Recent advances in the theory of transfer operators arising in statistical mechanics

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Abstract. We first make explicit the analytic properties of transfer operators due to Mayer and Ruelle that generalize the classical Perron-Frobenius operator. The purpose of this paper is to give and discuss two generalizations of them. We mostly focus on the analysis of what we call generalized Mayer-Ruelle operators depending on two complex parameters.

Mathematics Subject Classification 2000: 11K50, 49G10, 70G60.

Key words: Perron-Frobenius operator, Mayer-Ruelle operator, spectral properties.

1. Introduction

Statistical mechanics problems motivated the consideration of a class of functional operators - transfer operators - due to Mayer ([5], [6]) and Ruelle [9]. This class of operators including as a special case the Perron-Frobenius operator is the basic ingredient, well-developed in dynamical systems theory [1].

This paper surveys the main properties of Mayer-Ruelle operators and gives two generalizations of them.

The analytic properties of transfer operators associated to continued fractions have been investigated by Mayer and Roepstorff in a series of papers ([3], [4], [7], [8]) that provide the technical background for the functional analysis aspects of our paper.

2. The main properties of transfer operators

Let $D_1 = \{z \in \mathbb{C} | \; |z-1| < 3/2\}$ and consider the collection $A_\infty(D_1)$ of all holomorphic functions in D_1 which are continuous in $\bar{D}_1$; $A_\infty(D_1)$ is a Banach space under the spremum norm

$$||f|| = \sup_{z \in \bar{D}_1} |f(z)|, \quad f \in A_\infty(D_1).$$

The transfer operators of Mayer-Ruelle are defined by

$$R_s f(z) = \sum_{i \in \mathbb{N}_+} \frac{1}{(z+i)^s} f\left(\frac{1}{z+i}\right), \quad z \in \bar{D}_1,$$

Proceedings of The 2-nd International Colloquium of Mathematics in Engineering and Numerical Physics (MENP-2), April 22-27, 2002, University Politehnica of Bucharest, Romania.
BSG Proceedings 8, pp. 140-146, Geometry Balkan Press, 2003.

for s a complex number satisfying Re$s > 1$ and $f \in A_\infty(D_1)$. It is easy to check that R_s is a bounded linear operator on $A_\infty(D_1)$. R_s is nuclear of order 0, and thus has a discrete spectrum.

For $s = 2$, R_s has the same analytical expression as the Perron-Frobenius operator $P_\lambda = P$ of τ under λ

$$Pf(x) = \sum_{i \in \mathbb{N}_+} \frac{1}{(x+i)^2} f\left(\frac{1}{x+i}\right), \quad f \in L^1,\ x \in I = [0,1],$$

where λ is the Lebesgue measure on I and τ is the continued fraction transformation on I defined as

$$\tau(x) = \begin{cases} \dfrac{1}{x} - \left[\dfrac{1}{x}\right] & \text{if} \quad x \neq 0 \\ \quad 0 & \text{if} \quad x = 0, \end{cases}$$

(here $[\cdot] : \mathbb{R} \to \mathbb{Z}$ is the greatest integer function).

In what follows we give without proofs the most important properties of the Mayer-Ruelle operator R_s for Re$s > 1$, which generalize those of P. For proofs we refer the reader to Mayer ([5], [6]), Flajolet and Vallée [2].

Theorem 1. *Let s be real, strictly greater than 1. Then the following results hold.*

(i) *The operator $R_s : A_\infty(D_1) \to A_\infty(D_1)$ has a positive dominant eigenvalue λ_s which is simple and strictly greater in absolute value than all other eigenvalues. The corresponding eigenfunction $g_s \in A_\infty(D_1)$ is strictly positive on $\bar{D}_1 \cap \mathbb{R} = \left[-\frac{1}{2}, \frac{5}{2}\right]$.*

