SOME PROPERTIES OF QUASICONFORMAL MAPPINGS IN RIEMANNIAN MANIFOLDS

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Abstract

In this note we give some geometrical and analytical properties which caracterise the quasiconformal maps in Riemannian manifolds.

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We denote by M a n-dimensional, connected, paracompact, C^∞- differentiable manifold, with the metric tensor g, the geodesic metric d and Lebesgue measure τ.

If Γ is a family of curves on M, we define the p- modulus of Γ, $M_p(\Gamma)$, by

$$M_p(\Gamma) = \begin{cases} \inf\{\int_M \rho^p d\tau, \ \ \rho \in \mathcal{F}(\Gamma)\}, \ \text{for } \ \mathcal{F}(\Gamma) \neq \emptyset \quad \infty, \ \text{for } \ \mathcal{F}(\Gamma) = \emptyset \\ \infty, \ \text{for } \ \mathcal{F}(\Gamma) = \emptyset \end{cases},$$

where

$$\mathcal{F}(\Gamma) = \{\rho/\rho : M \to I\!R_+, \ \ \rho \ \text{ is Borel measurable and} \\ \int_\gamma \rho \, ds \geq 1, \ \text{ for every locally rectifiable arc } \ \gamma \in \Gamma\},$$

$$\int_\gamma \rho \, ds \geq 1, \ \text{ for every locally rectifiable arc } \ \gamma \in \Gamma\}.$$

We shall drop the index p if $p = n$. Clearly, $M_p(\Gamma) = M_p(\Gamma_0)$, where $\Gamma_0 \subset \Gamma$ is the family of all locally rectifiable curves.

On prove that M_p is an outer measure in the space of all curves on M.

We consider the geodesic ball $\mathcal{B}(x, r)$ where $\exp_x^{-1}$ is $(1+\varepsilon)$-isometry and $W_x = \mathcal{B}(x, r) \cap D$, $U_x = \exp_x^{-1}(W_x)$, D, $\tilde{D}$-domains in M.

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For a homeomorphism $f : D \to \tilde{D}$ we define *the maximal dilatation* of $F_x = \exp^{-1}_{f(x)} \circ f \circ \exp_x$, by

$$K(F_x) = max\left\{\sup_{\{\Gamma'\}} \frac{M(\Gamma')}{M(F_x(\Gamma'))},\ \sup_{\{\Gamma'\}} \frac{M(F_x(\Gamma'))}{M(\Gamma')}\right\},$$

where $\{\Gamma'\}$ is the set of all families of curves $\Gamma' \subset U_x$, for which $M_p(\Gamma')$ and $M_p(F_x(\Gamma'))$ are not simultaneously 0 or ∞.

A map $h : E \subset I\!R \to M$ is called *absolutely continuous (AC)* on E, if for any $\varepsilon > 0$, there exists $\delta > 0$ such that, for every sequence of non-overlapping intervals $\{[a_n, b_n]\}$ whose end-points $a_n, b_n \in E$, the inequality $\sum_n (b_n - a_n) < \delta$, implies that $\sum_n d(f(b_n), f(a_n)) < \varepsilon$.

A map $f : D \subset M \to M$ is said to be AC along of $\gamma : I \subset I\!R \to D$, γ-rectifiable, if $f \circ \gamma$ is AC. The map $f : D \to M$ is called *absolutely continuous on arcs (ACA)*, if $M(\Gamma) = 0$, where Γ is the family of all locally rectifiable arcs in D, which contain a compact subarc on which f is not AC.

For a homeomorphism $f : D \to \tilde{D}$ we define the *generalized Jacobian* of f, by

$$J_f^G = \limsup_{r\to 0} \frac{\tau[f(\overline{\mathcal{B}}(x,r))]}{\tau(\overline{\mathcal{B}}(x,r))}.$$

If f is differentiable at x, then $J_f^G(x) = \mid J_f(x) \mid$.

The function $D^+f,\ D^-f : M \to [0,\infty]$ defined by

$$D^+f(x) = \limsup_{y\to x} \frac{d(f(x),f(y))}{d(x,y)},\quad D^-f(x) = \liminf_{y\to x} \frac{d(f(x),f(y))}{d(x,y)},$$

are called *the scalar derivatives* (upper, respectively lower) of f. We have:

$$D^+f(x) = D^+F_x(0_x),\quad D^-f(x) = D^-F_x(0_x).$$

If f is differentiable at x, then $D^+f(x) = \|T_xf\|$ and $D^-f(x) = \inf\{\|(T_xf)X\|/\|X\| = 1\}$.

