SOME RELATIONS IN Osc^3M

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Abstract

R. Miron and Gh. Atanasiu in [15], [16], [17] studied the geometry of Osc^kM. Among many various problems which was solved, they introduced the adapted basis, the d-connection and gave its curvature theory. Different structures as almost product structure, metric structure was determined.

Here the attention on $E = Osc^3M$ will be restricted, but in this space some generalizations will be made.

1. With respect to the adapted basis the decomposition of $T(Osc^3M)$ and the integrability conditions will be given.
2. Instead of d-connection the generalized connection will be defined and its torsion and curvature tensor will be determined.
3. The Bianchi equations will be established.

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1 Adapted basis in $T(Osc^3M)$ and $T^*(Osc^3M)$

Let $E = Osc^3M$ be a $4n$ dimensional C^∞ manifold. In some local chart (U, φ) some point $u \in E$ has coordinates

$$(x^a, y^{1a}, y^{2a}, y^{3a}) = (y^{0a}, y^{1a}, y^{2a}, y^{3a}) = (y^{\alpha a}),$$

where $x^a = y^{0a}$ and

$$a, b, c, d, e, \ldots = 1, 2, \ldots, n, \quad \alpha, \beta, \gamma, \delta, \kappa, \ldots = 0, 1, 2, 3.$$

If in some other chart (U', φ') the point $u \in E$ has coordinates $(x^{a'}, y^{1a'}, y^{2a'}, y^{3a'})$, then in $U \cap U'$ the allowable coordinate transformation are given by:

$$(1) \qquad \text{(a)} \quad x^{a'} = x^{a'}(x^1, x^2, \ldots, x^n)$$

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$$\text{(b)}\quad y^{1a'} = \frac{\partial x^{a'}}{\partial x^{a}} y^{1a} = \frac{\partial y^{0a'}}{\partial y^{0a}} y^{1a}$$

$$\text{(c)}\quad 2y^{2a'} = \frac{\partial y^{1a'}}{\partial y^{0a}} y^{1a} + 2\frac{\partial y^{1a'}}{\partial y^{1a}} y^{2a}$$

$$\text{(d)}\quad 3y^{3a'} = \frac{\partial y^{2a'}}{\partial y^{0a}} y^{1a} + 2\frac{\partial y^{2a'}}{\partial y^{1a}} y^{2a} + 3\frac{\partial y^{2a'}}{\partial y^{2a}} y^{3a}.$$

Determination of the group of allowable coordinate transformations is the first step to construct some geometry. The second important step is the construction of the adapted basis in $T(E)$, which depends on the choice of the coefficients of the nonlinear connections, here denoted by N and M.

The following abbreviations

$$\partial_{\alpha a} = \frac{\partial}{\partial y^{\alpha a}},\ \alpha = 1,2,3, \text{ and } \partial_a = \partial_{0a} = \frac{\partial}{\partial x^a} = \frac{\partial}{\partial y^{0a}} \tag{2}$$

will be used.

The natural basis $\bar{B}$ of $T(E)$ is

$$\bar{B} = \{\partial_{0a}, \partial_{1a}, \partial_{2a}, \partial_{3a}\} = \{\partial_{\alpha a}\}. \tag{3}$$

The elements of $\bar{B}$ with respect to (1.1) are not transformed as d-tensors.

The natural basis $\bar{B}^*$ of $T^*(E)$ is

$$\bar{B}^* = \{dx^a, dy^{1a}, dy^{2a}, dy^{3a}\} = \{dy^{\alpha a}\}. \tag{4}$$

The adapted basis B^* of $T^*(E)$ is given by (as in [19])

$$B^* = \{\delta y^{0a}, \delta y^{1a}, \delta y^{2a}, \delta y^{3a}\}, \tag{5}$$

where

$$\begin{aligned} \delta y^{0a} &= dx^a = dy^{0a} \\ \delta y^{1a} &= dy^{1a} + M^{(1)a}_{\ \ b} dy^{0b} \\ \delta y^{2a} &= dy^{2a} + M^{(1)a}_{\ \ b} dy^{1b} + M^{(2)a}_{\ \ b} dy^{0b} \\ \delta y^{3a} &= dy^{3a} + M^{(1)a}_{\ \ b} dy^{2b} + M^{(2)a}_{\ \ b} dy^{1b} + M^{(3)a}_{\ \ b} dy^{0b} \end{aligned} \tag{6}$$

Theorem 1.1. *The necessary and sufficient conditions that $\delta y^{\alpha a}$ are transformed as d-tensor field, i.e.*

$$\delta y^{\alpha a'} = \frac{\partial x^{a'}}{\partial x^a} \delta y^{\alpha a},\ \alpha = 0,1,2,3,$$

are the following equations:

$$\begin{aligned} &(a)\quad M^{(1)a}_{\ \ b} \partial_a x^{a'} = M^{(1)a'}_{\ \ b'} \partial_b x^{b'} + \partial_b y^{1a'} \\ &(b)\quad M^{(2)a}_{\ \ b} \partial_a x^{a'} = M^{(2)a'}_{\ \ b'} \partial_b x^{b'} + M^{(1)a'}_{\ \ b'} \partial_b y^{1b'} + \partial_b y^{2a'} \\ &(c)\quad M^{(3)a}_{\ \ b} \partial_a x^{a'} = M^{(3)a'}_{\ \ b'} \partial_b x^{b'} + M^{(2)a'}_{\ \ b'} \partial_b y^{1b'} + M^{(1)a'}_{\ \ b'} \partial_b y^{2b'} + \partial_b y^{3a'}. \end{aligned} \tag{7}$$

