A BANACH SPACE OF NON BOUNDED FUNCTIONS

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Abstract

In this note a new norm is introduced on spaces of generalized Lipschitz functions. Completeness of these spaces is proved and relations between metric properties of the metric space and the corresponding normed space of Lipschitz functions are given.

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1 Introduction

Let S be any set, E any Banach space and ρ any nonnegative function on $S \times S$; let $L_E(S,\rho)$ denotes the set of all functions $x : S \to E$, such that $\|x(s) - x(t)\| \leq M\rho(s,t)$, $(s,t) \in S \times S$, for some $M \geq 0$, and let t_0 be any point in S. If $\|x\|_\rho$ denotes the smallest nonnegative constant M for which $\|x(s) - x(t)\| \leq M\rho(s,t)$, for all $(s,t) \in S \times S$ holds, then $\|x\|_\rho^{t_0}$ is defined to be

$$\|x\|_\rho^{t_0} = \|x(t_0)\| + \|x\|_\rho . \tag{1}$$

It is routine to show that $\|\cdot\|_\rho^{t_0}$ is a norm on $L_E(S,\rho)$ and this normed space is denoted by $L_E(S,\rho)_{t_0}$.

J.A. Johnson introduced in [2], the normed space of bounded functions x on S such that $\|x(s) - x(t)\| \leq M\rho(s,t)$ is endowed with the norm $\|x\|_\rho^\infty = \max\{\|x\|_\infty , \|x\|_\rho\}$, where $\|\cdot\|_\infty$ denotes the sup-norm. For distinction this last normed space will be denoted by $BL_E(S,\rho)_\infty$, while the same space, endowed with the norm (1) will be denoted by $BL_E(S,\rho)_{t_0}$.

The above introduced norm (1) permits us to study the more general space of all functions of this type. When S is a metric space with the metric d, taking $\rho = d$ we

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have the normed space of all Lipschitz functions $x : S \to E$ denoted by $Lip_E(S,d)_{t_0} = L_E(S,d)_{t_0}$ and the normed spaces $BL_E(S,d)_{t_0} = BLip_E(S,d)_{t_0}$ and $BL_E(S,d)_\infty = BLip_E(S,d)_\infty$. This last one normed space has been introduced in [3] and has studied mainly in [2], [4].

Taking ρ to be a modulus of continuity, we have the space of all functions satisfying the modulus of continuity. The space of bounded functions satisfying a modulus of continuity condition, endowed with the norm $\|\cdot\|_\rho^\infty$, has been introduced in [1].

2 The equivalence of norms

The above defined norm (1) is independent of t_0 up to equivalence.

Proposition 2.1. *For $t_1 \neq t_0$ in S, the norms $\|\cdot\|_\rho^{t_0}$ and $\|\cdot\|_\rho^{t_1}$ are equivalent.*

Proof. For any $x \in L_E(S,\rho)$, we have

$$\|x(t_0)\| - \|x(t_1)\| \leq \|x(t_0) - x(t_1)\| \leq \|x\|_\rho \, \rho(t_0,t_1),$$

hence

$$\|x(t_0)\| \leq \|x(t_1)\| + \|x\|_\rho \, \rho(t_0,t_1),$$

or

$$\|x(t_0)\| + \|x\|_\rho \leq (1 + \rho(t_0,t_1))(\|x(t_1)\| + \|x\|_\rho)$$

i.e.

$$\|x\|_\rho^{t_0} \leq (1 + \rho(t_0,t_1)) \, \|x\|_\rho^{t_1} .$$

In the same way, we have

$$\|x\|_\rho^{t_1} \leq (1 + \rho(t_0,t_1)) \, \|x\|_\rho^{t_0} ,$$

so that the two norms are equivalent. □

Remark. The norms $\|\cdot\|_\rho^{t_0}$ and $\|\cdot\|_\rho^\infty$ on $BL_E(S,\rho)$ are not in general equivalent as can be shown by taking $E = R$, i.e. the space of real numbers, $\rho(s,t) = |s-t|$, i.e. the natural metric on R and the family of functions

$$x_n(t) = \min\left\{\frac{|t|}{n}, 1\right\} - \frac{1}{n}, \ t \in R$$

for all natural numbers n, and $t_0 = 0$.

It can not be found a positive K such that $\|x_n\|_d^\infty \leq K \|x_n\|_d^{t_0}$ for all n.

However, when $\sup\{\rho(s,t) : s,t \in S\} = \delta_\rho(S) < \infty$ then $L_E(S,\rho) = BL_E(S,\rho)$, and the following holds.

Proposition 2.2. *If $\delta_\rho(S) < \infty$, $\|\cdot\|_\rho^{t_0}$ and $\|\cdot\|_\rho^\infty$ are equivalent on $L_E(S,\rho)$.*

Proof. It is evident that $\|x\|_\rho^{t_0} \leq \|x\|_\rho^\infty$, for every $x \in L_E(S,\rho)$.

