Stochastic algorithms solving nonlinear Boltzmann models

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Abstract. We present some new results on the convergence of sequences of sums of stochastic measures, solving numerically a general class of nonlinear kinetic equations for reacting fluids.

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Key words: Boltzmann, integro-partial differential equations, theoretical approximation of solutions.

1 Introduction

The last years have been marked by an increased interest in nonlinear Boltzmann-type equations modeling transport phenomena. This is mainly due to the existence of various far from equilibrium processes, which require a more realistic description, as that provided by generalized Boltzmann models [2].

The numerical method considered in our analysis solves, iteratively, time (and space) discretized versions of the exact kinetic equations. Specifically, the initial data are approximated by finite sums of Dirac measures. Due to the properties of the discretized models, the iteration results also in sums of Dirac measures approximating the true solution of the model. However, because of the nonlinearities, the iteration produces a power-like increasing number of terms. To render the method useful to applications, one diminishes the number of terms by means probabilistic selection. This is achieved following ideas introduced by Nambu [6] in the framework of the classical Boltzmann equation. The convergence of the method is proved in [5], by extending the results of [1] on Nambu's scheme.

In this article we introduce new, more efficient, convergent probabilistic selection schemes for the above iteration scheme.

As in the case of the classical Boltzmann equation, in the numerical solving of the kinetic models for reacting systems, the main difficulty comes form the treatment of the nonlinear integral terms. For this reason, for the sake of simplicity, we limit our considerations to the space-homogeneous case.

The paper is organized as follows. In Section 2 we present the Boltzmann model for space-homogeneous reacting gas mixtures. Section 3 introduces the iterated equations for measures. Our main results on the stochastic selection are formulated in Section 4.

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2 The Kinetic Model

We consider a space-homogeneous reacting gas in the absence of external fields, and without radiative processes, composed of $N \geq 1$, distinct species of mass-points, with one-state internal energy. The gas particles have free classical motion between instant (reactive) collisions. There are at most, $M \geq 2$ identical partners in each in (out) collision channel. By collisions, the particles may change their chemical nature (in particular, mass and internal energy) and velocity. It is supposed that the collisions occur with the conservation of the total mass, momentum and energy, respectively, according to the laws of classical mechanics. The internal energies of the particles enter in the energy balance.

Let $\mathcal{M} := \left\{ \gamma = (\gamma_n)_{1 \leq n \leq N} \mid \gamma_n \in \{0, 1, \ldots, M\},\ 2 \leq \sum_{n=1}^{N} \gamma_n \right\}$. Each kind of possible gas (reactive) collision can be represented by a couple $(\alpha, \beta) \in \mathcal{M} \times \mathcal{M}$. Here $\alpha = (\alpha_1, \ldots, \alpha_N)$ is the "in" collision channel. It refers to the pre-collisional configuration, with $\alpha_n \in \{0, 1, \ldots, M\}$ participants of the species $n = 1, ..., N$. Further, $\beta = (\beta_1, \ldots, \beta_N)$ is the "out" channel corresponding to the post-collisional configuration, with $\beta_n \in \{0, 1, \ldots, M\}$ participants of the species $n = 1, ..., N$. The total numbers of the particles in some channel γ is $\mid \gamma \mid := \sum_{n=1}^{N} \gamma_n$. The family of the species present in this channel is $\mathcal{N}(\gamma) := \{n \mid 1 \leq n \leq N,\ \gamma_n \geq 1\}$. For each $n \in \mathcal{N}(\gamma)$, there are γ_n identical particles of the species n in the channel γ. Their velocities are denoted by $\mathbf{w}_{n,1}, ..., \mathbf{w}_{n,\gamma_n} \in \mathbb{R}^3$. Also set $\mathbf{w} := ((\mathbf{w}_{n,i})_{1 \leq i \leq \gamma_n})_{n \in \mathcal{N}(\gamma)}$, understanding that $\mathbf{w} \in \mathbb{R}^{3|\gamma|}$. By $m_n > 0$ and $E_n \in \mathbb{R}$, denote the mass and the internal energy, respectively of a mass-point of the species $n = 1, ..., N$.

