Weak Gravitational Models Based on Beil Metrics

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Abstract

In §1 we consider a certain weak pseudo-Riemannian Beyl-type metric on a space-time manifold M and its associated Berwald-type nonlinear connection N on TM. For TM endowed with an (h, v)-metric structure, we outline the procedure of obtaining the canonic N-connection, its d-torsions and curvatures. In §2 we apply a Finslerian perturbation of Beil type to the weak metric, which yields a pseudo-Riemann - Finslerian $(h, v)-$metric structure on TM and determine the Einstein equations of the obtained model. This standpoint can provide the possibility of producing gravitational waves using two electromagnetic fields. In §3 and §4 are determined the equations of the stationary curves and of their deviations. Their analytic solutions are derived, and the special cases of $h-$ and $v-$paths are emphasized.

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1 Introduction

In a recent work, P.C.Stavrinos [21] introduced the concept of gravitational waves in a Finsler space. The study was extended by the authors in the framework of vector bundles endowed with $(h, v)-$metrics introduced by R.Miron, and M.Anastasiei [14], [16] and the one of osculator spaces $Osc^k(M)$ of higher-order geometries [13], [15] for the case $k = 1$. Thus, an initial deformed gravitational model produced the FWDM and CWDM models for General Relativity [5] on the tangent bundle (TM, π, M) of a given space-time M.

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In this work we study the geometrical structure of an $(h, v)-$metric produced by a Beil-type deformation of a weak pseudo-Riemannian metric γ_{ij} defined on a real 4-dimensional differentiable manifold M. The weakness of the gravitational field is expressed by the decomposition of the metric γ_{ij} into the flat Minkowski metric $n_{ij} = diag(-1, 1, 1, 1)$ and a small perturbation $\varepsilon^{(1)}_{ij}(x)$ [5], [21], [22]

$$\gamma_{ij}(x) = n_{ij} + \varepsilon^{(1)}_{ij}(x), \tag{1}$$

where $\varepsilon_{ij} = \varepsilon^{(1)}_{ij}$ is a symmetric tensor field with $|\varepsilon^{(1)}_{ij}(x)| << 1$. Throughout the paper, the indices are raised in the linearized approach via the flat metric n_{ij}, e.g., $\varepsilon^{rs} = n^{ri}n^{sj}\varepsilon_{ij}$. This permits to consider the linearized version of a given model of General Relativity, in which the symmetric tensor field $\varepsilon^{(1)}_{ij}$ propagates in the weak pseudo-Riemannian gravitational field γ_{ij}.

In our case, both the weak perturbation and the Finslerian deformation originate in the Beil *Finslerian fundamental function* defined on the tangent bundle [9]

$$F(x, y) = [(n_{ij} + \lambda b_i b_j)y^i y^j]^{1/2}, \tag{2}$$

where $\lambda \in \mathbb{R}$, the 1-form $\{b_i(x, y)\}$ is 0-homogeneous by y and we denoted by (x^i, y^a) the local coordinates in a chart $\tilde{U} \subset TM$. Then F provides the *Finsler metric tensor field*

$$\tilde{\varepsilon}_{ij}(x, y) = \frac{1}{2}(\partial^2 F^2/\partial y^i \partial y^j) = n_{ij} + \lambda\tilde{b}_i\tilde{b}_j + \lambda(b_s y^s)(\partial\tilde{b}_i/\partial y^j). \tag{3}$$

with $\tilde{b}_i = \partial(b_s y^s)/\partial y^i$. If one considers the case when $\tilde{b}_i$ satisfies the relation

$$\partial\tilde{b}_i/\partial y^j = 0 \;\Leftrightarrow\; (\partial^2 b_s/\partial y^i \partial y^j)y^s + \partial b_i/\partial y^j + \partial b_j/\partial y^i = 0, \tag{4}$$

i.e., is position-independent, a somewhat tedious but straightforward computation shows, using the first equality of (3), that in this case the fundamental function (2) satisfies $b_i = b_i(x) = \tilde{b}_i(x)$. Hence, in case that (4) holds, the Finsler space is pseudo-Riemannian. Then we may consider the weak Beil-type perturbation of n given by

$$\gamma_{ij}(x) = n_{ij} + b_i(x)b_j(x). \tag{5}$$

Then, the canonic *non-linear connection* on TM provided by $\gamma_{ij}(x)$ is

$$N^a_i(x, y) = \gamma^a_{jb}y^b, \tag{6}$$

where γ^i_{jk} are the Christoffel symbols of the metric; this produces on $\mathcal{X}(\tilde{U})$ the *local adapted basis*

$$\{\delta_i = \partial_i - N^b_i\partial_b,\ \dot{\partial}_a\}_{i,a=\overline{1,4}} \equiv \{\delta_\beta\}_{\beta=\overline{1,8}}, \tag{7}$$

with $\partial_i = \frac{\partial}{\partial x^i}$ and $\dot{\partial}_a = \frac{\partial}{\partial y^a}$, as well as *the dual local basis*

$$\{d^i = dx^i,\ \delta^a = \delta y^a = dy^a + N^a_j dx^j\}_{i,a=\overline{1,4}} \equiv \{\delta^\beta\}_{\beta=\overline{1,8}}. \tag{8}$$

Throughout the paper, the Latin indices $i, j, k, \ldots, a, b, c, \ldots$ run in the range $\overline{1,4}$, while the Greek ones $\alpha, \beta, \gamma, \ldots$, in the range $\overline{1,8}$.

The Finslerian deformation of the weak metric γ_{ij} considered in the next section will provide a certain $(h,v)-$metric on TM. Generally, given a $(h,v)-$metric on the tangent bundle (TM, π, M) [14],

$$G = g_{ij}(x,y)dx^i \otimes dx^j + h_{ab}(x,y)\delta y^a \otimes \delta y^b, \tag{9}$$

this provides a *canonical* $N-$*connection* $\mathbf{D}$, with the coefficients

$$\{L^i_{jk}, \tilde{L}^a_{bk}, \tilde{C}^i_{ja}, C^a_{bc}\} \equiv \{\Gamma^\alpha_{\beta\gamma}\}$$

explicitely given by [14]

$$\begin{array}{ll} L^i_{jk} = \frac{1}{2}g^{is}(\delta_{\{j}g_{sk\}} - \delta_s g_{jk}), & \tilde{L}^a_{bk} = \dot{\partial}_b N^a_k + \frac{1}{2}h^{ac}(\delta_k h_{bc} - h_{c\{d}\dot{\partial}_{b\}}N^d_k) \\ \tilde{C}^i_{ja} = \frac{1}{2}g^{ih}\dot{\partial}_a g_{jh} & C^a_{bc} = \frac{1}{2}h^{ad}(\dot{\partial}_{\{b}h_{dc\}} - \dot{\partial}_d h_{bc}), \end{array} \tag{10}$$

which preserves the $h-v$ splitting produced by N, is metrical, $h-$ and $v-$ symmetrical, and depends on N and G only.