On the other hand

$$\|x(s) - x(t_0)\| \leq \|x\|_\rho \, \rho(s,t_0) \leq \|x\|_\rho \, \delta_\rho(S)$$

or
$$\|x(s)\| \leq \|x(t_0)\| + \|x\|_\rho \, \delta_\rho(S), \ \forall s \in S;$$
hence
$$\|x\|_\infty \leq \|x(t_0)\| + \|x\|_\rho \, \delta_\rho(S)$$
or
$$\|x\|_\infty + \|x\| \, \delta_\rho(S) \leq (1 + \delta_\rho(S))(\|x(t_0)\| + \|x\|_\rho)$$
i.e.
$$\|x\|_\rho^\infty \leq (1 + \delta_\rho(S)) \, \|x\|_\rho^{t_0} .$$
□

3 The Banach space $L_E(S, \rho)_{t_0}$

Lemma 3.1. *Let $\{x_n\}$ be a sequence in $L_E(S, \rho)$. If for every $s \in S$, $\lim\limits_{n\to\infty} x_n(s) = x(s)$ then*
$$\|x\|_\rho \leq \varliminf_{n\to\infty} \|x_n\|_\rho .$$

Proof. Let s, t be in S, such that $\rho(s, t) \neq 0$; it holds
$$\|x(s) - x(t)\| \leq \|x(s) - x_n(s)\| + \|x_n(s) - x_n(t)\| + \|x_n(t) - x(t)\| .$$

Given $\varepsilon > 0$, there is a $n_0 \in N$ such that
$$\|x(s) - x_n(s)\| \leq \frac{\varepsilon}{2}\rho(s, t)$$
for every $n \geq n_0$; hence for $n \geq n_0$
$$\|x(s) - x(t)\| \leq (\varepsilon + \|x_n\|_\rho)\rho(s, t)$$
and thus
$$\|x(s) - x(t)\| \leq (\varepsilon + \varliminf_{n\to\infty} \|x_n\|_\rho)\rho(s, t).$$

Since ε can be taken arbitrarily small, it follows
$$\|x(s) - x(t)\| \leq (\varliminf_{n\to\infty} \|x_n\|_\rho)\rho(s, t).$$

This last inequality obviously holds for any (s, t) in $S \times S$ hence $\|x\|_\rho \leq \varliminf\limits_{n\to\infty} \|x_n\|_\rho$ ..
□

Now we can prove the following result.

Theorem 3.2. *If E is a Banach space then $L_E(S, \rho)_{t_0}$ is a Banach space too.*

Proof. We have proved completeness. Let $\{x_n\}$ be a Cauchy sequence in $L_E(S, \rho)_{t_0}$; given $\varepsilon > 0$, there is a n_0 in N such that
$$\|x_p - x_q\|_\rho^{t_0} < \frac{\varepsilon}{2} \qquad \forall p, q \geq n_0;$$

it follows that for any $s \in S$, $\{x_n(s)\}$ is a Cauchy sequence in E and hence it has a limit, denoted by $x(s)$.

Since $\{x_n\}$ is a Cauchy sequence in $L_E(S,\rho)_{t_0}$, the subset $\{\|x_n\|_\rho^{t_0} : n \in N\}$ is bounded; hence $\varliminf_{n\to\infty} \|x_n\|_\rho < \infty$ and after lemma 3.1, $\|x\|_\rho < \infty$, i.e. $x \in L_E(S,\rho)_{t_0}$.

On the other hand, for a fixed $k \in N$, $\{x_k - x_n\}$ is a Cauchy sequence in $L_E(S,\rho)_{t_0}$ and it is convergent pointwise to $x_k - x$; hence

$$\|x_k - x\|_\rho \leq \lim_{n\to\infty} \|x_k - x_n\|_\rho .$$

Choosing a $k \geq n_0$, it holds

$$\|x_k - x\|_\rho < \frac{\varepsilon}{2}$$

and $\|x_k - x\|_\rho^{t_0} < \varepsilon$; thus $\lim_{k\to\infty} \|x_k - x\|_\rho^{t_0} = 0$ and the space $L_E(S,\rho)_{t_0}$ is complete. □

4 The case of metric spaces

Now, let S be any metric space. Two metrics d_1 and d_2 on a space S are called *boundedly equivalent* if and only if there exist positive numbers K_1 and K_2 such that

$$K_1 d_1(s,t) \leq d_2(s,t) \leq K_2 d_1(s,t),\ \ s,t \in S$$

Choosing $\rho(s,t)$ to be a metric d on S, $L_E(S,d)_{t_0}$ denotes the space $Lip_E(S,d)_{t_0}$ of all Lipschitz functions on S. In the next few results, relations between metric properties of the metric space S and the normed space $Lip_E(S,d)_{t_0}$ are given,