Lemma 1. *If $T : I\!R^n \to I\!R^n$ is a selfadjoint positive definite operator, then $detT \le \|T\|^n$.*

Theorem 1. *Let $f : D \to \tilde{D}$ be a homeomorfism. If:*

$$\sup_{x\in D} K(F_x) < \infty \tag{1}$$

$$\exists K \ge 1 \text{ such that } K(F_x) \le (1+\varepsilon)K \text{ a. e. in } D, \text{ for every } \varepsilon > 0, \tag{2}$$

then we have :

$$\limsup_{r\to 0} \frac{\tau[f(\exp_x(\overline{B}(Y,r)))]}{\tau(\exp_x(\overline{B}(Y,r)))} = J_f^G(y) \tag{3}$$

$$\limsup_{r\to 0} \frac{m[\exp_{f(x)}^{-1}(f(\exp_x(\overline{B}(Y,r))))]}{\tau(f(\exp_x(\overline{B}(Y,r))))} = J^G_{exp^{-1}_{f(x)}}(f(y)) \tag{4}$$

where $Y = \exp_x^{-1}(y)$ *and* m *is Lebesgue measure on* $T_{f(x)}M$*.*

Lemma 2. *If* $f : M \to I\!\!R^n$ *is a differentiable* C*-isometry (i.e.* $\frac{1}{C}d(x,y) \leq \|f(x) - f(y)\| \leq Cd(x,y),\ \ \forall x, y \in M$*), then*

$$J_f \leq C^n, \ \text{ where }\ J_f(x) = \mid det\ DF_x(0_x) \mid .$$

Proof. For $F_x = f \circ exp_x : T_xM \to I\!\!R^n$, we have:

$$\|f(x) - f(y)\| = \|F_x(\exp_x^{-1} x) - F_x(\exp_x^{-1} y)\|, \ \ \exp_x^{-1} x = 0_x, \quad d(x,y) = \|\exp_x^{-1} y\|,$$

$$d(x,y) = \|\exp_x^{-1} y\|$$

and so

$$\frac{1}{C} \leq \frac{\|F_x(\exp_x^{-1} y) - F_x(0_x)\|}{\|\exp_x^{-1} y\|} \leq C.$$

It results

$$\begin{aligned}\|DF_x(0_x)\| = \sup\left\{\tfrac{\|F_x(0_x)Y\|}{\|Y\|}/Y \in T_xM - \{0_x\}\right\} = D^+F_x(0_x) = \\ = \limsup_{Y\to 0_x}\frac{\|F_x(Y) - F_x(0_x)\|}{\|Y\|} \leq C \\ = \limsup_{Y\to 0_x}\frac{\|F_x(Y) - F_x(0_x)\|}{\|Y\|} \leq C.\end{aligned}$$

Let us consider the selfadjoint positive definite operator $G = [DF_x(0_x)]^* \cdot DF_x(0_x)$, where $[DF_x(0_x)]^*$ is the adjoint of $DF_x(0_x)$. By using Lemma 1, we obtain

$$det\ G = (det\ DF_x(0_x))^2 = J_f^2(x) \leq \|G\|^n = \|DF_x(0_x)\|^{2n} \leq C^{2n}.$$

Lemma 3. *If* $f : M \to I\!\!R^n$ *is a differentiable* C*-isometry, then*

$$C^{-2n}M(\Gamma) \leq M(\tilde{\Gamma}) \leq C^{2n}M(\Gamma) \tag{5}$$

for every family of curves Γ *in* M*,* $\tilde{\Gamma} = f(\Gamma)$*.*

Proof. Let us consider $\rho \in \mathcal{F}(\Gamma)$ and $\tilde{\rho} = \rho \circ f^{-1}$.
Since

$$\int_{\tilde{\gamma}} C\tilde{\rho} d\tilde{s} = \int_{\gamma} C\rho\frac{d\tilde{s}}{ds}ds \geq \int_{\gamma}\rho ds \geq 1$$

it follows that $C\tilde{\rho} \in \mathcal{F}(\tilde{\Gamma})$. For every $\rho \in \mathcal{F}(\Gamma)$ we have

$$M(\tilde{\Gamma}) \leq \int_{R^n} C^n \tilde{\rho}^n dm = C^n \int_M \rho^n \frac{dm}{d\tau} d\tau \leq C^{2n} \int_M \rho^n d\tau$$

hence $M(\tilde{\Gamma}) \leq C^{2n} M(\Gamma)$.