From (1.7) it is obvious that we can take

$$
\begin{aligned}
M^{(1)a}_{\ \ b} &= M^{(1)a}_{\ \ b}(y^{0a}, y^{1a}),\\
M^{(2)a}_{\ \ b} &= M^{(2)a}_{\ \ b}(y^{0a}, y^{1a}, y^{2a}),\\
M^{(3)a}_{\ \ b} &= M^{(3)a}_{\ \ b}(y^{0a}, y^{1a}, y^{2a}, y^{3a}),
\end{aligned} \tag{8}
$$

From the choice of M depends the adapted basis B^* ((1.5)).

Let us denote the adapted basis of $T(E)$ by B, where

$$
B = \{\delta_{0a}, \delta_{1a}, \delta_{2a}, \delta_{3a}\} = \{\delta_{\alpha a}\}, \tag{9}
$$

and

$$
\begin{array}{lclclclcl}
\delta_{0a} &=& \partial_{0a} &-& N^{(1)b}_{\ \ a}\partial_{1b} &-& N^{(2)b}_{\ \ a}\partial_{2b} &-& N^{(3)b}_{\ \ a}\partial_{3b},\\
\delta_{1a} &=& & & \partial_{1a} &-& N^{(1)b}_{\ \ a}\partial_{2b} &-& N^{(2)b}_{\ \ a}\partial_{3b}\\
\delta_{2a} &=& & & & & \partial_{2a} &-& N^{(1)b}_{\ \ a}\partial_{3b}\\
\delta_{3a} &=& & & & & & & \partial_{3a}.
\end{array} \tag{10}
$$

Theorem 1.2. *The necessary and sufficient conditions that B ((1.9)) be dual to B^* ((1.5)), (when $\bar{B}$ ((1.3)) is dual to $\bar{B}^*$ ((1.4)) i.e.*

$$
< \delta_{\alpha a}\delta y^{\beta b} >= \delta^{\beta}_{\alpha}\delta^{b}_{a}
$$

are the following relations:

$$
\begin{aligned}
N^{(1)b}_{\ \ a} &= M^{(1)b}_{\ \ a}\\
N^{(2)b}_{\ \ a} &= M^{(2)b}_{\ \ a} - N^{(1)c}_{\ \ a}M^{(1)b}_{\ \ c} = M^{(2)b}_{\ \ a} - M^{(1)c}_{\ \ a}M^{(1)b}_{\ \ c}\\
N^{(3)b}_{\ \ a} &= M^{(3)b}_{\ \ a} - N^{(1)c}_{\ \ a}M^{(2)b}_{\ \ c} - N^{(2)c}_{\ \ a}M^{(1)b}_{\ \ c} =\\
&\quad M^{(3)b}_{\ \ a} - M^{(1)c}_{\ \ a}M^{(2)b}_{\ \ c} - M^{(2)c}_{\ \ a}M^{(1)b}_{\ \ c} + M^{(1)d}_{\ \ a}M^{(1)c}_{\ \ d}M^{(1)b}_{\ \ c}.
\end{aligned} \tag{11}
$$

From (1.10) and (1.11) it follows

Theorem 1.3. *The necessary and sufficient conditions that $\delta_{\alpha a}$ with respect to (1.1) are transformed as d-tensors, i.e.*

$$
\delta_{\alpha a'} = \frac{\partial x^a}{\partial x^{a'}}\delta_{\alpha a},\ \alpha = 0, 1, 2, 3, \tag{12}
$$

are the following formulae:

$$
\begin{aligned}
N^{(1)b'}_{\ \ a'}\partial_a x^{a'} &= N^{(1)b}_{\ \ a}\partial_b x^{b'} - \partial_a y^{1b'}\\
N^{(2)b'}_{\ \ a'}\partial_a x^{a'} &= N^{(2)b}_{\ \ a}\partial_b x^{b'} + N^{(1)b}_{\ \ a}\partial_b y^{1b'} - \partial_a y^{2b'}\\
N^{(3)b'}_{\ \ a'}\partial_a x^{a'} &= N^{(3)b}_{\ \ a}\partial_b x^{b'} + N^{(2)b}_{\ \ a}\partial_b y^{1b'} + N^{(1)b}_{\ \ a}\partial_b y^{2b'} - \partial_a y^{3b'}.
\end{aligned} \tag{13}
$$