Its *torsion tensor field* $\mathcal{T} \in \mathcal{T}^1_2(TM)$ has the coefficients

$$\mathcal{T}_{\beta\ \alpha}^{\ \kappa} = \Gamma^\alpha_{[\beta\kappa]} + B^\alpha_{[\beta\kappa]}, \qquad \mathcal{T}(\delta_\alpha, \delta_\beta) = \mathcal{T}_{\beta\ \alpha}^{\ \kappa}\delta_\kappa, \tag{11}$$

where we denoted $\tau_{[\alpha\beta]} = \tau_{\alpha\beta} - \tau_{\beta\alpha}$ and where *the non-holonomy coefficients* $B_{\alpha\ \beta}^{\ \gamma}$ are defined by $[\delta_\alpha, \delta_\beta] = B_{\alpha\ \beta}^{\ \gamma}\delta_\gamma$. The $h, v-$splitting of $\mathcal{T}$ provides the *torsion* $N-$*tensor fields* [14]

$$\begin{array}{ll} T^i_{jk} = d^i\mathcal{T}(\delta_k, \delta_j) = L^i_{[jk]}, & R^a_{kl} = \delta^a\mathcal{T}(\delta_l, \delta_k) = \delta_{[l}N^a_{k]}, \\ P^i_{ja} = d^i\mathcal{T}(\dot{\partial}_a, \delta_j) = \tilde{C}^i_{ja}, & P^a_{bk} = \delta^a\mathcal{T}(\delta_k, \dot{\partial}_b) = \dot{\partial}_b N^a_k - \tilde{L}^a_{bk}, \\ S^a_{bc} = \delta^a\mathcal{T}(\dot{\partial}_c, \dot{\partial}_b) = C^a_{[bc]}, & \end{array} \tag{12}$$

Similarly, the *curvature tensor field* $\mathcal{R} \in \mathcal{T}^1_3(TM)$ of the $N-$connection $\mathbf{D}$ has the coefficients given by

$$\mathcal{R}_{\beta\ \gamma\theta}^{\ \alpha} = \delta_{[\theta}-^{\alpha}_{\beta\gamma]} + -^{\phi}_{\beta[\gamma}-^{\alpha}_{\phi\theta]} + -^{\alpha}_{\beta\phi}\mathcal{B}^{\phi}_{\gamma\theta}, \qquad \mathcal{R}(\delta_\alpha, \delta_\beta)\delta_\gamma = \mathcal{R}_{\gamma\ \beta\alpha}^{\ \lambda}\delta_\lambda, \tag{13}$$

and its $h, v-$splitting of $\mathcal{R}$ provides the *curvature* $N-$*tensor fields*

$$\begin{array}{l} R_{j\ kl}^{\ i} = d^i\mathcal{R}(\delta_\updownarrow, \delta_\|)\delta_| = \delta_{[\updownarrow}\mathcal{L}^{\rangle}_{|||]} + \mathcal{L}^{\langle}_{|[||}\mathcal{L}^{\rangle}_{\langle\updownarrow]} + \mathcal{C}^{\rangle}_{|\dashv}\mathcal{R}^{\dashv}_{\|\updownarrow} \\ \tilde{R}_{b\ kl}^{\ a} = \delta^a\mathcal{R}(\delta_\updownarrow, \delta_\|)\partial_\lfloor = \delta_{[\updownarrow}\mathcal{L}^{\dashv}_{\lfloor||]} + \mathcal{L}^{\rfloor}_{\lfloor[||}\mathcal{L}^{\dashv}_{\rfloor\updownarrow]} + \mathcal{C}^{\dashv}_{\lfloor\rfloor}\mathcal{R}^{\rfloor}_{\|\updownarrow} \\ P_{j\ kc}^{\ i} = d^i\mathcal{R}(\partial_\rfloor, \delta_\|)\delta_| = \partial_\rfloor\mathcal{L}^{\rangle}_{|||} - (\delta_\|\mathcal{C}^{\rangle}_{|\rfloor} + \mathcal{L}^{\rangle}_{\langle||}\mathcal{C}^{\langle}_{|\rfloor} - \mathcal{L}^{\langle}_{|||}\mathcal{C}^{\rangle}_{\langle\rfloor} - \mathcal{L}^{\lfloor}_{\rfloor||}\mathcal{C}^{\rangle}_{|\lfloor}) + \mathcal{C}^{\rangle}_{|\lfloor}\mathcal{P}^{\lfloor}_{||\rfloor} \\ \tilde{P}_{b\ kc}^{\ a} = \delta^a\mathcal{R}(\partial_\rfloor, \delta_\|)\partial_\lfloor = \partial_\rfloor\mathcal{L}^{\dashv}_{\lfloor||} - (\delta_\|\mathcal{C}^{\dashv}_{\lfloor\rfloor} + \mathcal{L}^{\dashv}_{\lceil||}\mathcal{C}^{\lceil}_{\lfloor\rfloor} - \mathcal{L}^{\lceil}_{\lfloor||}\mathcal{C}^{\dashv}_{\lceil\rfloor} - \mathcal{L}^{\lceil}_{\rfloor||}\mathcal{C}^{\dashv}_{\lceil\lfloor}) + \mathcal{C}^{\dashv}_{\lfloor\lceil}\mathcal{P}^{\lceil}_{||\rfloor} \\ \tilde{S}_{j\ bc}^{\ i} = d^i\mathcal{R}(\partial_\rfloor, \partial_\lfloor)\delta_| = \partial_{[\rfloor}\mathcal{C}^{\rangle}_{|\lfloor]} + \mathcal{C}^{\langle}_{|[\lfloor}\mathcal{C}^{\rangle}_{\langle\rfloor]} \\ S_{b\ cd}^{\ a} = \delta^a\mathcal{R}(\partial_\lceil, \partial_\rfloor)\partial_\lfloor = \partial_{[\lceil}\mathcal{C}^{\dashv}_{\lfloor\rfloor]} + \mathcal{C}^{\rceil}_{\lfloor[\rfloor}\mathcal{C}^{\dashv}_{\rceil\lceil]}. \end{array} \tag{14}$$

These geometrical objects are basic for infering the Einstein equations of the linearized deformed models defined in the following section.