The first half in (5) follows similarly if we change $M(\Gamma)$ and $M(\tilde{\Gamma})$ between them.

Theorem 2. *A homeomorphism $f : D \to \tilde{D}$ satisfies the conditions (1) and (2) from Theorem 1, iff*

$$f \text{ is } ACA \text{ in } D \tag{6}$$

$$K^{-1}(D^+ f(x))^n \leq J_f^G(x) \leq K(D^- f(x))^n \text{ a.e. in } D. \tag{7}$$

Proof. For the necessity we consider the family Γ of all locally rectifiable arcs in D, which contain a compact subarc on which f is not AC, and the sequence $(x_n) \subset D$ such that $\{\mathcal{B}(x_n, r)\}_{n \in I\!N}$ is a covering of D with normal charts. For every $\gamma \in \Gamma$, there exists a compact subarc $\beta \subset \gamma$ on which f is not AC, hence there exists $n \in N$ such that $W_{x_n} = D \cap \mathcal{B}(x_n, r)$ contains a compact subarc $\alpha \subset \beta$ on which f is not AC. Let Γ_0 be the family of the arcs α with this property. Since $\Gamma_0 < \Gamma$, we have that

$$M(\Gamma) \leq M(\Gamma_0). \tag{8}$$

If we denote by $\Gamma_n = \{\alpha / \alpha \in \Gamma_0,\ \alpha \subset W_{x_n}\}$, then $\Gamma_0 = \bigcup_n \Gamma_n$ and so $M(\Gamma_0) \leq \sum_n M(\Gamma_n)$.

Because F_{x_n} is ACA in $U_{x_n} = exp_x^{-1}(W_{x_n})$, it follows that $M(\Gamma'_n) = 0$, where $\Gamma'_n = \exp_x^{-1}(\Gamma_n)$.

By Lemma 3 we have that $M(\Gamma_n) = 0$ and from (8) it results $M(\Gamma) = 0$, hence f is ACA in D.

The condition (2) implies that there exists $E_1 \subset D$ with $\tau(E_1) = 0$, such that $K(F_x) \leq (1 + \varepsilon)K$ for every $x \in D - E_1$.

Let us consider $x \in D - E_1$, $Y = \exp_x^{-1} y$ and $y, y' \in W_x$.

We have:

$$(D^+ f(y))^n = \left[\limsup_{y' \to y} \frac{d(f(y), f(y'))}{d(y, y')}\right]^n =$$

$$= \left[\limsup_{y' \to y} \frac{d(\exp_{f(x)}(F_x(Y)), \exp_{f(x)}(F_x(Y'))}{\|F_x(Y) - F_x(Y')\|} \frac{\|F_x(Y) - F_x(Y')\|}{\|Y - Y'\|} \frac{\|Y - Y'\|}{d(y, y'}\right]^n \leq$$

$$= \left[\limsup_{y' \to y} \frac{d(\exp_{f(x)}(F_x(Y)), \exp_{f(x)}(F_x(Y'))}{\|F_x(Y) - F_x(Y')\|} \frac{\|F_x(Y) - F_x(Y')\|}{\|Y - Y'\|} \frac{\|Y - Y'\|}{d(y, y'}\right]^n \leq$$

$$\leq (1 + \varepsilon)^{2n} (D^+ F_x(Y))^n \leq (1 + \varepsilon)^{2n+1} K J^G_{F_x}(Y)$$

$$\leq (1 + \varepsilon)^{2n} (D^+ F_x(Y))^n \leq (1 + \varepsilon)^{2n+1} K J^G_{F_x}(Y)$$

a.e. in U_x. By using Theorem 1, we obtain:

$$J^G_{F_x}(Y) = \limsup_{r \to 0} \frac{m(F_x(\overline{B}(Y, r)))}{m(\overline{B}(Y, r))} =$$

$$= \limsup_{r \to 0} \frac{m(\exp^{-1}_{f(x)}(f(\exp_x(\overline{B}(Y, r)))))}{\tau(f(\exp_x(\overline{B}(Y, r))))} \frac{\tau(f(\exp_x(\overline{B}(Y, r)))}{\tau(\exp_x(\overline{B}(Y, r)))} \frac{\tau(\exp_x(\overline{B}(Y, r)))}{m(\overline{B}(Y, r))}$$
$$\leq (1 + \varepsilon)^{2n} J^G_f(y) \text{ in } W_x.$$

$$= \limsup_{r \to 0} \frac{m(\exp^{-1}_{f(x)}(f(\exp_x(\overline{B}(Y, r)))))}{\tau(f(\exp_x(\overline{B}(Y, r))))} \frac{\tau(f(\exp_x(\overline{B}(Y, r)))}{\tau(\exp_x(\overline{B}(Y, r)))} \frac{\tau(\exp_x(\overline{B}(Y, r)))}{m(\overline{B}(Y, r))}$$
$$\leq (1 + \varepsilon)^{2n} J^G_f(y) \text{ in } W_x.$$

