An equivalency condition of the invexity and preinvexity notions in optimization theory

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Abstract. In this paper it is shown that the notions of invexity, pseudoinvexity and quasiinvexity, respectively of preinvexity, prepseudoinvexity and prequasiinvexity are equivalently for the real differentiable functions defined on nonempty open sets in $\mathbb{R}^n$. The invexity and preinvexity notions are not equivalent for real functions defined on closed sets.

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1 Introduction

In 1981 Hanson [6] introduced in the differentiable optimization theory the invex, pseudoinvex and quasiinvex functions, which generalize the convex, pseudoconvex and quasiconvex functions, respectively.

In 1987 Hanson and Mond [7] introduced in the optimization theory the preinvex functions, while in the differentiable case form a subclass of invex functions. In 1991 Pini [13] generalized the preinvex functions through the pre-pseudoinvex and pre-quasiinvex functions.

Later on, these functions have known numerous developments and applications. E.g., in nonsmooth framework, Craven defined in 1986 the invex functions [3] and in 1993 Mititelu defined the pseudoinvex and quasiinvex functions ([5], [11]).

Moreover, in 1985 Craven and Glover [4] showed that a *differentiable function is invex iff each its stationary point is a global minimum point* (rephrased by Ben Israel and Mond [1]). Also, in 1997 Mititelu [10] showed that *for all generalized (strict) preinvex functions, every (strict) local minimum point is a (strict) global minimum point*, excepting the semistrict quasiinvex functions.

These properties were reasons to introduce invex and preinvex functions and their generalizations in optimization theory.

Equivalency relations among invexity types and preinvexity for open sets were also set materialized by Mititelu [10] in 1997. The aim of this paper is to establish precise correlations between the invexity types and the preinvexity types for the real differentiable functions defined on open sets and on closed sets. Similar correlations between the generalized types of these notions with respect to open sets are established.

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2 Invex functions and preinvex functions on open sets and on closed sets

Let A be a nonempty open set in $\mathbb{R}^n$ and a function $f : A \to \mathbb{R}$, differentiable on A.

Definition 2.1. The function f is said to *invex* (on A) if there exists a vector function $\eta : A \times A \to \mathbb{R}^n$ such that

$$(I) \qquad \forall\, x, u \in A,\ f(x) - f(u) \geq \eta^t(x, u)\nabla f(u).$$

where $'$ is the transposed sign and ∇f is the gradient of f.

This notion introduced by Hanson [6] in 1981 and its name was given by Craven [2] in the same year, as an abbreviation of "*invariant convex*". Indeed, Martin [9] in 1995 showed that if f satisfies the property on and $\varphi : f(A) \to \mathbb{R}^n$ is a differentiable convex function, then the composed function $\varphi \circ f$ satisfies (I) too. Therefore, the invex function remains "invariant through convexity".

The invex functions generalize the convex ones obtained in particular for $\eta(x, u) = x - u$.

The characteristic property of the invex functions is given by

Theorem 2.1 (Craven, Glover [4], Ben Israel, Mond [11]). *The function f is invex (on A) iff every stationary point is a global minimum point.*

In 1987 Hanson and Mond [7] introduced the notion of preinvex function (name given by Jeyakumar [16]) by the following

Definition 2.2. The function f is said to be *preinvex* (*on* A) if there exists a vector function $\eta : A \times A \to \mathbb{R}^n$ such that

$$(pI) \qquad \forall\, x, y \in A,\ \forall \lambda \in [0, 1],\ f(u + \lambda\eta(x, u)) \leq \lambda f(x) + (1 - \lambda)f(u),$$

where $u + \lambda\eta(x, u) \in A$.

The preinvex functions are used in optimization theory because of the following

Theorem 2.2 (Weir, Mond [16]). *For any preinvex function, each local minimum point is a global minimum.*

The domains of invex functions and of preinvex functions are invex sets [9]. A correlation between invex and preinvex functions is given by the following

Theorem 2.3 (Ben Israel, Mond [1]). *Any differentiable preinvex function is an invex function (with the same η).*

Coming now to Theorem 2.1, for a real function f with open domain A, we ascertain that the logical proposition

$$\langle \textit{every stationary point} \to \textit{a global minimum point} \rangle$$

determines the logical proposition

$$\langle \textit{each local minimum point} \to \textit{a global minimum point} \rangle,$$

that is, according to Theorem 2.2, f is a preinvex function.

Corollary 2.1. *For real differentiable functions defined on open sets, the invexity and preinvexity notions are equivalent.*

Differences between the two notions appear on the frontier of the domain, e.g., for closed sets, as shows the following.