(ii) *The map $s \mapsto \lambda_s$ defines on $(1, \infty)$ a strictly decreasing and logconcave function with*

$$\lim_{s \downarrow 1} \lambda_s = \infty,\ \lambda_{s=2} = 1,\ \lim_{s \to \infty} \frac{\log \lambda_s}{s} = \log \frac{\sqrt{5}-1}{2}.$$

Moreover, $\lambda_{s+u} \leq \left(\frac{\sqrt{5}-1}{2}\right)^u \lambda_s,\ u \in \mathbb{R}_+.$

(iii) *There exists a linear functional l_s on $A_\infty(D_1)$ with $l_s(g_s) = 1$ and $l_s(f) > 0$ for any $f \in A_\infty(D_1)$ such that $f|_{[-1/2,5/2]} > 0$. If Π_{1s} denotes the projection defined as $\Pi_{1s} f = l_s(f) g_s$, $\quad f \in A_\infty(D_1)$, then $R_s = \lambda_s \Pi_{1s} + T_{0s}$ with $\Pi_{1s} T_{0s} = T_{0s} \Pi_{1s} = 0$. Hence*

$$R_s^n = \lambda_s^n \Pi_{1s} + T_{0s}^n, \quad n \in \mathbb{N}_+.$$

(iv) *The spectral radius ρ_s of the linear operator $T_{0s} : A_\infty(D_1) \to A_\infty(D_1)$ is strictly smaller than λ_s, and for any $f \in A_\infty(D_1)$ such that $f|_{[-1/2,5/2]} > 0$ we have*

$$\frac{R_s^n f(z)}{\lambda_s^n l_s(f) g_s(z)} = 1 + O\left(\left(\frac{\rho_s}{\lambda_s}\right)^n\right)$$

as $n \to \infty$, where the constant implied in O is independent of $z \in \bar{D}_1$.

(v) *There exists $\varepsilon = \varepsilon(s) > 0$ such hat for any $t \in \mathbb{C}$ satisfying $|s - t| \leq \varepsilon$ the dominant spectral properties of $R_s : A_\infty(D_1) \to A_\infty(D_1)$ transfer to $R_t : A_\infty(D_1) \to A_\infty(D_1)$: quantities λ_t, ρ_t, g_t, l_t (thus Π_{1t}) and T_{0t} can be defined to represent the dominant spectral objects associated with R_t, and all of them are analytical with respect to t. Moreover, let $a \in (\rho_s/\lambda_s, 1)$. For any $f \in A_\infty(D_1)$ such that $f|_{[-1/2,5/2]} > 0$ we have*

$$\frac{R_t^n f(z)}{\lambda_t^n l_t(f) g_t(z)} = 1 + O(a^n)$$

as $n \to \infty$, where the constant implied in O is independent of $z \in \bar{D}_1$ and t satisfying $|s - t| \leq \varepsilon$, but depends on a, f and s. Finally, $\rho_{s+\mathrm{i}u} < \rho_s$ for $u \in [-\varepsilon, \varepsilon]$, $u \neq 0$.

The Mayer-Ruelle operators enjoy better properties when they operate on suitable Hilbert spaces.

Let $\mathrm{Re} s > 1$. Consider the collection $H^{(s)}$ of functions f which are holomorphic in the half-plane $\mathrm{Re} z > -\frac{1}{2}$, bounded in any half-plane $\mathrm{Re} z > -\frac{1}{2} + \varepsilon$, $\varepsilon > 0$, and can be represented in the form

$$f(z) = \int_{\mathbb{R}_+} e^{-zu} \varphi(u)^{(s-1)/2} m'(du), \quad \mathrm{Re}\, z > -\frac{1}{2}, \tag{1}$$

where m' is the measure on $\mathcal{B}_{\mathbb{R}_+}$ with density

$$\frac{dm'}{du} = \begin{cases} \dfrac{1}{e^u - 1} & \text{if} \quad u > 0 \\ 0 & \text{if} \quad u = 0, \end{cases}$$