2 The Beil - Finslerian deformed weak model

The Beil-type deformation of the weak metric γ_{ij} is produced by a Finslerian perturbation of the pseudo-Riemannian gravitational field γ_{ij}, which leads to the generalized Finslerian metric [21]

$$f_{ij}(x,y) = \gamma_{ij}(x) + \varepsilon^{(2)}{}_{ij}(x,y) = \eta_{ij} + \varepsilon^{(1)}_{ij}(x) + \tilde{\varepsilon}_{ij}(x,y), \tag{15}$$

where the Beil-type Finslerian perturbation $\varepsilon^{(2)}{}_{ij}(x,y)$ coincides with the Finslerian metric tensor field $\tilde{\varepsilon}$ in (3), assumed to satisfy the condition $|\varepsilon^{(2)}{}_{ij}(x,y)| \ll 1$ in order that f_{ij} be non-degenerate. Moreover, the tensor

$$\varepsilon^{*}{}_{ij}(x,y) = \varepsilon^{(1)}_{ij}(x) + \tilde{\varepsilon}_{ij}(x,y) \tag{16}$$

provides a weak Finslerian perturbation of the Minkowski metric n_{ij}, and vanishes iff γ_{ij} is flat. This point of view permits us to consider $(h,v)-$metric $v-$Finslerian or $v-$Lagrangian approaches. Note that for $\tilde{\varepsilon}_{ij}$ one can consider the Finsler metric tensors provided by different physically significant choices for $b_i(x,y)$ in (2) [9], like

$$b_a \in \left\{ \frac{y_a}{s_b y^b}, \frac{A_b y^b s_a}{s_b v^b}, \frac{\sqrt{y_b y^b} s_a}{s_b y^b}, \frac{y_b y^b s_a}{(s_b y^b)^2}, \frac{y_a}{\sqrt{y_b y^b}}, \frac{A_b y^b y_a}{(s_b y^b)^2} \right\}, \tag{17}$$

where $A_a(x)$ and $s_a(x)$ are vectors to be specified and $y_a = \tilde{\varepsilon}_{ab} y^b$. From the physical point of view, the weak Finslerian gravitational field f_{ij} appears as a Finslerian perturbation $\varepsilon^{*}{}_{ij}$ of a Minkowski space-time (M, n_{ij}). A pseudo-Riemannian weak gravitational field can constitute a first order perturbation of the Minkowski space-time. Although the Beil metric can be applied for the strong gravity acting at the hadronic level, here we deal with the Beil-type deformation of the weak metric γ_{ij}. This is produced, in the classical point of view, by the interaction of two electromagnetic potentials $b_i(x), b_j(x)$. By the predictions of the conventional general relativity, when an electromagnetic (e.m.) wave passes through an electromagnetic field, it produces a gravitational wave of the same frequency. This gravitational wave, when propagating through another electromagnetic field, creates an e.m. wave [20].

We considered in (15) the perturbation $\tilde{\varepsilon}_{ij}$ within the geometrical framework developed by R.G.Beil [9]. But valid models seem to provide as well the Kaluza-Klein ansatz or the one of the Randers-type Yang-Mills theory [7], [8]; in these cases the Finslerian perturbation of the pseudo-Riemannian metric is provided by the electromagnetic field, or by a gauge or spinor extension of the pseudo-Riemannian gravitational field. In each of this models, the original pseudo-Riemannian model appears as a limiting case. Hence the correspondence principle between the Finslerian and pseudo-Riemannian structures depends basically on the type of the generalized Finsler or Lagrange space associated to the deformed metric.

Note that the deformed metric $f_{ij}(x,y)$ is of Finsler type itself, providing on TM a particular case of a generalized Lagrange structure $GL^n = (M, f_{ij})$ in the sense of R.Miron [14]. Then one might consider the *almost Hermitian model* of GL^n, given by the $N-$lift of f_{ij} to TM and by the canonic adapted complex structure $J \in End(\mathcal{X}(M))$ having locally the associated matrix $[J] = \begin{pmatrix} 0 & -I_4 \\ I_4 & 0 \end{pmatrix}$. This yields an *almost Kahler structure*, which is Kahler in case that $\varepsilon^{(1)}_{ij} = const.$ and $\tilde{\varepsilon}_{ij}(x,y) = \tilde{\varepsilon}_{ij}(y)$, i.e., when $b_i(x) = b_i = const$ and the Finsler metric is Minkowski.

As an alternative approach which we follow hereafter, we build on TM the $(h,v)-$metric provided by the two adjusted components $n + \varepsilon^{(1)}$ and $\varepsilon^{(2)} = \frac{1}{F^2}\tilde{\varepsilon}$ of the weak Finslerian metric (15),

$$G = (n_{ij} + b_i(x)b_j(x))\, dx^i \otimes dx^j + \varepsilon^{(2)}{}_{ab}(x,y)\delta y^a \otimes \delta y^b, \tag{18}$$

with

$$\varepsilon^{(2)}{}_{ij}(x,y) = \frac{1}{F^2(x,y)}\tilde{\varepsilon}_{ij}(x,y) = \frac{1}{F^2(x,y)}\left(n_{ij} + \lambda\tilde{b}_i\tilde{b}_j + \lambda(b_s y^s)(\partial\tilde{b}_i/\partial y^j)\right),$$

where in view of preserving the dimensionality in the directional variables, the Beil metric $\tilde{\varepsilon}_{ij}(x,y)$ is scaled by the conformal factor $F^{-2}(x,y)$. We call the metric structure (TM, G), *the Beil-Finslerian deformed weak model* (B-FDWM).