Example. Let the set

$$X = \left\{ (x,y) \in \mathbb{R}^2 \,\middle|\, x \geq -\frac{1}{2},\ y \geq 1 \right\}$$

and the function

$$f : X \to \mathbb{R},\ f(x,y) = y(x^2-1)^2.$$

The stationary point set of f is $S = \{(1,y) \in \mathbb{R}^2 | y \geq 1\}$. Each point $(1,y) \in S$ satisfies the relation

$$f(1,y) = 0 \leq f(x,y), \quad \forall (x,y) \in X.$$

Therefore any stationary point of f is a global minimum point, that is, f is a invex function. We remark that $\left(-\frac{1}{2}, 1\right) \in Fr\ X$ is a local minimum point of f, but it is not a global minimum for f because

$$f\left(-\frac{1}{2}, 1\right) = \frac{9}{16} > 0 = f(1,y).$$

Therefore, f is not a preinvex function.

3 Pseudoinvex functions and prepseudoinvex functions on open sets

In what follows are necessary some properties of the directional derivative of the function $f : A \to \mathbb{R}$ at a point $u \in A$ in the direction $d \in \mathbb{R}^n$, namely,

$$f'(u;d) = \lim_{t\to 0} \frac{f(u+td) - f(u)}{t}.$$

We mention that

$$f'(u;d) = f'_r(u;d) = f'_l(u;d), \tag{3.1}$$

where

$$\begin{aligned} &f'_r(u;d) = \lim_{t \searrow 0} \tfrac{f(u+td)-f(u)}{t} \\ &f'_l(u;d) = \lim_{t \nearrow 0} \tfrac{f(u+td)-f(u)}{t} \\ &f'_r(u;\lambda d) = \lambda f'_r(u:d), \quad \forall \lambda \geq 0. \end{aligned} \tag{3.2}$$

(3.3) If f is differentiable at u, then $f'(u;d) = d'\nabla f(u)$.

Definition 3.1 (Hanson [6, 1981]). The differentiable function f is said to be *pseudoinvex* (on A) if there exists a vector function $\eta : A \times A \to R^n$ such that

$$(PI) \qquad \forall x, u \in A,\ \eta^t(x,u)\nabla f(u) \geq 0 \ \to\ f(x) \geq f(u).$$

Definition 3.2. (Mititelu [10, 1007]). The function f is said to be *pseudoinvex (on A)* if there exists a vector function $\eta : A \times A \to \mathbb{R}^n$ such that

$$(PI') \qquad \forall x, u \in A,\ \forall \lambda \in [0,1] : f(u + \lambda\eta(x,u)) \geq f(u) \to f(x) \geq f(u).$$

Shortly, f satisfying these definitions is named *η-pseudoinvex* [13].

Theorem 3.1. *If the function f is differentiable, then the definitions 3.1 and 3.2 are equivalent.*

Proof. Def. 3.1 $\to$ Def. 3.2. Assume that f is η-pseudoinvex on A; hence the implication (PI) in Definition 3.1 is true. According to (3.1) and (3.3), this becomes

$$\forall x, u \in A,\ f'_r(u; \eta(x,u)) \geq 0 \to f(x) \geq f(u),$$

or equivalently,

$$\forall x, u \in A,\ f(x) < f(u) \to f'_r(u; \eta(x,u)) < 0.$$

For $t > 0$, it results $f'_r(u; t\eta(x,u)) < 0$, or equivalently,

$$\lim_{\lambda \downarrow 0} \frac{f(u + \lambda t\eta(x,u)) - f(u)}{\lambda} < 0.$$

The function $\phi(\lambda) = [f(u + \lambda t\eta(x,u)) - f(u)]/\lambda$ is continuous for $\lambda > 0$. Then there exists $\lambda_0 > 0$, sufficiently small, such that

$$\frac{f(u + \lambda t\eta(x,u)) - f(u)}{\lambda} < 0, \quad \text{for any} \quad \lambda \in (0, \lambda_0],$$

or

$$f(u + \lambda t\eta(x,u)) < f(u)], \quad \text{for any} \quad \lambda \in (0, \lambda_0 t].$$

Denoting $\mu = \lambda t$ and taking $t = 1/\lambda_0$, it results

$$f(u + \mu\eta(x,u)) < f(u), \quad \forall \mu \in (0,1].$$

Therefore,

$$\forall x, u \in A,\ \forall \mu \in (0,1],\ f(x) < f(u) \to f(u + \mu\eta(x,u)) < f(u),$$

and finally

$$\forall x, u \in A,\ \forall \mu \in [0,1],\ f(u + \mu\eta(x,u)) \geq f(u) \to f(x) \geq f(u).$$

