations in which it can be applied which seems to be endless: the observable quantities must be continuos, the statistic involved must be complete and sufficient, the pivotal function must be monotonic. He wrote to Barnard in 1962 (Bennet, 1990):

> *"A pivotal quantity is a function of parameters and statistics, the distribution of which is independent of all parameters. To be of any use in deducing probability statements about parameters, let me add*
> *a) it involves only one parameter*
> *b) the statistics involved are jointly exhaustive for that parameter*
> *c) it varies monotonically with the parameter*
> *For sets of pivotals then I add*
> *d) the joint distribution is independent of parameters*
> *e) all are monotonic, uniformly for variations of parameters."*

Later, other statisticians (among others Dempster, 1964) have proved that Fisher imposed far too many restrictions and that one can renounce at a part of them or at least replace them with something more convenient. For instance, he asserted that Fisher put the condition that a proper fiducial distribution must depend on the data only through sufficient statistics and parameters, without explaining beyond reasonable doubt why only this kind of dependence remains valid for post-data prescription.

2.2 Fiducial inference without pivotal variables

Should it not be possible to find a pivotal quantity, one needs an alternative solution. Some authors (among others Sprott, 1963) have developped a theory to support the fiducial inference in the cases in which one has no pivotal quantity.

The experiment is constructed as following: from a population X, with the density function $f(x,\theta)$, a value x is yielded. Is this value a sufficient statistic for θ, then the probability associated with the experiment is:

$$F(x,\theta) = \int_{-\infty}^{x} f(t;\theta)dt.$$

If the experiment is repeated K times, yielding $x_1, x_{2,\ldots,} x_K$, then one can actually find a θ_k for each x_k, so that the relation

$$(**) \qquad F(x_k,\theta) = F(x,\theta_k)$$

holds, irrespective of the value that θ might have.

In other words, the sample $x_1, x_{2,\ldots,} x_K$ (who represents a series of future observations) is generating a sample $\theta_1, \theta_{2,\ldots,} \theta_K$ and as a consequence is transformed into the original observations while preserving probabilities. So the uncertainty in these future observations for a given θ is being transferred to θ in the sense that $\theta_{1,}\theta_{2,...,}\theta_K$ would all give rise to the original x with the same probability that θ gave rise to $x_{1,}x_{2,...,}x_K$. Since X_k is a random variable, it follows that for the original fixed x, $\theta_k = \theta_k(\theta, x_{k,}x)$ defined by $(**)$ can be expressed as a random variable. This reasoning leads us to the obtaining of the usual fiducial distribution, so that $\theta_{1,}\theta_{2,...,}\theta_K$ is a sample generated by $x_{1,}x_{2,...,}x_K$ from the fiducial distribution $f(x, \theta)$.

2.3 Characteristics of fiducial inference

The first thing to say about the fiducial probability is that it isn't an ordinary frequency distribution -in the Neyman-Pearson sense-, but, as implied in the above sections, something new.

In the continuos one-dimensional case the fiducial distribution is unique and the α-fiducial interval coincides with the α-confidence interval. In other words, an α-confidence interval consists of those values of the parameter which,-taken as null hypothesis-, are not rejected at the 1-α level by a test, while an α-fiducial interval asserts that the odds are $\frac{1-\alpha}{100}$ to $\frac{\alpha}{100}$ that the parameter value, when established, will lie in the respective interval.

In the multidimensional case, the non-uniqueness of pivotal quantities can produce several fiducial distributions, which cannot be considered as a inconsistency of fiducial inference. Sprott (1963) and other authors tried to provide supplementary conditions which should ensure uniqueness, but they only obtained a particular frame in which the resulting fiducial distributions are the same. For example, for a bivariate normal distributed random vector $\mathbf{X} \sim N(\mu = (\mu_1, \mu_2), \sigma, \rho)$ the fiducial distributions that can be obtained for $\frac{\mu_1}{\mu_2}$ are different for different pivotals.

Another problem that puzzled many statisticians and was regarded as an inconsistency of the fiducial argument was the fact that each sample provides us a different fiducial distribution, although this should not be surprising, because different samples reflect different knowledge.

The incorporation of additional information in the fiducial distribution is also possible. If an inference has to be made about the unknown mean μ of a normal distribution with variance 1, in the absence of any prior information, then the fiducial argument provides an $N(x, 1)$ as fiducial distribution for μ. Now, suppose that it is known additionally that the mean is nonnegative , i.e. $\mu \in [0, \infty)$.

Applying the fiducial inference in this new situation provides as distribution function for μ

$$F(\mu) = \begin{cases} 0, & \text{if } \mu < 0, \\ \Phi(\mu - x), & \text{if } \mu \geq 0. \end{cases}$$

2.4 Comparison with other methods

There has been a tendency to claim the universality of such methods as the use of the likelihood function or the use of Bayes theorem. As pointed out by Rao and continually emphasized by Fisher, different situations require different approaches depending on what is known. Sprott (1967) concludes that the fiducial argument, in the cases to which it applies, produces objective probability measures of uncertainty arising solely from the observations at hand. In particular, unwanted parameters can then be integrated out, and fiducial probability provides the only non-Bayesian method where this is possible. The apparent inconsistencies should be cleared up by more careful reasoning.