We note that though the deformation of type

$$\tilde{f}_{ij}(x,y) = \gamma_{ij}(x) + \varepsilon^{(1)}_{ij}(x) + \frac{1}{F^2(x,y)}\tilde{\varepsilon}_{ij}(x,y)$$

is nolonger proper Finslerian (due to the lack of 0-homogeneity in the last term), the adjusted lift G on TM satisfies this property.

In particular, if $\varepsilon^{(2)}$ depends on y only, then G is a *pseudo-Riemann - locally Minkowski* $(h,v)-$metric, and the gravitational field of this space is called *weak Riemannian-locally Minkowski gravitational field*. In the *linear approach*, the Christoffel symbols γ^i_{jk} of the weak metric γ_{ij} will take the linearized form $\bar{\gamma}^i_{jk}$ given by ([21], [5])

$$\bar{\gamma}^i_{jk} = \frac{1}{2}n^{is}(\partial_{\{j}\varepsilon_{sk\}} - \partial_s\varepsilon_{jk}) = \frac{1}{2}n^{is}(b_{\{jk\}}b_s + b_{[sk]}b_j + b_{[sj]}b_k) \approx \gamma^i_{jk}, \tag{19}$$

where we denote $\tau_{\{ij\}} = \tau_{ij} + \tau_{ji}$ and $b_{sj} = \partial_j b_s$. The nonlinear connection is also approximated by the *weak nonlinear connection*

$$\bar{N}^a_i = \varepsilon^a_{ib}y^b = \frac{1}{2}n^{ac}(b_{\{i0\}}b_s + b_{[s0]}b_i + b_{[si]}b_0) \approx N^a_i, \tag{20}$$

where the null index denotes the transvection with y.

We also note that in particular, if $b = b_i(x)dx^i$ is a *potential* 1-form with $b_i = \partial_i b$, then

$$\bar{\gamma}^i_{jk} = \varepsilon^i b_{jk}, \qquad \bar{N}^a_i = \varepsilon^a b_{j0}, \tag{21}$$

where $\varepsilon^i = 1 - 2\delta^i_1,\ i = \overline{1,4}$.

For obtaining the Einstein equations of the deformed model, we set first the following

Lemma 1. *a) The coefficients of the canonic linear $N-$connection $\mathbf{D}$ of the linearized B-FDWM are*

$$L^i_{jk} = \tilde{L}^i_{jk} = \bar{\gamma}^i_{jk} \approx \gamma^i_{jk}, \qquad \tilde{C}^i_{ja} = 0;\ C^a_{bc} = \frac{1}{2}\tilde{\varepsilon}^{ad}C_{dbc}, \tag{22}$$

where $C_{abc} = \dot{\partial}_a\tilde{\varepsilon}_{bc}$ is the Cartan tensor field associated to $\tilde{\varepsilon}_{ij}$.

b) The $N-$fields of torsion of the linearized B-FDWM are null

$$T^i{}_{jk} = 0, \quad \tilde{C}^i{}_{ja} = 0, \quad P^a{}_{kb} = 0, \quad S^a_{bc} = 0, \tag{23}$$

except the curvature d-tensor field of N,

$$R^a{}_{jk} = r_c{}^a{}_{jk}y^c = \frac{1}{2}n^{is}\left(b_s\partial^2_{[k0}(b_sb_{j]}) + \partial^2_{[js}(b_0b_{k]})\right). \tag{24}$$

c) The $N-$fields of curvature of the linearized B-FDWM are

$$\begin{array}{rcl} R_j{}^i{}_{kl} &=& r_j{}^i{}_{kl}, \quad \tilde{R}_b{}^a{}_{kl} = r_b{}^a{}_{kl}, \quad P_j{}^i{}_{kc} = 0, \\ \tilde{P}_b{}^a{}_{kc} &=& -(\delta_k C^a_{bc} + \bar{\gamma}^a_{dk}C^d_{bc} - \bar{\gamma}^d_{\{bk}C^a_{dc\}}) \\ S_j{}^i{}_{bc} &=& 0, \quad \tilde{S}_b{}^a{}_{cd} = C^s_{b[d}C^a_{c]s}, \end{array}$$

where $r_j{}^i{}_{kl}$ is the linearized weak curvature,

$$\begin{aligned} r_j{}^i{}_{kl} &= \partial_{[l}\bar{\gamma}^i_{jk]} = \frac{1}{2}n^{is}(\partial^2_{[lj}\varepsilon_{sk]} + \partial^2_{[ks}\varepsilon_{jl]}) = \\ &= \frac{1}{2}n^{is}(b_s\partial^2_{[lj}b_{k]} + \partial_{[l}b_s\cdot\partial_jb_{k]} + \\ &\quad + \partial_jb_s\cdot\partial_{[l}b_{k]} + b_{[k}\partial^2_{l]j}b_s + \partial^2_{[ks}(b_jb_{l]})). \end{aligned} \tag{25}$$

In the linearized potential case, the horizontal curvature simplifies:

$$r_j{}^i{}_{kl} = \frac{1}{2}n^{is}\partial_{[l}b_s\cdot\partial_jb_{k]}.$$

By straightforward computation, the $hh-$Ricci $N-$tensor field and the horizontal scalar of curvature are [5]

$$\begin{array}{l} R_{ij} \equiv R_i{}^k{}_{jk} = r_i{}^k{}_{jk} = r_{jk} = \frac{1}{2}(\Box\varepsilon_{ij} + \partial^2_{ij}\varepsilon - \partial^2_{\{js}\varepsilon^s_{i\}}), \\ R = r = \Box\varepsilon - \partial^2_{ij}\varepsilon^{ij}, \end{array} \tag{26}$$

where $\varepsilon = n^{ij}\varepsilon_{ij}$, and "$\Box$" denotes the d'Alambertian

$$\Box = -\partial^2_{00} + \partial^2_{11} + \partial^2_{22} + \partial^2_{33} \equiv -\partial^2_{tt} + \partial^2_{xx} + \partial^2_{yy} + \partial^2_{zz}.$$