Let $V_\gamma(\mathbf{w})$ and $W_\gamma(\mathbf{w})$ be the classical mass center velocity and the total energy, respectively, for the particles in channel γ, i.e.

$$V_\gamma(\mathbf{w}) := (\sum_{n=1}^{N} \gamma_n m_n)^{-1} \sum_{n \in \mathcal{N}(\gamma)} \sum_{i=1}^{\gamma_n} m_n \mathbf{w}_{n,i},$$

$$W_\gamma(\mathbf{w}) := \sum_{n \in \mathcal{N}(\gamma)} \sum_{i=1}^{\gamma_n} (2^{-1} m_n \mathbf{w}_{n,i}^2 + E_n).$$

According to the conservation assumptions, in the description of the gas kinetics, one has to take into account only the reactions $(\alpha, \beta) \in \mathcal{M} \times \mathcal{M}$, for which

$$\sum_{n=1}^{N} m_n(\alpha_n - \beta_n) = 0, \tag{2.1}$$

$$V_\alpha(\mathbf{w}) = V_\beta(\mathbf{u}), \quad W_\alpha(\mathbf{w}) = W_\beta(\mathbf{u}), \tag{2.2}$$

with $\mathbf{w} = ((\mathbf{w}_{n,i})_{1 \leq i \leq \gamma_n})_{n \in \mathcal{N}(\alpha)}$ and $\mathbf{u} = ((\mathbf{u}_{n,i})_{1 \leq i \leq \beta_n})_{n \in \mathcal{N}(\beta)}$ the velocities of the particles in the channels α and β, respectively.

For each $(\alpha, \beta) \in \mathcal{M} \times \mathcal{M}$, $k = 1, 2, ..., N$, denote

$$g_\gamma(\mathbf{w}) := \prod_{n \in \mathcal{N}(\gamma)} \prod_{i=1}^{\gamma_n} g_n(\mathbf{w}_{n,i}), \quad \forall \gamma \in \mathcal{M}. \tag{2.3}$$

Relations (2.1) and (2.2) exclude reactions with *at most* one particle in some reaction channel (because in the absence of radiative processes, (2.1) and (2.2) cannot be simultaneously fulfilled). This explains the restriction $|\gamma| \geq 2$ in the definition of $\mathcal{M}$. The conservation of the total energy stated in (2.2) implies the existence of reaction thresholds and shows what happens with the internal energies of the particles participating in reactions: for instance in dissociation, the binding energy is gained by the resulting reaction products as kinetic energy.

The abstract transcription of the WCUB and LH equations or the standard Boltzmann procedure (based on the molecular chaos assumption) lead to the following system of space-homogeneous kinetic equations

$$\partial_t f_k = P_k(f) - S_k(f), \qquad 1 \leq k \leq N. \tag{2.4}$$

The unknowns are the functions $f_k : \mathbb{R}_+ \times \mathbb{R}^3 \to \mathbb{R}_+$, $1 \leq k \leq N$, where $\mathbb{R}_+ := [0, \infty)$. Here $f_k = f_k(t, \mathbf{v})$ (t-time, $\mathbf{v}$ -velocity) is the one-particle distribution function for species $k = 1, ..., N$ of particles, and $f := (f_1, \ldots, f_N)$.

For some $m \in \mathbb{N}$, let $C_b(\mathbb{R}^m)$ be the space of the bounded elements of $C(\mathbb{R}^m; \mathbb{R})$, with the usual sup norm. Denote by $C_c(\mathbb{R}^m)$ the subset of the functions of $C_b(\mathbb{R}^m)$ with compact support.