for some $\varphi \in L^2_{m'}(\mathbb{R}_+)$ the Hilbert space of m'-square integrable functions $\varphi : \mathbb{R}_+ \to \mathbb{C}$ with inner product $(\cdot, \cdot)_{m'}$ defined as

$$(\varphi, \psi)_{m'} = \int_{\mathbb{R}_+} \varphi \psi^* dm', \quad \varphi, \psi \in L^2_{m'}(\mathbb{R}_+)$$

and norm

$$||\varphi||_{2,m'} = \left(\int_{\mathbb{R}_+} |\varphi|^2 dm' \right)^{\frac{1}{2}}, \quad \varphi \in L^2_{m'}(\mathbb{R}_+).$$

Introducing the inner product

$$(f_1, f_2)_{(s)} = (\varphi_1, \varphi_2)_{m'},$$

where φ_i is associated with f_i, $i = 1, 2$, by (1), $H^{(s)}$ is made a Hilbert space with norm

$$|f|_{(s)} = ||\varphi||_{2,m'}, \quad f \in H^{(s)},$$

where f and φ are again associated by (1).

Theorem 2. *Let* $\mathrm{Re}\, s > 1$. *Then the following results hold.*
(i) *The linear operator* R_s *takes boundedly* $H^{(s)}$ *into itself.*
(ii) *For any* $f \in H^{(s)}$ *we have*

$$R_s f(z) = \int_{\mathbb{R}_+} e^{-zu} K_s \varphi(u) u^{(s-1)/2} m'(du), \ \mathrm{Re}\, z > -\frac{1}{2},$$

where $K_s : L^2_{m'}(\mathbb{R}_+) \to L^2_{m'}(\mathbb{R}_+)$ *is a symmetric integral operator defined as*

$$K_s \varphi(u) = \int_{\mathbb{R}_+} J_{s-1}(2\sqrt{uv}) \varphi(v) m'(dv), \quad \varphi \in L^2_{m'}(\mathbb{R}_+), \ u \in \mathbb{R}_+.$$

Here J_{s-1} *is the Bessel function of order* $s-1$ *defined as*

$$J_{s-1}(u) = \left(\frac{u}{2}\right)^{s-1} \sum_{k \in \mathbb{N}} \frac{(-1)^k}{k! \Gamma(k+s)} \left(\frac{u}{2}\right)^{2k}, \ u \in \mathbb{R}_+.$$

Hence $R_s : H^{(s)} \to H^{(s)}$ *can be diagonalized in an orthonormal basis of* $H^{(s)}$. *Moreover, if* $s \in \mathbb{R}$, *then* R_s *is self-adjoint and its spectrum is real.*

(iii) *The spectra of the operators* $R_s : A_\infty(D_1) \to A_\infty(D_1)$, $R_s : H^{(s)} \to H^{(s)}$ *and* $K_s : L^2_{m'}(\mathbb{R}_+) \to L^2_{m'}(\mathbb{R}_+)$ *are identical. Hence, for* $s \in \mathbb{R}$, *these spectra are all real.*

3. A first generalization

For any subset M of $\mathbb{N}_+$ define

$$R_{M,s} f(z) = \sum_{i \in M} \frac{1}{(z+i)^s} f\left(\frac{1}{z+i}\right), \quad z \in \bar{D}_1,$$

whatever $s \in \mathbb{C}$ with $\mathrm{Re}\, s > 1$ and $f \in A_\infty(D_1)$. Clearly, $R_{M,s}$ is a bounded linear operator on $A_\infty(D_1)$, hence a nuclear one of trace-class, which coincides with R_s when $M = \mathbb{N}_+$. Now, for an arbitrarily fixed $k \in \mathbb{N}_+$, let M_i, $1 \le i \le k$, be subsets of $\mathbb{N}_+$ and write $\mathcal{M} = (M_1, \ldots, M_k)$. Consider the linear operator $R_{\mathcal{M},s} : A_\infty(D_1) \to A_\infty(D_1)$ defined as

$$R_{\mathcal{M},s} = R_{M_k,s} \circ \ldots \circ R_{M_1,s}$$

which is nuclear of trace-class, too.