In the considered model, we have

Lemma 2. *a) The Ricci N−tensor fields of the linearized B-FDWM are*

$$\begin{aligned}
R_{ij} &= \tfrac{1}{2}\varepsilon^s[b_s\partial^2_{[si}b_{j]} + \partial_{[s}b_s \cdot \partial_i b_{j]} + \\
&\quad +\partial_i b_s \cdot \partial_{[s}b_{j]} + b_{[j}\partial^2_{s]i}b_s + \partial^2_{[js}(b_i b_{s]})]; \\
P_{jb} &\equiv P_{j\ kb}^{\ k} = 0 \qquad \tilde{P}_{bk} \equiv \tilde{P}_{b\ kd}^{\ d} = -(\delta_k C^a_{ba} - \varepsilon^d_{bk}C^a_{da}) \\
S_{ab} &\equiv S_{a\ bd}^{\ d} = C^e_{a[d}C^d_{b]e},
\end{aligned} \tag{27}$$

In the linearized potential case we have

$$r_{jk} = \frac{\lambda}{2}\varepsilon^i(\partial_i b_i \cdot \partial_j b_k - \partial_k b_i \cdot \partial_j b_i).$$

b) The Ricci scalars of curvature of the linearized B-FDWM are

$$\begin{aligned}
R &= \tfrac{1}{2}\varepsilon^i\varepsilon^j\left[b_i\partial^2_{[ij}b_{j]} + \partial_{[i}b_i \cdot \partial_j b_{j]} + \partial_j b_i \cdot \partial_{[i}b_{j]} + b_{[j}\partial^2_{i]j}b_i + \partial^2_{[ji}(b_j b_{i]})\right], \\
S &= C^e_{b[d}C^d_{c]e}\tilde{\varepsilon}^{bc},
\end{aligned} \tag{28}$$

and in the linearized potential case we have

$$r = \frac{\varepsilon^i\varepsilon^j}{2}\left(b_{jk[k}b_{j]} + b_{[jk}b_{jk]}\right),$$

where we denoted $b_{ijk} = \partial^2_{jk}b_i = \partial^3_{ijk}b$.

Theorem 1. *The Einstein equations of the linearized B-FDWM are*

$$\begin{aligned}
R_{ij} - \tfrac{1}{2}(R+S)n_{ij} &\equiv \tfrac{1}{2}(\Box\varepsilon_{ij} + \partial^2_{ij}\varepsilon - \partial^2_{\{js}\varepsilon^s_{i\}}) - n_{ij}(R+S) = \\
&= \kappa T_{ij} \\
S_{ab} - \tfrac{1}{2}(R+S)\tilde{\varepsilon}_{ab} &\equiv C^e_{a[d}C^d_{b]e} - \tfrac{1}{2}\tilde{\varepsilon}_{ab}(R+S) = \kappa T_{ab} \\
\tilde{P}_{jb} &\equiv 0 = -\kappa T_{jb}, \\
P_{bk} &\equiv -(\delta_k C^a_{ba} - \varepsilon^d_{bk}C^a_{da}) = \kappa T_{bk},
\end{aligned} \tag{29}$$

where $T_{ij}, T_{ab}, T_{jb}, T_{bk}$ *are the energy-momentum N−tensor fields, and* κ *is a constant.*

Theorem 2. *The conservation laws for the Einstein equations of the linearized B-FDWM are*

$$\begin{aligned}
&\frac{\varepsilon^i}{2}(\Box\varepsilon_{ij}+\partial^2_{ij}\varepsilon-\partial^2_{\{js}\varepsilon^s_{i\}})|_i-\\
&-\delta_{ij}(R+S)\tilde{\varepsilon}^{ac}(\delta_j C^b_{cb}-\bar{\gamma}^d_{cj}C^b_{db})\Big|_a = \kappa\left(\varepsilon^i T_{ij}|_i+\varepsilon^a T_{aj}\Big|_a\right),\\
&\tilde{\varepsilon}^{ca}\left(C^e_{a[d}C^d_{b]e}-\tfrac{1}{2}\tilde{\varepsilon}_{ab}(R+S)\right)\Big|_c = \kappa\left(\varepsilon^i T_{ib}|_i+\varepsilon^c T_{cb}\Big|_c\right),
\end{aligned} \tag{30}$$

where $|_i$ *and* $\Big|_a$ *are respectively the* $h-$ *and the* $v-$*covariant derivations of* $N-$*tensor fields induced by the* $N-$*connection* ∇.

3 The stationary curves of the linearized B-FDWM

Let $c : I = [a, b] \subset \mathbb{R} \to TM$ be a smooth curve, such that its image lies in a chart $\tilde{U} \subset TM$,

$$c(t) = (x^i(t), y^a(t)) \equiv (y^\alpha(t)), \forall t \in I,$$

and let $\mathbf{D}$ be a linear $N-$connection on TM.

Definitions. a) We shall call the *covariant velocity* field and respectively the *covariant force* on the curve c, the fields defined on c by

$$\begin{aligned}
&\mathcal{V} = \mathcal{V}^\alpha\delta_\alpha, && \mathcal{V}^\alpha = \frac{\delta y^\alpha}{dt}\\
&\mathcal{F} = \frac{\mathbf{D}\mathcal{V}}{dt} = \mathcal{F}^\alpha\delta_\alpha, && \mathcal{F}^\alpha = \frac{\delta\mathcal{V}^\alpha}{dt} + \Gamma^\alpha_{\beta\kappa}\mathcal{V}^\beta\mathcal{V}^\kappa,\ \alpha = \overline{1,8},
\end{aligned} \tag{31}$$

b) We shall say that c is a *stationary curve* with respect to $\mathbf{D}$ iff $\mathcal{F} = 0$ along the curve.

c) The curve c is called $h-$*curve*, if $\pi_v(\mathcal{V}) = 0$, and $v-$*curve*, if $\pi_h(\mathcal{V}) = 0$, where by π_h and π_v we denote respectively the $h-$ and $v-$projectors of the canonic splitting induced by N. If a h-/v-curve satisfies also the extra condition $\mathcal{F} = 0$, then it is called *h-/v-path*, respectively.