The collision processes are described by the gain and loss terms, $P_k(f)$ and $S_k(f)$, respectively. Following [4], one can see that the collision terms have the form

$$\begin{aligned} &P_k(g)(\mathbf{v}) = \\ &= \sum_{\alpha,\beta \in \mathcal{M}} \alpha_k \left[\int_{\mathbb{R}^{3|\alpha|-3} \times \mathbf{\Omega}_\beta} p_{\beta,\alpha}(\mathbf{w}, \mathbf{n})\, g_\beta(\mathbf{u}_{\beta,\alpha}(\mathbf{w}, \mathbf{n}))\, d\tilde{\mathbf{w}}_k\, d\mathbf{n} \right]_{\mathbf{w}_{k,\alpha_k} = \mathbf{v}}, \end{aligned} \tag{2.5}$$

$$S_k(g)(\mathbf{v}) = \sum_{\alpha,\beta \in \mathcal{M}} \alpha_k \left[\int_{\mathbb{R}^{3|\alpha|-3} \times \mathbf{\Omega}_\beta} r_{\beta,\alpha}(\mathbf{w}, \mathbf{n})\, g_\alpha(\mathbf{w})\, d\tilde{\mathbf{w}}_k\, d\mathbf{n} \right]_{\mathbf{w}_{k,\alpha_k} = \mathbf{v}}, \tag{2.6}$$

for all $g \in C_c^N(\mathbb{R}^3)$.

It is known [3] that, if $r_{\alpha;\beta}(\mathbf{v}, \mathbf{w}, \mathbf{n}) \leq C(1 + |\mathbf{v}|^2 + |\mathbf{w}|^2)$ for some constant, $C > 0$, then the Cauchy problem associated to the system (2.4) has, in some sense, unique positive global solutions $f_i(t) \in \mathcal{L}^1(\mathbb{R}^3; d\mathbf{v})-$ real, $1 \leq i \leq N$, provided that the initial data are positive and their 4^{th} order momenta are well defined. In addition, the solutions satisfy the global mass, momentum and energy (kinetic + internal) conservation (for more details the interested reader is referred to [3], [4], [5]).

3 Time Discretization and Weak Form of Equations

We introduce a discretized version of (2.4). Consider some time interval $[0, T]$ and let $0 < \Delta t < T$ be a fixed time step. Denote by $[[x]]$ the integer part of x. Consider the following iteration scheme

$$\begin{aligned} f_k^{\,j} &= f_k^{\,j-1} + \Delta t \cdot \left[P_k(\mathbf{f}^{\,j-1}) - S_k(\mathbf{f}^{\,j-1}) \right], \\ f_k^0 &= f_{0,k} \geq 0, \quad a.e., \quad j = 1, ..., \left[\left[\tfrac{T}{\Delta t}\right]\right]; \quad k = 1, ..., N, \end{aligned} \tag{3.7}$$

where $\mathbf{f}^{\,j} = (f_1^{\,j}, ..., f_N^{\,j})$ and $f_k^{\,j} = f_k^{\,j}(\mathbf{v})$.

If Δt is small and if $\int_{\Omega_\beta} r_{\beta,\alpha}$ is bounded, it is shown [5], that $\mathbf{f}^{\,j}$ is close to the solution $\mathbf{f}(t)$ of (2.4), on $((j-1)\Delta t,\, j\Delta t]$. The resulting scheme (3.7) might destroy the positivity of the functions f_k^j for $j \geq 1$. However, if Δt is sufficiently small, then f_k^j remains positive [5].

Since the macroscopic quantities are evaluated as integrals with respect to measures having one-particle distribution functions as densities, for numerical purposes, it is convenient to write (3.7) in the weak form. The functions f_k^j are replaced by the measures μ_k^j, given by $d\mu_k^j\,(\mathbf{v}) := f_k^j\,(\mathbf{v})\,d\mathbf{v}$.

For some $\gamma \in \mathcal{M}$ and $j = 0, 1, ..., \left[\left[\frac{T}{\Delta t}\right]\right]$, define the measure μ_γ^j on $\mathbb{R}^{3|\gamma|}$ by

$$d\mu_\gamma^j(\mathbf{w}) = \otimes_{k \in \mathcal{N}_\gamma} \otimes_{i=1}^{\gamma_k} d\mu_k^j(\mathbf{w}_{k,i}). \tag{3.8}$$