The operators $R_{\mathcal{M},s}$ for various $\mathcal{M}$ control the dynamics of continued fraction expansions of irrationals subject to periodical constraints. Their spectral properties are entirely similar to those of R_s.

4. A second generalization

This generalization has been motivated by the study of the transformation

$$z \mapsto \frac{1}{z} - \left[\mathrm{Re}\, \frac{1}{z}\right], \ 0 \neq z \in \mathbb{C},$$

which extends to the complex domain the continued fraction transformation τ defined in Section 2. For a detailed account we refer the reader to [10].

Let $D_2 = \{z| \ |z-1| < 5/4\}$ and consider the collection $B_\infty(D_2)$ of all functions F which are holomorphic in D_2^2 and continuous in $\bar{D}_2^2$. Under the supremum norm

$$||F|| = \sup_{(z,w)\in\bar{D}_2^2} |F(z,w)|,$$

$B_\infty(D_2)$ is a Banach space. Then, for any $(s,t) \in \mathbb{C}^2$ with $\mathrm{Re}(s+t) > 1$, a linear bounded operator

$$R_{s,t} : B_\infty(D_2) \to B_\infty(D_2)$$

is defined by

$$R_{s,t}F(z,w) = \sum_{i\in\mathbb{N}_+} \frac{1}{(z+i)^s(w+i)^t} F\left(\frac{1}{z+i}, \frac{1}{w+i}\right)$$

for any $F \in B_\infty(D_2)$ and $(z,w) \in D_2^2$. The spectral properties of $R_{s,t}$ which is positive and nuclear of trace-class, are strongly related to those of R_{s+t+2l}, $l \in \mathbb{N}$.

Theorem 3. *For any* $(s,t) \in \mathbb{C}^2$ *with* $\mathrm{Re}(s+t) > 1$, $\mathrm{Re}(s) \geq 1$ *and* $\mathrm{Re}(t) > -1$ *the following results hold.*

(i) *The operator* $R_{s,t} : B_\infty(D_2) \to B_\infty(D_2)$ *has a unique dominant eigenvalue* $\lambda_{s,t}$ *which is equal to the dominant eigenvalue* λ_{s+t} *of* R_{s+t}. *The corresponding eigenfunction* $G_{s,t}$ *of* $R_{s,t}$ *is defined by*

$$(2) \qquad G_{s,t}(z,w) = \int_0^1 \beta_{t,s}(y) g_{s+t}(z+(w-z)y)dy,$$

where g_{s+t} *is the eigenfunction of* R_{s+t} *and* $\beta_{t,s}$ *is the classical density* β

$$\beta_{t,s}(y) = \frac{\Gamma(s+t)}{\Gamma(s)\Gamma(t)} y^{t-1}(1-y)^{s-1};$$

moreover, $G_{s,t}$ *satisfies* $G_{s,t}(z,z) = g_{s+t}(z)$. *The adjoint operator* $R^*_{s,t}$ *has a dominant eigenfunction* $G^*_{s,t}$ *satisfying* $G^*_{s,t}(F) = g^*_{s+t}(f)$, *for all* $F \in B_\infty(D_2)$ *whose diagonal function is* f. *If* $\Pi_{s,t}$ *denotes the projection on the dominant eigensubspace,* $\Pi_{s,t} = g^*_{s+t} \otimes G_{s,t}$, *then* $R_{s,t}$ *has the representation* $R_{s,t} = \lambda_{s+t}\Pi_{s,t} + T_{s,t}$, *where* $\Pi_{s,t} \circ T_{s,t} = T_{s,t} \circ \Pi_{s,t} = 0$. *Hence, for any* $F \in B_\infty(D_2)$ *we have*

$$(3) \qquad R^n_{s,t}F(z,w) = \lambda^n_{s+t} g^*_{s+t}(f) G_{s,t}(z,w) + T^n_{s,t}F(z,w),$$

for all $(z,w) \in D_2^2$ *and* $n \in \mathbb{N}_+$.