It results

$$(D^+(f(x))^n \leq (1 + \varepsilon)^{4n+1} K J^G_f(x) \text{ a.e. in } W_x. \tag{9}$$

Because ε was arbitrarily and $\{\mathcal{B}(x_n, r)\}$ is a countable covering, it follows that the left side in (7) is true a.e. in D. Analogously we obtain the right side in (7).

For the sufficiency, let Γ' be the family of all locally rectifiable arcs in U_x, which contain a compact subarc on which F_x is not AC, and $\Gamma = \exp_x(\Gamma')$. We have $M(\Gamma) = 0$ and by Lemma 3, $M(\Gamma') = 0$, hence F_x is ACA in U_x. By using (7) it results:

$$D^+ F_x(Y))^n \leq (1 + \varepsilon)^{2n} (D^+(f(y))^n \leq (1 + \varepsilon)^{2n} K J^G_f(y) \leq$$
$$(1 + \varepsilon)^{4n} K J^G_{F_x}(Y) \text{ a.e. in } U_x.$$

Analogously, we obtain:

$$(D^- F_x(Y))^n \geq \frac{(D^- f(x))^n}{(1 + \varepsilon)^{2n}} \geq \frac{J^G_f(y)}{K(1 + \varepsilon)^{2n}} \geq \frac{1}{K(1 + \varepsilon)^{4n}} J^G_{F_x}(Y)$$

a.e. in U_x. It follows that

$$\sup_{x \in D} K(F_x) < \infty \text{ and } K(F_x) \leq (1 + \varepsilon)^{4n} K \text{ a.e. in } D.$$

Theorem 3. *A homeomorphism $f : D \to \tilde{D}$ satisfies the conditions (1) and (2) from Theorem 1, iff*

$$K^{-1}M(\Gamma) \le M(\tilde{\Gamma}) \le K(M(\Gamma)) \tag{10}$$

for every family of curves $\Gamma \subset D$, $\tilde{\Gamma} = f(\Gamma)$.

Proof. Suppose that (10) holds for every family of arcs Γ in D. By Lemma 3 and (7), for $\varepsilon > 0$ fixed, we have:

$$M(\Gamma_1') \le (1+\varepsilon)^{2n} M(\tilde{\Gamma}) \le K(1+\varepsilon)^{2n} M(\Gamma) \le K(1+\varepsilon)^{4n} M(\Gamma_1),$$

$$M(\Gamma_1') \ge \frac{M(\tilde{\Gamma})}{(1+\varepsilon)^{2n}} \ge \frac{M(\Gamma)}{K(1+\varepsilon)^{2n}} \ge \frac{M(\Gamma_1)}{K(1+\varepsilon)^{4n}}$$

for each arc family $\Gamma_1 \subset U_x$, where $\Gamma_1' = F_x(\Gamma_1)$.

It follows that F_x is qc and its maximal dilatation $K(F_x)$ satisfies:

$$\sup_{x\in D} K(F_x) < \infty \ \text{ and } \ K(F_x) \le (1+\varepsilon)K \ \text{ a.e. in } \ D.$$

Conversely, we suppose that f satisfies the conditions (1) and (2). Let Γ be the family of all arcs in D, Γ_1 the family of arcs in Γ which are locally rectifiable, and Γ_2 the family of arcs in Γ_1 such that, on each compact subarc, f is AC. We have:

$$M(\Gamma) = M(\Gamma_1) = M(\Gamma_2).$$

Let us consider $\tilde{\rho} \in \mathcal{F}(\tilde{\Gamma})$, $\tilde{\Gamma} = f(\Gamma)$ and we define the function $\rho(x) = (\tilde{\rho} \circ f)(x)$. $D^+ f(x)$, for all $x \in D$ and $\gamma \in \Gamma_2$. We prove that $\rho \in \mathcal{F}(\Gamma_2)$.