The **Likelihood** approach uses, just like the fiducial inference, only the actual sample and the information about the form of the population distribution, but doesn't need an additional information - e.g. about a pivotal quantity. On the other hand, it is quite clear that the statements obtained with in terms of fiducial probabilities are stronger than the corresponding likelihood statements although they both use the same degree of knowledge.

Birnbaum pointed out (1960) that the fiducial method has some similarities with the likelihood method. Let us consider the case of two parameters θ_1 and θ_2 and distributions that admit symmetry of the likelihood ratio, in fact the case with an unique decomposition into simple experiments. Within that simple experiment the permutation group on two elements can be used and a fiducial distribution on the two parameter values derived. The conclusion is that the ratio of fiducial probabilities turns out to be the ratio of the likelihoods.

The Bayesian claim that the **Baye**s theorem is always applicable, and obtain a prior distribution for the parameter subjectively or axiomatically. In basic scientific work it seems, however, that one rarely has any precise empirical knowledge concerning the prior distribution. Furthermore, no method of assigning objective and consistent prior distributions is known. In this frame it can't be considered as an inconsistency of the fiducial distribution the fact that it doesn't necessarily obey the Bayes theorem. Still, there is possible to make one connection between fiducial distributions and priors. It was shown by Hora and Buehler (1966) that for any location and scale parameter problem fiducial densities are exactly the same as Bayesian posterior densities obtained using certain prior distributions, namely the Jeffreys priors. They also computed in a few cases the apriori distributions which give rise to the same distribution as the fiducial distribution.

Fraser has pointed out that the information about θ extracted by means of fiducial argument in the pivotal case can be combined in a logical manner with a prior density function with respect to Haar measure and be entirely consistent with the Bayes approach.

Lindley proved that in the one-dimensional case a fiducial distribution is Bayes posterior if and only if the c.d.f. is invariant under a continuos one-parameter group of transformations. It is also known that if in the multi-parameter case is invariant under a particular type of group, then the fiducial distribution is also Bayes'posteriori. However, Brillinger provided an example that contradicts the converse affirmation - more precisely, that a bivariate fiducial distribution may be Bayes without possesing strong group invariance properties.

Dempster explored a generalization of Bayesian inference which also uses fiducial-like arguments, but belongs without any doubt to the Bayesian theory because a certain prior information is incorporated in the model. His inferences assign upper and lower probabilities to our uncertainties rather then precise probabilities as given by Bayesian or fiducial argument. Walley (1996) proposed another method for making inferences in cases where there's no or there is only little prior information. He expresses also the model in the language of upper and lower probabilities but his reasoning is different from the fiducial method because he resorts to the Bayes theorem, too.

The **significance** intervals of tests may be also used to generate probability statements concerning parameters of statistical models. One should recognize however that inferential statements on the basis of significance tests are directed not only at the uncertainity of a special parameter of a statistic model -as it is the case with the fiducial inference- but at the model itself and at its auxiliary hypotheses frequently not specified.

Sprott (1967) has provided a parallel between the significance intervals and fiducial intervals, his result being that there are cases in which the two kinds of intervals coincide mathematically, but there also cases in which this doesn't occur.

Dempster (1963) states "I should not wish to claim that **confidence** statements have no relevance to scientific inference, but I find it desirable to be clear that fiducial statements are logically distinct from confidence statements". This happens because in the post-data situation a fiducial probability is intended for predictive interpretation while a confidence coefficient is susceptible only to postdictive interpretation.

The fiducial approach, just like the Neyman-Pearson approach, uses no prior distribution to describe the uncertainty of the parameter, but the Neyman-Pearson confidence interval covers the true value with a given probability, so that the random variable is in this case an interval. Another difference between the fiducial and the frequentist inference is that the fiducial approach uses exclusively the actual concrete sample, whereas the Neyamn-Pearson approach takes into account the whole sample space. In many cases, the fiducial interval can be identical to a confidence interval, but in some examples -as the Behrens-Fisher problem- they differ.

In 1995, Barnard synthetized as following the spirit of the fiducial argument:

> "Fisher never claimed universal applicability for his fiducial argument, only that the form of statement to which it leads is particularly easy to understand and that its domain of application is sufficiently wide to make it of considerable interest".

3 Fiducial distribution - the discrete case

Although Fisher limited his fiducial inference to parameters of continuous variables, this restriction may be proved-as some others- to be unnecessary. Kalbfleisch and Sprott (1967), illustrating the idea of fiducial inference, put forward the following example:

Example 3.

> Suppose that θ is a fixed unknown integer, and that x can take values $\theta+1$, $\theta+2$, $\theta+3$, each with probability 1/3. The quantity $u = x - \theta$ may then assume values 1 ,2, 3, each with probability 1/3. The statement
>
> $$\Pr(U = u) = \frac{1}{3}, \quad for\ u \in \{1, 2, 3\}$$
>
> does not depend upon x or θ. For any specified θ, this relation may be used to calculate the probabilities of various possible x values. After x has been observed, such probability stements are no longer relevant because x is now known. However the distribution of u does not depend upon x, and observing x provides no information concerning which value of u was actually obtained. Consequently, the above equality is still true, and the particular observed value of x could have arisen with $u = 1$, $u = 2$,or $u = 3$, these three probabilities being equally probable. Thus
>
> $$\Pr(\Theta = x - u) = \frac{1}{3}, \quad for\ u \in \{1, 2, 3\}$$
>
> and this is called the fiducial distribution of θ.