(ii) *The spectral radius* ρ_{s+t} *of the linear operator* T_{s+t} *is strictly smaller than* λ_{s+t}.

(iii) *Let* $a \in (\nu_{s+t}, 1)$, *where* $\nu_{s+t} = \dfrac{\rho_{s+t}}{\lambda_{s+t}} < 1$. *Moreover,*

$$\nu_{s+t} = \frac{1}{\lambda_{s+t}} \max(\lambda_{s+t+2}, \rho_{s+t}). \tag{4}$$

For any $F \in B_\infty(D_2)$ *such that* $F|_{[-1/4,9/4]^2} > 0$ *we have*

$$\frac{R^n_{s,t}F(z,w)}{\lambda^n_{s+t}} = g^*_{s+t}(f)G_{s,t}(z,w)(1 + O(||F||a^n)) \tag{5}$$

as $n \to \infty$, *where the constant implied in* O *is independent of* $(z, w) \in D_2^2$, *but depends on* a.

Proof. Since, by Theorem 1 (ii), the map $s \mapsto \lambda_s$ defines a strictly decreasing function of s, it follows that λ_{s+t} is the dominant eigenvalue and ν_{s+t} satisfies (4). With the change of variable $u = z + (w - z)y$ in (2), we get the expression of $G_{s,t}$

$$G_{s,t}(z,w) = \frac{\Gamma(s+t)}{\Gamma(s)\Gamma(t)} \int_\gamma g_{s+t}(u) \frac{(u-z)^{t-1}(w-u)^{s-1}}{(w-z)^{s+t-1}} dz, \tag{6}$$

where γ is the interval $[z, w]$. Now, let $\tilde{h}(z)$ be the holomorphic function that coincides with $\sqrt{|h'(z)|}$ on D_2, for any homography of depth 1, $h(z) = h_i(z) = \dfrac{1}{i+z}$, $i \in \mathbb{N}_+$. If F is defined by (6), to obtain $R_{s,t}F$ we evaluate the expression

$$\frac{\tilde{h}(z)^s \tilde{h}(w)^t}{[h(w) - h(z)]^{s+t-1}} \int_\delta f(u)[u - h(z)]^{t-1}[h(w) - u]^{s-1} du, \tag{7}$$

for any simple path δ that links $h(z)$ to $h(w)$. Put $\delta = h(\gamma)$, where γ is a simple path that links z to w. Using the change of variable $u = h(r)$ in (7) and relations $du = h'(r)dr = -\tilde{h}(r)^2 dr$, $h(a) - h(b) = -\tilde{h}(a)\tilde{h}(b)(a - b)$, for any a and b, we can rewrite (7) as

$$\frac{1}{(w-z)^{s+t-1}} \int_\gamma \tilde{h}(r)^{s+t} f \circ h(r)(r-z)^{t-1}(w-r)^{s-1} dr.$$

If F is defined by (6), we get

$$R_{s,t}F = \frac{\Gamma(s+t)}{\Gamma(s)\Gamma(t)} \int_\gamma R_{s+t}(f)(r) \frac{(r-z)^{t-1}(w-r)^{s-1}}{(w-z)^{s+t-1}} dr.$$

If f is the eigenfunction of R_{s+t} relative to eigenvalue λ, then F is the eigenfunction of $R_{s,t}$ relative to λ. Since $R^n_{s,t}F(z,z) = R^n_{s+t}f(z)$, it is clear that $G^*_{s,t}$ is expressible in terms of g^*_{s+t}.

(iii) The proof is similar to that of (iv) in Theorem 1. □

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