In applications, the covariant force determines the non-linear connection. E.g., for the covariant force $\mathcal{F}^\alpha = E^\alpha_\beta y^\beta$, with $E_{\alpha\beta} = b_{[\alpha\beta]}$ a Beil-type field, the corresponding non-linear connection is

$$N^a_j = \gamma^a_{ij}y^j - E^a_j. \tag{32}$$

In the linearized approach, the $h-$paths project onto geodesics of M, and are solutions of the Volterra-Hamilton-type second-order differential system

$$\begin{cases}
\dfrac{dy^a}{dt} + N^a_j(x(t), y(t))\dfrac{dx^j}{dt} = 0\\
\dfrac{d^2x^i}{dt^2} + L^i_{jk}(x(t), y(t))\dfrac{dx^j}{dt}\dfrac{dx^k}{dt} = 0, \quad a, i = \overline{1,n}\ (n = 4),
\end{cases} \tag{33}$$

which in the linearized B-FDWM rewrites as the first-order system

$$\begin{cases} \dfrac{dy^i}{dt} = -\bar{\gamma}^i_{jk}(x(t)) \cdot y^j(t) z^k(t), & \dfrac{dx^i}{dt} = z^i(t) \\ \dfrac{dz^i}{dt} = -\bar{\gamma}^i_{jk}(x(t)) \cdot z^j(t) z^k(t), & i = \overline{1,n}. \end{cases} \tag{34}$$

The equations of motion described above can physically interpret the gravitational interaction caused by electromagnetic fields. The system (33) with the unknowns $x^i = x^i(t), y^i = y^i(t), z^i = z^i(t), i = \overline{1,n}$ provides for initial conditions

$$x^i(0) = x^i_0, \quad y^i(0) = y^i_0, \quad z^i(0) = z^i_0,$$

a *Cauchy problem* which is tractable numerically (e.g., using a Runge-Kutta type algorithm). Note that in the case of Beil type weak non-linear connection by (32), the above system takes the form

$$\frac{dy^a}{dt} + \varepsilon^a_{jk} y^j y^k = E^a_j(x) y^j, \quad a = \overline{1,n}. \tag{35}$$

Moreover, in the analytic case of the linearized B-FDWM, the coefficients of the system (34) decompose in Taylor series

$$\bar{\gamma}^i_{jk} = f^i_{[u]jk} x^u, \tag{36}$$

where we denoted $[u] = (u_1, \ldots, u_n) \in \mathbb{N}^n, x = (x^1, \ldots, x^n), x^u = (x^1)^{u_1} \cdot (x^2)^{u_2} \cdot \ldots \cdot (x^n)^{u_n}$, and consequently (34) rewrites

$$\begin{cases} \dfrac{dx^i}{dt} - z^i(t) = 0, \\ \dfrac{dy^i}{dt} + f^i_{[u]jk} x^u y^j z^k = 0, \\ \dfrac{dz^i}{dt} + f^i_{[u]jk} x^u z^j z^k = 0, \qquad i = \overline{1,n}. \end{cases} \tag{37}$$

We remark that (37) is of the type

$$\begin{cases} \dfrac{dx^i}{dt} + A^i_{[u][v][w]} x^u y^v z^w = 0, \\ \dfrac{dy^i}{dt} + B^i_{[u][v][w]} x^u y^v z^w = 0, \\ \dfrac{dz^i}{dt} + C^i_{[u][v][w]} x^u y^v z^w, \quad i = \overline{1,n}, \end{cases} \tag{38}$$

with the coefficients

$$\begin{aligned} A^i_{[u][v][w]} &= -\delta^{[0]}_{[u]} \delta^{[0]}_{[v]} \delta^{\langle i \rangle}_{[w]}, \\ B^i_{[u][v][w]} &= f^i_{[u]jk} \delta^{\langle j \rangle}_{[v]} \delta^{\langle k \rangle}_{[w]}, \\ C^i_{[u][v][w]} &= f^i_{[u]jk} \delta^{[0]}_{[v]} \delta^{\langle jk \rangle}_{[w]}, \end{aligned} \tag{39}$$

where we denoted

$$[0] = (0, \ldots, 0), \quad \langle j \rangle = e_j, \langle jk \rangle = e_j + e_k, \quad j, k \in \overline{1, n},$$

and $e_j = (0, \ldots, 0, 1, 0, \ldots, 0)$, with 1 on the $j-$th position. Then applying the general theory described in [25], we obtain the following

Theorem 3. *The analytic solutions of the system (34) are given by*

$$\begin{aligned} x^i &= \sum_{m=0}^{\infty} \frac{(-t)^m}{m!} P^m_{\langle i \rangle [0][0][u][v][w]} a^u b^v c^w \\ y^i &= \sum_{m=0}^{\infty} \frac{(-t)^m}{m!} P^m_{[0]\langle i \rangle [0][u][v][w]} a^u b^v c^w \\ z^i &= \sum_{m=0}^{\infty} \frac{(-t)^m}{m!} P^m_{[0][0]\langle i \rangle [u][v][w]} a^u b^v c^w, \quad i = \overline{1, n} \end{aligned} \tag{40}$$

where $a = x(0), b = y(0), c = z(0) \in \mathbb{R}^n$, $P^0_{[\alpha][\beta][\gamma][u][v][w]} = \delta^{[u]}_{[\alpha]} \delta^{[v]}_{[\beta]} \delta^{[w]}_{[\gamma]}$ *and*

$$\begin{aligned} P^{m+1}_{[\alpha][\beta][\gamma][u][v][w]} &= \frac{d}{dt} P^m_{[\alpha][\beta][\gamma][u][v][w]} + \sum_{[\eta],[\mu],[\nu] \in \mathbb{N}} P^m_{[\eta][\mu][\nu][u][v][w]} \cdot \\ &\cdot \left(\sum_{s=1}^{n} \alpha_s A^s_{[\eta]-[\alpha]+\langle s \rangle, [\mu]-[\beta], [\nu]-[\gamma]} + \right. \\ &+ \sum_{s=1}^{n} \beta_s B^s_{[\eta]-[\alpha], [\mu]-[\beta]+\langle s \rangle, [\nu]-[\gamma]} + \\ &+ \left. \sum_{s=1}^{n} \gamma_s C^s_{[\eta]-[\alpha], [\mu]-[\beta], [\nu]-[\gamma]+\langle s \rangle} \right), \quad m \in \mathbb{N}. \end{aligned} \tag{41}$$