If β is any compact subarc of $\gamma \in \Gamma_2$, then β is rectifiable. Since f is AC along β, it follows that $\tilde{\beta} = f \circ \beta$ is rectifiable. We represent β and $\tilde{\beta}$ by means of their arc-length s, t:

$$(\beta) \quad x = x(s), \ \ 0 \le s \le 1; \quad (\tilde{\beta}) \ \ x = x(t), \ \ 0 \le t \le 1$$

and choose the orientation so that t is an increasing function of s. Since $\tilde{\beta}$ is rectifiable we have $\dfrac{dt}{ds} = D(f \circ \beta)(s)$ a.e. in D (see Th. 4.9 [3]).

In a point of differentiability $\tilde{x_0} = (f \circ \beta)(s_0)$ and using the normal chart in $\tilde{x}_0$, we obtain:

$$D(f\circ\beta)(s_0) = \lim_{s\to s_0} \frac{(\exp_{\tilde{x}}^{-1} \circ f \circ \beta)(s) - (\exp_{\tilde{x}_0}^{-1} \circ f \circ \beta)(s_0)}{s - s_0} =$$

$$= \lim_{s\to s_0} \frac{d((f\circ\beta)(s), (f\circ\beta)(s_0))}{d(\beta(s),\beta(s_0))} \cdot \frac{d(\beta(s),\beta(s_0))}{s-s_0} \le D^+ f(x_0)$$

$$= \lim_{s\to s_0} \frac{d((f\circ\beta)(s), (f\circ\beta)(s_0))}{d(\beta(s),\beta(s_0))} \cdot \frac{d(\beta(s),\beta(s_0))}{s-s_0} \le D^+ f(x_0).$$

Hence, $\dfrac{dt}{ds}(s_0) \le D^+ f(x_0)$, and

$$\int_\gamma \rho ds \ge \int_\beta \rho ds = \int_\beta (\tilde\rho \circ f) D^+ f ds \ge \int_\beta (\tilde\rho \circ f)\frac{dt}{ds} ds = \int_{\tilde\beta} \tilde\rho dt.$$

Since this inequality holds for all such β, it follows that

$$\int_\gamma \rho ds \ge \sup_{\tilde\beta} \int_{\tilde\beta} \tilde\rho dt = \int_{\tilde\gamma} \tilde\rho dt \ge 1.$$

Because ρ is Borel measurable, we conclude that $\rho \in \mathcal{F}(\Gamma_2)$. We have:

$$M(\Gamma_2) \le \int_D \rho^n d\tau = \int_D (\tilde\rho \circ f)^n (D^+ f)^n d\tau \le K \int_D (\tilde\rho \circ f)^n J_f^G d\tau \le K \int_{\tilde D} \tilde\rho d\tau$$

for any $\tilde\rho \in \mathcal{F}(\tilde\Gamma)$, hence $M(\Gamma_2) \le K M(\tilde\Gamma)$.
The same argument applied to f^{-1} yields the right-hand side of (10).

Definition. A homeomorphism $f : D \to \tilde D$ is said to be K-*quasiconformal* ($K - qc$), $1 \le K < \infty$, in D if
(i) δ_f is bounded in D
(ii) $\delta_f(x) \le K$ a.e. in D,
where

$$\delta_f(x) = \limsup_{r\to 0} \frac{\sup\{d(f(x), f(y))/d(x,y) = r\}}{\inf\{d(f(x), f(y))/d(x,y) = r\}}.$$

Theorem 4. *A homeomorphism $f : D \to \tilde D$ is $K^{2/n} - qc$ in D, iff for every family of curves $\Gamma \subset D$, the inequality (10) holds.*

For the proof see [1].

By using Theorems 3 and 4, we obtain
Theorem 5. *A homeomorphism $f : D \to \tilde D$ is $K^{2/n} - qc$ in D, iff f satisfies the conditions (1) and (2) of Theorem 1.*
From Theorems 2, 3 and 4 it follows

Theorem 6. *A homeomorphism $f : D \to \tilde D$ is $K^{2/n} - qc$ in D, iff f satisfies the conditions (6) and (7) of Theorem 2.*

References

[1] Borcea, V.T., Neagu, A., *The deformation of the angles by quasiconformal mappings*, Algebras Groups and Geometries, Hadronic Press, Vol. 16, Nr. 1 (1999).

[2] Caraman, P., n-Dimensional quasiconformal mappings, Ed. Acad., Bucureş ti, Romania and Abacus Press Tunbridge Wells, Kent, England, 1974.

[3] Suominen, K., Quasiconformal maps in manifolds. Ann. Acad. Sci. Fenn., **393** (1966) 5-39.

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