Define $\mathcal{U} := \{\gamma = (\gamma_k)_{1 \leq k \leq N} \quad |\gamma_k \in \{0, 1, ..., NM\},\ |\gamma| \geq 2\}$. Following [5], one can put (3.7) in the form

$$\int_{\mathbb{R}^3} \varphi(\mathbf{v})\, d\mu_k^j(\mathbf{v}) = \sum_{l=1}^{L} \int_{\mathbb{R}^{3|\alpha(l)|+q(l)}} R_{k,l}(\mathbf{z})(\varphi \circ h_{k,l})(\mathbf{z})\, d(\mu_{\alpha(l)}^{j-1} \otimes \pi_l)(\mathbf{z}), \tag{3.9}$$

with $R_{k,l} \in C(\mathbb{R}^{3|\alpha(l)|+q(l)}; \mathbb{R}_+)$ and $h_{k,l} \in C(\mathbb{R}^{3|\alpha(l)|+q(l)}; \mathbb{R}^3)$. Here, $L \in \mathbb{N}^*$, $\{\alpha(l)\}_{1 \leq l \leq L} \subset \mathcal{U}$, $\{q(l)\}_{1 \leq l \leq L}$ is some given sequence of natural numbers and $\{\pi_l\}_{1 \leq l \leq N}$ is a family of known measures absolute continuous with respect to the Lebesgue measure on $\mathbb{R}^{q(l)}$.

4 Stochastic Algorithms

In order to solve numerically (3.9), one approximates the initial data by weakly convergent sums of Dirac measures.

The iteration scheme produces a weakly convergent sequence of approximating solutions, depending on the iteration order [5]. Every approximating solution is also sum of Dirac measures. However, because of the nonlinearity in (3.9), each iteration step produces a power-like growing number of terms in the sums of point measures.

As mentioned in Introduction, in order to keep this number of terms constant, one applies random selections.

To this end, first we construct two suitable families of random multi-indices.

Let $\Omega = [0,1)^{\infty}$ (in the countable sense) be endowed with the usual product Borel $\sigma-$algebra β_{Ω} and P the usual product probability induced on Ω by the uniform distribution of $[0,1)$.

For some given m_n and for some $l \in \mathbb{N}^*$, define $c_{n,l} : \Omega \to \{1,2,...,m_n\}$ by $c_{n,l}(\omega) := [[\omega_l \cdot m_n]] + 1$, where ω_l is the l^{th} component of $\omega = (\omega_1, \omega_2, ...) \in \Omega$. Obviously,

$$P(\{c_{n,l}(\omega) = j\}) = \frac{1}{m_n} \quad for\ all \quad j = 1,...,m_n.$$

Consequently, for some $q \in \mathbb{N}^*$ and distinct $l_1, l_2, ..., l_q \in \mathbb{N}^*$, the random variables $c_{n,l_1}, ..., c_{n,l_q}$ are i.i.d.. Then for each $n \in \mathbb{N}^*$, we introduce the following families of row-wise independent, random multi-indices.

a) Let $p_n \geq 2$, for all $n \in \mathbb{N}^*$. For each n, consider the family $(\mathbf{J}_{n,i})_{1 \leq i \leq m_n}$ of random vectors $\mathbf{J}_{n,i} : \Omega \to \{1,2,...,m_n\}^{p_n}$ defined on components by

$$\mathbf{J}_{n,i}(\omega) := (i, \ \ c_{n,\,(i-1)p_n+1}(\omega), \ \ c_{n,\,(i-1)p_n+2}(\omega), ..., \ \ c_{n,\,ip_n-1}(\omega)), \tag{4.10}$$

for all $\omega = (\omega_1, \omega_2, ...) \in \Omega$; $i \in \{1, ..., m_n\}$.

b) Let $p_n \geq 1$, for all $n \in \mathbb{N}^*$ and $\{k_n\}_{n \in \mathbb{N}^*}$ a given sequence of positive integers. For each n, consider the family $(\mathbf{C}_{n,i})_{1 \leq i \leq k_n}$ of random vectors $\mathbf{C}_{n,i} : \Omega \to \{1,2,...,m_n\}^{p_n}$ defined (on components) by

$$\mathbf{C}_{n,i}(\omega) := (c_{n,\,(i-1)p_n+1}(\omega), \ \ c_{n,\,(i-1)p_n+2}(\omega), ..., \ \ c_{n,\,ip_n}(\omega)), \tag{4.11}$$

for all $\omega = (\omega_1, \omega_2, ...) \in \Omega$; $i \in \{1, ..., k_n\}$.