Remark that for the linearized case system (34), the recurrence relation above becomes

$$\begin{aligned} P^{m+1}_{[\alpha][\beta][\gamma][u][v][w]} &= \frac{d}{dt} P^m_{[\alpha][\beta][\gamma][u][v][w]} + \\ &+ \sum_{s=1}^{n} \left(\alpha_s P^m_{[\alpha]-\langle s \rangle, [\beta][\gamma]+\langle s \rangle, [u][v][w]} + \right. \\ &\quad + \sum_{[\eta] \in \mathrm{N}} \beta_s f^s_{[\eta]-[\alpha], jk} \cdot P^m_{[\eta], [\beta]+\langle j \rangle, [\gamma]+\langle k \rangle, [u][v][w]} + \\ &\quad + \left. \sum_{[\eta] \in \mathrm{N}} \gamma_s f^s_{[\eta]-[\alpha], jk} \cdot P^m_{[\eta], [\beta], [\gamma]+\langle j \rangle + \langle k \rangle, [u][v][w]} \right). \end{aligned} \tag{42}$$

Regarding *the $v-$paths* of the linearized B-FDWM, these coincide with the $v-$paths of the Finsler space $(M, F(x, y))$, with $F^2 = \varepsilon^{(2)}{}_{ab}(x, y) y^a y^b$, and any $v-$path $c : I \subset$

$\mathbb{R} \to TM$, $c(t) = (x_0^i, y^a(t))$ is a solution of the second-order differential system

$$\frac{d^2y^a}{dt^2} + C^a_{bc}(x_0, y(t))\frac{dy^b}{dt}\frac{dy^c}{dt} = 0. \tag{43}$$

In the analytic case, a similar approach as for $h-$paths can be applied to this system as well. Since we have $x(t) = x_0 = const$, the coefficients of the system decompose in Taylor series

$$C^i_{jk} = g^i_{[u]jk}y^u, \tag{44}$$

and hence (43) rewrites

$$\frac{dy^i}{dt} - z^i(t) = 0, \qquad \frac{dz^i}{dt} + g^i_{[u]jk}y^u z^j z^k = 0, \quad i = \overline{1,n}, \tag{45}$$

Note that this is of the type

$$\frac{dy^i}{dt} + A^i_{[u][v]}y^u z^v = 0, \qquad \frac{dz^i}{dt} + B^i_{[u][v]}y^u z^v = 0, \quad i = \overline{1,n} \tag{46}$$

with the coefficients

$$A^i_{[u][v]} = -\delta^{[0]}_{[u]}\delta^{\langle i\rangle}_{[v]}, \quad B^i_{[u][v]} = g^i_{[u]jk}\delta^{\langle jk\rangle}_{[v]}, \quad i = \overline{1,n}. \tag{47}$$

Theorem 4. *The analytic solutions of the system (43) are given by*

$$\begin{aligned} y^i &= \sum_{m=0}^{\infty}\frac{(-t)^m}{m!}P^m_{\langle i\rangle[0][u][v]}a^u b^v \\ z^i &= \sum_{m=0}^{\infty}\frac{(-t)^m}{m!}P^m_{[0]\langle i\rangle[u][v]}a^u b^v, \quad i = \overline{1,n}, \end{aligned} \tag{48}$$

where $a = y(0), b = z(0) \in \mathbb{R}^n$, $P^0_{[\alpha][\beta][u][v]} = \delta^{[u]}_{[\alpha]}\delta^{[v]}_{[\beta]}$ *and*

$$\begin{aligned} P^{m+1}_{[\alpha][\beta][u][v]} &= \frac{d}{dt}P^m_{[\alpha][\beta][u][v]} + \sum_{[\eta],[\mu]\in N} P^m_{[\eta][\mu][u][v]} \cdot \\ &\cdot \sum_{s=1}^{n}(\alpha_s A^s_{[\eta]-[\alpha]+\langle s\rangle,[\mu]-[\beta]} + \beta_s B^s_{[\eta]-[\alpha],[\mu]-[\beta]+\langle s\rangle}), \quad m \in \mathbb{N}. \end{aligned} \tag{49}$$

It is worthy to note that in the linearized case (34), the recurrence relation (49) becomes much simpler,

$$\begin{aligned} P^{m+1}_{[\alpha][\beta][u][v]} &= \frac{d}{dt}P^m_{[\alpha][\beta][u][v]} + \sum_{s=1}^{n}\Bigg(\alpha_s P^m_{[\alpha]-\langle s\rangle,[\beta]+\langle s\rangle,[u][v]} + \\ &+ \sum_{[\eta]\in N}\beta_s P^m_{[\eta],[\beta]-\langle s\rangle+\langle j\rangle+\langle k\rangle,[u][v]}\Bigg), \quad m \in \mathbb{N}. \end{aligned} \tag{50}$$

4 Deviation of stationary curves in the linearized B-FDWM

Let now $c : I_1 \times I_2 \subset \mathbb{R}^2 \to \tilde{U} \subset TM$ be a family of stationary curves, having t as arc-length parameter, and u the deviation parameter [19], [10],

$$c(t,u) = (x^i(t,u), y^a(t,u)) = (y^\alpha(t,u)) \in \tilde{U}, \quad \forall (t,u) \in I_1 \times I_2.$$

where $\tilde{U} \subset TM$ is an open domain. Then let $\mathcal{Z} = \mathcal{Z}^\alpha \delta_\alpha$ be the *deviation vector field*, given by

$$\mathcal{Z}^i = \partial_u x^i, \quad \mathcal{Z}^a = \partial_u y^a + N^a_i \partial_u x^i,$$

and let $\mathcal{V} = \mathcal{V}^\alpha \delta_\alpha$ be the *velocity vector field*, where

$$\mathcal{V}^i = \partial_t x^i, \quad \mathcal{V}^a = \partial_t y^a + N^a_i \partial_t x^i.$$

For any vector field $\mathcal{W} = \mathcal{W}^\alpha \delta_\alpha$, defined on the family of curves $Im(c)$, we can consider the partial covariant derivatives

$$\delta_t \mathcal{W}^\alpha = \partial_t \mathcal{W}^\alpha + \Gamma^\alpha_{\beta\gamma} \mathcal{W}^\beta \mathcal{V}^\gamma, \quad \delta_u \mathcal{W}^\alpha = \partial_u \mathcal{W}^\alpha + \Gamma^\alpha_{\beta\gamma} \mathcal{W}^\beta \mathcal{Z}^\gamma. \tag{51}$$