Def. 3.2 → Def. 3.1. From the relation (PI') by Definition 3.2, for $\lambda > 0$, we subsequently have

$$\forall x, u \in A, \quad \lim_{\lambda \downarrow 0} \frac{f(u + \mu\eta(x,u)) - f(u)}{\lambda} \geq 0 \to f(x) \geq f(u),$$

$$\forall x, u \in A, \ \ f'_r(u; \eta(x,u)) \geq 0 \to f(x) \geq f(u),$$

$$\forall x, u \in A, \ \ \eta^t(x,u)\nabla f(u) \geq 0 \to f(x) \geq f(u),$$

Definition 3.3 (Pini [13]). The function f is said to be *prepseudoinvex* (on A) if there exists a vector function $\eta : A \times A \to \mathbb{R}^n$ and a strictly positive function $b : A \times A \to \mathbb{R}_+$ such that $\forall x, u \in A$, $\forall t \in (0,1)$

$$(pPI) \qquad f(x) < f(u) \to f(u + t\eta(x,u)) \leq f(u) + t(t-1)b(x,u).$$

Theorem 3.2. *Let A be a nonempty open set in $\mathbb{R}^n$, $f : A \to \mathbb{R}$ be a differentiable function and a continuous vector function $\eta : A \times A \to \mathbb{R}^n$. Then f is η-prepseudoinvex, iff it is η-pseudoinvex function.*

Proof. Necessity. Definition 3.3 is true and from the relation (pPI), it successively results

$$\frac{f(u + t\eta(x,u)) - f(u)}{t} \leq (t-1)b(x,u), \ t > 0, \ x \neq u,$$

$$\lim_{\lambda \downarrow 0} \frac{f(u + t\eta(x,u)) - f(u)}{t} - b(x,u) < 0, \ x \neq u,$$

$$f'_r(u; \eta(x,u)) < 0, \ x \neq u, \quad \eta^t(x,u)\nabla f(u) < 0, \ n \neq u.$$

Consequently,

$$\forall x, u \in A, \ x \neq u : f(x) < f(u) \to \eta^t(x,u)\varphi f(u) < 0,$$

or equivalently

$$(3.4) \qquad \forall x, u \in A, \ x \neq u : \eta^t(x,u)\nabla f(u) \geq 0 \to f(x) \geq f(u).$$

The vector function η being continuous, for $x \to u$ the implication (3.4) becomes

$$\forall u \in A, \ \eta^t(u,u)\nabla f(u) \geq 0 \to f(u) = f(u).$$

Therefore,

$$\forall x, u \in A, \ \eta^t(x,u)\nabla f(u) \geq 0 \to f(x) \geq f(u),$$

that is, f is a η-pseudoinvex function.

Sufficiency. Suppose that f is η-pseudoinvex on A. Then, according to Definition 3.1 and to relations (3.1) and (3.3), we have

$$\forall x, u \in A, \ f'_r(u; \eta(x,u)) \geq 0 \to f(x) \geq f(u),$$

or equivalently

$$\forall x, u \in A,\ f(x) < f(u) \rightarrow f'_r(u, \eta(x,u)) < 0.$$

From $f'_r(u, \eta(x,u)) < 0$ and $t > 0$ it results $f'_r(u, \eta(x,u)) < 0$, or equivalently

$$\lim_{\lambda \downarrow 0} \frac{f(u + \lambda + \eta(x,u)) - f(u)}{\lambda} < 0.$$

There exists $\lambda_0 > 0$, sufficiently small such that

$$\frac{f(u + \lambda t\eta(x,u) - f(u)}{\lambda} < 0, \quad \forall \lambda(0, \lambda_0]\ (\lambda_0 < 1).$$

We define the function $\varphi : A \times A \times (0,1) \times (0, \lambda_0] \rightarrow \mathbb{R}$ by

$$\frac{f(u + \lambda t\eta(x,u)) - f(u)}{\lambda} = t(\lambda t - 1)\varphi(x, u, t, \lambda). \tag{3.5}$$

It results

$$\varphi(x, u, t, \lambda) = \frac{f(u + \lambda t\eta(x,u)) - f(u)}{\lambda + (\lambda t - 1)}.$$

Having $t(\lambda t - 1) < 0$ and according to (3.5), it results $\varphi(x, u, t, \lambda) > 0$. φ is continuous on its domain and also it is continuous through prolongations on $[0,1] \times [0, \lambda_0]$, where

$$\varphi(x, u, t, 0) = \varphi(x, u, 0, \lambda) = \varphi(x, u, 0, 0) = -\eta^t(x,u)\nabla f(u).$$