The row-wise independence of $(\mathbf{J}_{n,i})_{1 \leq i \leq m_n}$ and $(\mathbf{C}_{n,i})_{1 \leq i \leq k_n}$, respectively, is immediate observing that $ip_n + j = i'p_n + j'$ if and only if $i = i'$ and $j = j'$, for all $i, i' \in \mathbb{N}^*$ and $j, j' = 1, 2, ..., p_n$.

Denote by $\mathcal{I}_n := \{1,2,...,m_n\}^{p_n}$. Let $((\mu_{n,\mathbf{j}})_{\mathbf{j} \in \mathcal{I}_n})_{n \in \mathbb{N}^*}$ be a bounded family of positive measures on $\mathbb{R}^m$, i.e. $\exists$ some constant $a > 0$ such that $|\mu_{n,\mathbf{j}}| \leq a$, for all $\mathbf{j} \in \mathcal{I}_n$, $n \in \mathbb{N}^*$ (where $|\mu| := \mu(\mathbb{R}^m)$ for some finite measure μ on $\mathbb{R}^m$). Suppose that there is a positive measure μ on $\mathbb{R}^m$, absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^m$ such that

$$\frac{1}{m_n^{p_n}} \sum_{\mathbf{j} \in \mathcal{I}_n} \mu_{n,\mathbf{j}} \xrightarrow{w} \mu, \quad as \quad n \to \infty, \tag{4.12}$$

where $\xrightarrow{w}$ denotes the weak convergence for measures.

Theorem 1 *a) Let $p_n \geq 2$, for all $n \in \mathbb{N}^*$ and define $\mu_{n,i}(\omega) := \mu_{n,\mathbf{j}} \mid_{\mathbf{j}=\mathbf{J}_{n,i}(\omega)}$, for all $\omega \in \Omega$. If $\sum_{n=1}^{\infty} m_n^{-2} < \infty$, then for $P-$almost all ω,*

$$\sigma_{1,n}(\omega) := \frac{1}{m_n} \sum_{i=1}^{m_n} \mu_{n,i}(\omega) \xrightarrow{w} \mu, \quad as \quad n \to \infty. \tag{4.13}$$

b) Let $p_n \geq 1$, for all $n \in \mathbb{N}^*$ and define $\mu_{n,i}(\omega) := \mu_{n,\mathbf{j}} \mid_{\mathbf{j}=\mathbf{C}_{n,i}(\omega)}$, for all $\omega \in \Omega$. If $k_n \leq m_n$ for all n and $\sum_{n=1}^{\infty} k_n^{-2} < \infty$, then for $P-$almost all ω,

$$\sigma_{2,n}(\omega) := \frac{1}{k_n} \sum_{i=1}^{k_n} \mu_{n,i}(\omega) \xrightarrow{w} \mu, \quad as \quad n \to \infty. \tag{4.14}$$

Theorem 1.a can be applied to reduce the number of terms in the product measures appearing in the r.h.s of (3.9). The result represent a generalization of the selection method used in the classical Boltzmann equation.

Since the r.h.s. in (3.9) has several terms of product measures, one way to reduce the terms (in the resulting sum of Dirac measures) can be performed by invoking the point (b) in Theorem 1. However, the functions $R_{k,l}$ in (3.9) destroy the homogeneity of the sums of Dirac measures in (4.12). Then the point (b) in Theorem 1 cannot be applied directly. One needs following homogenization result.