Let $\sigma : S \times S \to R$ be the function defined by

$$\sigma(s,t) = \sup\left\{\|x(s) - x(t)\| :\ x \in Lip_E(S,d)_{t_0},\ \ \|x\|_d^{t_0} \leq 1\right\};$$

this function defines a metric on S and we have:

Proposition 4.1. *The metrics d and σ are identical on S.*

Proof. Let x be in $Lip_E(S,d)_{t_0}$, such that $\|x\|_d^{t_0} \leq 1$; it follows $\|x\|_d \leq 1$ or $\|x(s) - x(t)\| \leq d(s,t)$ and hence $\sigma(s,t) \leq d(s,t)$ for every $s,t \in S$. Now, let e be in E with $\|e\| = 1$; then for an $s \in S$ the function $x_s : S \to E$ defined by

$$x_s(u) = [d(t_0,s) - d(u,s)]e,\ \ u \in S;$$

x_s is obviously in $Lip_E(S,d)_{t_0}$ and $\|x_s\|_d^{t_0} \leq 1$.

We have

$$\sigma(s,t) \geq \|x_s(s) - x_s(t)\| = |[d(t_0,s) - d(s,s)] - [d(t_0,s) - d(t,s)]|\, \|e\| = d(s,t),\ \ \forall s,t \in S.$$

Thus $\sigma(s,t) = d(s,t)$. □

Lemma 4.2. *Let d_1, d_2 be two metrics in S. If $Lip_E(S,d_1) = Lip_E(S,d_2)$ then $\|x\|_{d_1}^{t_0}$ and $\|x\|_{d_2}^{t_0}$ are equivalent norms.*

Proof. The space $L = Lip_E(S, d_1) = Lip_E(S, d_2)$ endowed with the norm $\|\cdot\|_{d_1}^{t_0}$, as well as with the $\|\cdot\|_{d_2}^{t_0}$, is complete; it follows that L is also complete, when is endowed with the norm$\|\cdot\|^{t_0} = \max\{\|\cdot\|_{d_1}^{t_0}, \|\cdot\|_{d_2}^{t_0}\}$.

Because of $\|\cdot\|_{d_1}^{t_0} \leq \|\cdot\|^{t_0}$ and $\|\cdot\|_{d_2}^{t_0} \leq \|\cdot\|^{t_0}$, $\|\cdot\|_{d_1}^{t_0}$ and $\|\cdot\|_{d_2}^{t_0}$ are equivalent with the $\|\cdot\|^{t_0}$, hence equivalent to each other. □

Theorem 4.3. *Let d_1, d_2 be metrics on S; then*

$$Lip_E(S, d_1) = Lip_E(S, d_2) \tag{2}$$

and

$$K_1 d_2 \leq d_1 \leq K_2 d_2, \tag{3}$$

where K_1, K_2 are positive constants, are equivalent.

Proof. Let L, $\|\cdot\|_{d_1}^{t_0}$ and $\|\cdot\|_{d_2}^{t_0}$ be as in lemma 4.2; let σ_1, σ_2 be the metrics of proposition 4.1 corresponding to d_1 and d_2 respectively. From (2) follows that there are K_1, K_2 such that

$$K_1 \|x\|_{d_1}^{t_0} \leq \|x\|_{d_2}^{t_0} \leq K_2 \|x\|_{d_1}^{t_0} \quad \forall x \in L.$$

We have

$$\begin{aligned} \sigma_2(s,t) &\leq \sup\left\{\|x(s) - x(t)\| : x \in L, K_1 \|x\|_{d_1}^{t_0} \leq 1\right\} \\ &\leq \tfrac{1}{K_1} \sup\left\{\|x(s) - x(t)\| : x \in L, \|x\|_{d_1}^{t_0} \leq 1\right\}, \end{aligned}$$

hence

$$K_1 \sigma_2(s,t) \leq \sigma_1(s,t).$$

In the same way we obtain

$$\sigma_1(s,t) \leq K_2 \sigma_2(s,t).$$

Thus

$$K_1 \sigma_2(s,t) \leq \sigma_1(s,t) \leq K_2 \sigma_2(s,t)$$

or

$$K_1 d_2 \leq d_1 \leq K_2 d_2.$$

The fact that (2) implies (3) is obvious. □

References

[1] Fraser R. B., *Banach spaces of functions satisfying a modulus of continuity condition.* Studia Math., 32 (1969) 277-283.

[2] Johnson J. A., *Banach spaces of Lipschitz functions and vector valued Lipschitz functions*, Trans.Amer.Math.Soc. 148 (1970) 147-169.

[3] de Leeuw K., *Banach spaces of Lipschitz functions*, Studia Math., 21 (1961/62) 55-66.

[4] Sherbert D. R., *Banach Algebras of Lipschitz Functions*, Pacific J. Math., 13 (1963) 1387-1399.

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