Besides being, in the authors' own words, "a very simple, somewhat artificial, example", it is also an example of fiducial inference about the parameters of a discrete variable, which they unfortunately fail to recognize or at least emphasize. This example taken alone, and it suffices to state that rejecting en masse the applicability of the fiducial inference to discrete random variables is fallacious.

There are discrete cases, however, in which the things are not cristal-clear. Let us consider the Bernoulli-distribution:

$$X \sim Ber\,(1; p)\,,$$

with the probability function

$$f\,(x; p) = p^x\,(1-p)^{1-x}\,,\ x \in \{0; 1\}\,.$$

A sufficient statistic for the parameter p is S, the number of successes in K Bernoulli-experiments $X_1, \ldots, X_k, \ldots, X_K$, $X_k \sim Ber\,(1; p)\,, k \in \{0, 1, \ldots, K\}$, i.e. $S = X_1 + \ldots + X_k + \ldots + X_K$. It's distribution is a binomial one, with the parameters K and p.

The probability function is

$$f\,(s; K, p) = \binom{K}{s} p^s\,(1-p)^{1-s}\,,\ for\ s = \sum_{k=1}^{K} x_k \in \{0, 1, \ldots, K\}\,.$$

and the corresponding cumulative distribution function is

$$F(x;K,p) = \sum_{k=0}^{s} \binom{K}{k} p^k (1-p)^{K-k}, \; for \; s-1 < x \le s.$$

Since a pivotal variable seems to be rather difficult to find, we use Sprott's approach. Using the well-known relationship between the binomial and the beta distributions:

$$\frac{1}{B(s, K-s+1)} \int_p^1 u^{s-1} (1-u)^{K-s} \, du = \sum_{k=0}^{s} \binom{K}{k} p^k (1-p)^{K-k}$$

it can be assigned a value of p to each potential observation s, thus obtaining

$$f(p;s) = \frac{p^{s-1}(1-p)^{K-s}}{B(s, K-s+1)}$$

as density of the fiducial distribution associated to the parameter p.

The probable reason for which Fisher refused to apply fiducial inference to discrete variables was the fact that the probability statements could not be preserved and only statements about inequalities were admissible. On one hand, this is not the case for all discrete random variables, as we've seen, and on the other hand this is actually not so relevant for fiducial inference, at least in the formulation of Sprott; since we do not know the value of the parameter anyway, this is the best thing we can fiducially obtain, given the sample information.

The above arguments enables us to state that the restriction of fiducial inference to parameters of continuous random variables is only a prejudice that survived from the last century without any solid ground .

4 Conclusions

Many methods of expressing uncertainty in the parameter value have been suggested. It is generally agreed that when the parameter is a random variable with an empirically verifiable prior frequency distribution, the use of Bayes theorem is appropriate, and then the fiducial argument is not applicable. The likelihood function may be used to answer the same sort of question as fiducial probability and it seems reasonable to use it when neither the the fiducial argument nor the Bayes theorem is applicable. Confidence intervals also provide a reasonable and useful measure of uncertainty in the long run, but they can fail in particular cases.

The claim of the fiducial inference towards being a universally applicable inferencial theory was not successfull. Though, it greatly influenced the development of the Neyman-Pearson approach and recent works prove an outgoing interest towards an inferencial theory that generates distributions for the unknown parameters without resorting to any priors.

The fiducial inference is by no means a ripe inferencial approach. The extensive list of caveats -or at least of what were supposed to be caveats-has caused many statisticians to shy away from fiducial inference. The justification for the fact that Fisher himself advocates different fiducial inferences in different situations and fails in describing a general theory lies in the fact that he was an applied statistician. For him, each problem deserved a carefully constructed framework which accounted for the details of the issues in question and should be solved separately.

Nevertheless, Dempster points out that this is a strength rather than a weakness of fiducial inference. In his opinion the criticism "The fiducial argument appears to me to consist of meaningless manipulations with derivatives of distribution functions" does not hold because these manipulations should in fact be regarded as a convenient device for calculating the fiducial distribution and while the definitions on which such devices are based are clearly defined.

By renouncing at some unnecessary and rigid conditions that the fiducial distributions were supposed initially to fulfill, the fiducial argument has been proved to work also in the discrete case, as it was shown above for the Bernoulli distribution.

Wallace concludes about the fiducial inference the following:

> "Of all Fishers's creations, fiducial inference was perhaps the most ambitious, yet least accepted. Ignored by a large part of the statistics community satisfied with the mathematically simpler confidence approach, and rejected as logically imperfect and inconsistent in general by those who recogneized the strength of the fiducial objectives, the fiducial argument continues under active and sympathetic study today only in a few lands of the statistical world."

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