Let $\left((a_{n,i})_{1\leq i\leq n}\right)_{n\in\mathbb{N}^*}$ be a family of positive numbers and $\left((\mu_{n,i})_{1\leq i\leq n}\right)_{n\in\mathbb{N}^*}$ a family of positive finite measures on $\mathbb{R}^m$. For some $c > 1$ fixed, set $s_n := \sum_{i=1}^{n} [[n^c \cdot a_{n,i} + 1]]$. Also define the function Λ: $\{1, ..., s_n\} \to \{1, ..., n\}$ as

$$\Lambda(i) := \min_{1\leq j\leq n} \left\{ j \,\middle|\, i \leq \left(\sum_{k=1}^{j} [[n^c \cdot a_{n,k} + 1]] \right) \right\}.$$

For each $n = 1, 2, ...$, assume that $|\mu_{n,1}| = |\mu_{n,2}| = ... = |\mu_{n,n}| := b_n$.

Lemma 2 *Suppose that there is some finite measure μ such that*

$$\sum_{i=1}^{n} a_{n,i} \cdot \mu_{n,i} \xrightarrow{w} \mu, \quad as \quad n \to \infty. \tag{4.15}$$

If $\frac{b_n}{n^{c-1}} \to 0$ as $n \to \infty$, then

$$\frac{1}{n^c} \sum_{i=1}^{s_n} \mu_{n,\Lambda(i)} \xrightarrow{w} \mu, \quad as \quad n \to \infty. \tag{4.16}$$

Instead of applying Theorem 1.b, numerical tests suggested that more accurate results can be obtained if the selected terms in (4.12) are only one time extracted. This implies a conditional selection.

For simplification, we consider $p_n = 1$ in (4.12). Let $(\Omega, \beta_\Omega, P)$ be the same probability space defined before and let $c_{n,l} : \Omega \to \{1, 2, ..., m_n\}$ be random variables defined as follows.

$$c_{n,l}(\omega) := [[\omega_l \cdot (m_n - l + 1)]] + 1, \tag{4.17}$$

with m_n a sequence of natural numbers such that $m_n \to \infty$ as $n \to \infty$.

Using the independent random variables $c_{n,l}$ defined in the beginning of this Section, we introduce, recursively, the (non independent) random variables

$d_{n,l,j} : \Omega \to \{1, 2, ..., m_n\}$, with $d_{n,0,j} := j$, for all $j \in \{1, 2, ..., m_n\}$,

$$d_{n,l+1,j}(\omega) := \begin{cases} d_{n,l,j}(\omega), & \text{if} \quad j \in \{1, 2, ..., m_n\} \setminus \{c_{n,l}(\omega)\} \\ \\ m_n - l + 1, & \text{if} \quad j = c_{n,l}(\omega) \end{cases} . \tag{4.18}$$

Denote by $s_{n,i}(\omega) := d_{n,i,m_n-k_n+1}(\omega)$.

Let μ and ν measures on $\mathbb{R}^m$. For some bounded, continuous real valued function ϕ on $\mathbb{R}^m$ define the mean error $e_\phi(\nu, \mu)$ between μ and ν by

$$e_\phi(\nu, \mu) := \left\langle \left| \int_{\mathbb{R}^m} \phi d\nu - \int_{\mathbb{R}^m} \phi d\mu \right| \right\rangle, \tag{4.19}$$

where $\langle \quad \rangle$ designates the mean with respect to the probability of selection P.

Let $((\mu_{n,j})_{1 \le j \le m_n})_{n \in \mathbb{N}^*}$ be a bounded family of positive measures on $\mathbb{R}^m$. Suppose that there exists a positive measure μ on $\mathbb{R}^m$, absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^m$ such that

$$\frac{1}{m_n} \sum_{j=1}^{m_n} \mu_{n,j} \xrightarrow{w} \mu, \quad as \quad n \to \infty. \tag{4.20}$$

Theorem 3 *If $m_n \ge k_n$, for all $n \in \mathbb{N}^*$ and $k_n \to \infty$ as $n \to \infty$ and if $\sum_{n=1}^{\infty} k_n^{-2} < \infty$ then*

$$e_\phi \left(\frac{1}{k_n} \sum_{i=1}^{k_n} \mu_{n,s_{n,i}}(\omega), \mu \right) \to 0, \quad as \quad n \to \infty. \tag{4.21}$$

The selection + homogenization by means of Theorem 1.b and Lemma 2 can be replaced by the following type of weighted selection.