The equations of deviations of the family with respect to the connection $\mathbf{D}$ characterize the tidal force $\mathcal{Z}$, and have the form ([1], [5], [6])

$$\delta_t^2 \mathcal{Z}^\alpha + \delta_t \mathcal{T}^\alpha = \rho^\alpha + \delta_u \mathcal{F}^\alpha, \quad \alpha = \overline{1, 2n}, \tag{52}$$

where we denoted $\mathcal{T}^\alpha = \mathcal{T}_{\beta\ \gamma}^{\ \alpha} \mathcal{V}^\beta \mathcal{Z}^\gamma$ and $\rho^\alpha = \mathcal{R}_{\beta\ \gamma\lambda}^{\ \alpha} \mathcal{V}^\beta \mathcal{Z}^\gamma \mathcal{V}^\lambda$. These equations can be rewritten as

$$\partial_t^2 \mathcal{Z}^\alpha + X^\alpha_\gamma \partial_t \mathcal{Z}^\gamma + Y^\alpha_\gamma \mathcal{Z}^\gamma + L^\alpha = 0, \quad \alpha = \overline{1, 2n}, \tag{53}$$

where

$$X^\alpha_\gamma = (\mathcal{T}^\alpha_{\beta\gamma} + 2\Gamma^\alpha_{\gamma\beta}) \mathcal{V}^\beta, \qquad L^\alpha = -\delta_u \mathcal{F}^\alpha$$

and

$$\begin{aligned} Y^\alpha_\gamma \ &= \delta_t [(\mathcal{T}^\alpha_{\beta\gamma} + \Gamma^\alpha_{\gamma\beta}) \mathcal{V}^\beta] + (\mathcal{T}^\alpha_{\sigma\beta} + \Gamma^\alpha_{\beta\sigma}) \Gamma^\beta_{\gamma\mu} \mathcal{V}^\sigma \mathcal{V}^\mu - \\ &\qquad - \mathcal{R}_{\beta\ \gamma\lambda}^{\ \alpha} \mathcal{V}^\beta \mathcal{V}^\lambda, \quad \alpha, \gamma = \overline{1, 2n}. \end{aligned}$$

Then, denoting $\{\xi^A\}_{A=\overline{1,4n}} = \{\mathcal{Z}^\alpha, \partial_t \mathcal{Z}^\beta\}$ and $\{L^A\}_{A=\overline{1,4n}} = \{0, \partial_u \mathcal{F}^\alpha\}$ as $4n-$column vectors, and considering the $2n \times 2n-$matrices X and Y of entries X^α_γ and Y^α_γ respectively, the system (53) rewrites in matrix form

$$\partial_t \xi + P\xi + L = 0 \qquad \Leftrightarrow \qquad \partial_t \xi^A + P^A_B \xi^B + L^A = 0, \quad A = \overline{1, 4n}, \tag{54}$$

where $P = \begin{bmatrix} 0 & -I_{2n} \\ Y & X \end{bmatrix}$. Denoting by $\{\xi^A(0)\} = \{\mathcal{Z}^\alpha(0), \partial_t \mathcal{Z}^\beta(0)\}$ the initial condition column $4n-$vector, we define the subsequent vectors

$$Q^{A,m} = \{Q_1^{\alpha,m}, Q_2^{\beta,m}\}, \quad m \in \mathbb{N}$$

inductively as follows: $Q^{A,0} = \xi^A(0)$, i.e., $Q_1^{\alpha 0} = \mathcal{Z}^\alpha(0)$, $Q_2^{\beta 0} = \partial_t \mathcal{Z}^\beta(0)$, and

$$Q^{A,m+1} = \partial_t Q^{A,m} + P_B^A Q^{B,m} + L^A, \quad A = \overline{1,4n}, \;\; m \in \mathbb{N},$$

i.e.

$$\begin{cases} Q_1^{\alpha,m+1} = \partial_t Q_1^{\alpha,m} - Q_2^{\alpha,m}, \\ Q_2^{\beta,m+1} = \partial_t Q_2^{\beta,m} + Y_\gamma^\beta Q_1^{\gamma,m} + X_\gamma^\beta Q_2^{\gamma,m} - \partial_t F^\beta, \quad \alpha,\beta \in \overline{1,2n}, \;\; m \in \mathbb{N}. \end{cases}$$

Applying the procedure described in [26] for systems of nonhomogeneous PDE in the special case when the number of variables is 1, we finally obtain

Theorem 5. *The analytic solution of the system of PDEs (54) is given by*

$$\xi^A = \sum_{m=0}^{\infty} \frac{(-t)^m}{m!} Q^{A,m}, \quad A = \overline{1,4n}$$

and as consequence the solution of (52) is

$$\mathcal{Z}^\alpha = \sum_{m=0}^{\infty} \frac{(-t)^m}{m!} Q_1^{\alpha,m}, \quad \alpha = \overline{1,2n}.$$

Remark also that the equations (52) split

$$\delta_t^2 \mathcal{Z}^i + \delta_t \mathcal{T}^i = \rho^i + \delta_u \mathcal{F}^i, \qquad \delta_t^2 \mathcal{Z}^a + \delta_t \mathcal{T}^a = \rho^a + \delta_u \mathcal{F}^a, \tag{55}$$

and hence for *paths* these considerably simplify. E.g., if c is an h-path, (55) become

$$\delta_t^2 \mathcal{Z}^i + \delta_t \mathcal{T}^i = \rho^i, \qquad \delta_t^2 \mathcal{Z}^a + \delta_t \mathcal{T}^a = 0. \tag{56}$$

The equations of deviations of paths presented above are particular cases of the ones in ([4], [1], [5], [6]), of the extended Finslerian case developed in [19]. Alternatively, the study of deviation of geodesics for the Finslerian case $n+\varepsilon^{(1)}+\varepsilon^{(2)}$ was performed in [21], [23].

Conclusions. The weak pseudo-Riemannian gravitational model was extended by considering a Beil–type deformation of the weak pseudo-Riemannian metric γ_{ij} of the 4-dimensional base space M which provides an $(h,v)-$metric on TM. The considered model fits in the general theory of $(h,v)-$metric structures on vector bundles developed in [14], [16], [17], [5], [4]. In this framework, the explicit Einstein equations and the equations of stationary curves and of their deviations were determined for the canonic linear $N-$connection, with the Berwald-type nonlinear connection N considered in linearized approach. The ability of revelation of the gravitational waves in these spaces is possible under the study of the weak field, since the wave vectors of the weak field theory are intrinsically incorporated in spaces in which the elements depend on the position and the direction.

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