Hence, from (3.5) we obtain

$$f(u + \lambda t\eta(x,u)) - f(u) \leq \lambda t(\lambda t - 1) \min_{\substack{0 \leq \lambda \leq \lambda_0 \\ 0 \leq t \leq 1}} \varphi(x, u, t, \lambda). \tag{3.6}$$

From $0 < \lambda < \lambda_0$ and $0 < t < 1$ it results $0 < \lambda t < 1$. We denote $\mu = \lambda t$ and $b(x,u) = \min\limits_{0 \leq \lambda \leq \lambda_0} \varphi(x, u, t, \lambda) \geq 0$. If $b(x,u) = 0$, then it is replaced by $b(x,u) = 1 > 0$. Then (3.6) becomes

$$f(u + \mu\eta(x,u)) - f(u) \leq \mu(1_\mu)b(x,u), \quad \forall \mu \in (0,1),$$

that is, f is η-pseudoinvex function on A.

4 Quasiinvex functions and prequasiinvex functions on open sets

The quasiinvexity and prequasiinvexity notions were introduced in optimization theory as follows.

Definition 4.1 (Hanson [6]). The differentiable function f is said to be *quasi-invex (on A)*, if there exists a vector function $\eta : A \times A \to \mathbb{R}^n$ such that

$$(QI) \qquad \forall x, u \in A,\ f(x) \le f(u) \to \eta^t(x,u)\nabla f(u) \le 0.$$

Definition 4.2 (Pini [13]). The function $f : A \to \mathbb{R}$ is said to be *prequasiinvex (on A)* if there exists a vector function $\eta : A \times A \to \mathbb{R}^n$ such that

$$(pQI) \qquad \forall x, u \in A,\ \forall t \in (0,1),\ f(x) \le f(u) \to f(u + t\eta)) \le f(u).$$

Theorem 4.1. *Let A be a nonempty open set in $\mathbb{R}^n$, $f : A \to \mathbb{R}$ a differentiable function and $\eta : A \times A \to \mathbb{R}^n$ a vector function. Then f is η-prequasiinvex iff it is η-quasiinvex.*

Proof. Necessity. f is η-prequasiinvex on A and then, the implication (pQI) is true. It results

$$\lim_{\lambda \downarrow 0} \frac{f(u + \lambda\eta(x,u)) - f(u)}{\lambda} \le 0,$$

that is $f'_r(u; \eta(x,u)) \le 0$. Therefore,

$$\forall x, u \in A,\ f(x) \le f(u) \to \eta^t(x,u)\nabla f(u) \le 0$$

and hence, f is a η-quasiinvex function on A.

Sufficiency. Suppose that f is η-quasiinvex on A. Then, according to Definition 4.1 and relation (3.2), it successively results

$$\forall x, u \in A,\ f(x) \le f(u) \to \eta^t(x,u)\nabla f(u) \le 0,$$

$$\forall x, u \in A, \forall t > 0,\ f(x) \le f(u) \to f'_r(u; t\eta(x,u)) \le 0,$$

or equivalently,

$$\forall x, u \in A,\ \forall t > 0,\ \lim_{\lambda \downarrow 0} \frac{f(u + \lambda t\eta(x,u)) - f(u)}{\lambda} > 0 \to f(x) > f(u).$$

But the function $\phi(\lambda) = \dfrac{[f(u + \lambda t\eta(x,u)) - f(u)]}{\lambda}$ is continuous for $\lambda > 0$ and then there exists a $\lambda_0 > 0$, sufficiently small, such that, $\forall x, u \in A,\ \forall t > 0$

$$\frac{f(u + \lambda t\eta(x,u)) - f(u)}{\lambda} > 0 \to f(x) > f(u), \quad \text{for any} \quad \lambda \in (0, \lambda_0]$$

or $\forall x, u \in A,\ \forall t > 0$

$$f(u + \lambda t\eta(x,u)) - f(u) > 0 \to f(x) > f(u),\ \forall (0 < \lambda t \le \lambda_0 t) \quad \text{for any} \quad \lambda \in (0, \lambda_0 t].$$

Noting $\mu = \lambda t$ and taking $t = \dfrac{1}{\lambda_0}$, it successively results

$$\forall x, u \in A\ \ f(u + \mu\eta(x,u)) > f(u) \to f(x) > f(u), \quad \text{for any} \quad \mu(0,1),$$

$$\forall x, u \in A,\ \forall \mu \in (0,1)\ \ f(x) \le f(u) \to f((u + \mu\eta(x,u)) \le f(u);$$

therefore f is an η-prequasiinvex function on A.

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