Let $((a_{n,j})_{1 \le j \le m_n})_{n \in \mathbb{N}^*}$ be a positive sequence of real numbers and suppose that $\sum_{l=1}^{m_n} a_{n,l} < \infty$, for all $n \in \mathbb{N}^*$. We introduce the weights $p_{n,j} := a_{n,j} \cdot (\sum_{l=1}^{m_n} a_{n,l})^{-1}$ and denote by $q_{n,l} := \sum_{j=1}^{l} p_{n,j}$, for all $j \in \{1, 2, ..., m_n\}$.

For a given $m_n \in \mathbb{N}^*$ and $l \in \mathbb{N}^*$, we define the random variables $\tilde{c}_{n,l} : \Omega \to \{1, 2, ..., m_n\}$ by

$$\tilde{c}_{n,l}(\omega) := l, \quad if \quad \omega_l \in [q_{n,l-1}, q_{n,l}), \tag{4.22}$$

where ω_l is the l^{th} of $\omega = (\omega_1, \omega_2, ...) \in \Omega$ and $q_{n,0} := 0$.

Theorem 4 *Let $((\mu_{n,j})_{1\le j\le m_n})_{n\in\mathbb{N}^*}$ be a bounded family of positive measures on $\mathbb{R}^m$ and μ a finite positive measure on $\mathbb{R}^m$, absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^m$ such that*

$$\sum_{j=1}^{m_n} a_{n,j}\cdot\mu_{n,j} \xrightarrow{w} \mu, \quad as \quad n\to\infty. \tag{4.23}$$

Let $k_n\in\mathbb{N}^*$, $k_n\le m_n$ such that, $k_n\to\infty$ as $n\to\infty$. Define $\tilde{\mu}_{n,i}(\omega) := \mu_{n,\tilde{c}_{n,i}(\omega)}$, for all $\omega\in\Omega$. If $\sum_{n=1}^{\infty} k_n^{-2}<\infty$, then for $P-$almost all ω,

$$\sigma_{3,n}(\omega) := \left(\sum_{l=1}^{m_n} a_{n,l}\right)\frac{1}{k_n}\sum_{i=1}^{k_n}\tilde{\mu}_{n,i}(\omega) \xrightarrow{w} \mu, \quad as \quad n\to\infty. \tag{4.24}$$

The above theorems were successfully applied in numerical tests [7], [8] for space-homogeneous equations of type (2.4).

We end this section with the remark that, in the more complicated situation where the equations are space dependent [4], the algorithm may be improved by introducing an additional selection. This can be achieved considering the number of selected terms also as a random variable. To this end, the following convergence theorem is useful.

Let μ and $((\mu_{n,\mathbf{j}})_{\mathbf{j}\in\mathcal{I}_\setminus})_{n\in\mathbb{N}^*}$ be defined as in Theorem 1. Let $(k_n)_{n\in\mathbb{N}^*}$ be a sequence of random variables defined on Ω taking values in $\mathbb{N}^*$ such that, for each $n\in\mathbb{N}^*$, each random variable $\mathbf{C}_{n,i}$ (where $\mathbf{C}_{n,i}$ are defined as in (4.11)) to be independent with respect to k_n for each $i\in\{1,...,n\}$.

Theorem 5 *Let $p_n\ge 1$, for all $n\in\mathbb{N}^*$. Define $\mu_{n,i}(\omega) := \mu_{n,\mathbf{j}}\,|_{\mathbf{j}=\mathbf{C}_{n,i}(\omega)}$, for all $\omega\in\Omega$. If $m_n\ge\langle k_n\rangle$, for all $n\in\mathbb{N}^*$ and $\sum_{n=1}^{\infty}\langle k_n^{-2}\rangle<\infty$, then for $P-$almost all ω,*

$$\sigma_{4,n}(\omega) := \frac{1}{k_n(\omega)}\sum_{i=1}^{k_n(\omega)}\mu_{n,i}(\omega) \xrightarrow{w} \mu, \quad as \quad n\to\infty. \tag{4.25}$$

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