From (1.10) and (1.11) it follows

$$\begin{aligned}(14)\qquad &\partial_{3c} = \delta_{3c}\\ &\partial_{2c} = \delta_{2c} + M^{(1)d}_{\ \ c}\delta_{3d}\\ &\partial_{1c} = \delta_{1c} + M^{(1)d}_{\ \ c}\delta_{2d} + M^{(2)d}_{\ \ c}\delta_{3d}\\ &\partial_{0c} = \delta_{0c} + M^{(1)d}_{\ \ c}\delta_{1d} + M^{(2)d}_{\ \ c}\delta_{2d} + M^{(3)d}_{\ \ c}\delta_{3d}.\end{aligned}$$

Theorem 1.4. *The relations (1.11) which determine $N^{(1)}$, $N^{(2)}$, $N^{(3)}$ as functions of $M^{(1)}$, $M^{(2)}$, $M^{(3)}$ don't depend on coordinate system, i.e. (1.11) is valid if a, b, c, d is substituted by a', b', c', d' respectively.*

Proof. If we substitute $N^{(1)}$, $N^{(2)}$, $N^{(3)}$ from (1.11) into (1.13), further $M^{(1)}$, $M^{(2)}$, $M^{(3)}$ from (1.7) into such obtained formulae, after long calculation we get relations, which can be obtained from (1.11) if the indices a, b, c, d are substituted by a', b', c', d' respectively.

From (1.8) and (1.13) it follows

$$\begin{aligned}(15)\qquad &N^{(1)a}_{\ \ b} = N^{(1)a}_{\ \ b}(y^{0a}, y^{1a})\\ &N^{(2)a}_{\ \ b} = N^{(2)a}_{\ \ b}(y^{0a}, y^{1a}, y^{2a})\\ &N^{(3)a}_{\ \ b} = N^{(3)a}_{\ \ b}(y^{0a}, y^{1a}, y^{2a}, y^{3a}).\end{aligned}$$

The above formulae will be important at determination of integrability conditions.

In [5] the relations (1.8) and (1.15) were not taken into account. These formulae are restrictions on $M^{(\alpha)}$ and $N^{(\alpha)}$ $\alpha = 1, 2, 3$,but even in this case they satisfy all relations which appear by construction of adapted bases in $T(E)$ and $T^*(E)$.

2 Decomposition of $T(E)$. Integrability conditions

Let us denote by T_H, T_{V_1}, T_{V_2}, T_{V_3} the subspaces of $T(E)$ spanned by

$$\{\delta_{0a}\},\ \{\delta_{1a}\},\ \{\delta_{2a}\},\ \{\delta_{3a}\}$$

respectively. Then we have

$$T(E) = T_H \oplus T_{V_1} \oplus T_{V_2} \oplus T_{V_3}.$$

Proposition 2.1. *The horizontal distribution T_H is integrable if all $K_{0a\ 0b}^{\ \ \alpha c}$, $\alpha = 1, 2, 3$ determined by (2.2) and (2.4) are equal to zero.*

Proof. By direct calculation one obtains

$$(1)\qquad [\delta_{0a}, \delta_{0b}] = \bar{K}_{0a\ 0b}^{\ \ 1c}\partial_{1c} + \bar{K}_{0a\ 0b}^{\ \ 2c}\partial_{2c} + \bar{K}_{0a\ 0b}^{\ \ 3c}\partial_{3c},$$

where

$$(2)\qquad \bar{K}_{0a\ 0b}^{\ \ \alpha c} = \delta_{0b}N^{(\alpha)c}_{\ \ a} - \delta_{0a}N^{(\alpha)c}_{\ \ b},\ \alpha = 1, 2, 3.$$

The substitution of ∂_{1c}, ∂_{2c}, ∂_{3c} from (1.14) into (2.1) yields

(3) $$[\delta_{0a}, \delta_{0b}] = K_{0a}{}^{1c}{}_{0b}\delta_{1c} + K_{0a}{}^{2c}{}_{0b}\delta_{2c} + K_{0a}{}^{3c}{}_{0b}\delta_{3c},$$

where

(4) $$\begin{aligned} K_{0a}{}^{1c}{}_{0b} &= \bar{K}_{0a}{}^{1c}{}_{0b} \\ K_{0a}{}^{2c}{}_{0b} &= \bar{K}_{0a}{}^{2c}{}_{0b} + \bar{K}_{0a}{}^{1d}{}_{0b}M^{(1)c}_{\ \ d} \\ K_{0a}{}^{3c}{}_{0b} &= \bar{K}_{0a}{}^{3c}{}_{0b} + \bar{K}_{0a}{}^{2d}{}_{0b}M^{(1)c}_{\ \ d} + \bar{K}_{0a}{}^{1d}{}_{0b}M^{(2)c}_{\ \ d}. \end{aligned}$$

Proposition 2.2. *The distribution T_{V_1} is integrable if $K_{1a}{}^{\alpha c}{}_{1b}$ $\alpha = 2,3$ determined by (2.6) and (2.8) are equal to zero.*

Proof. We have

(5) $$[\delta_{1a}, \delta_{1b}] = \bar{K}_{1a}{}^{2c}{}_{1b}\partial_{2c} + \bar{K}_{1a}{}^{3c}{}_{1b}\partial_{3c},$$

where

(6) $$\begin{aligned} \bar{K}_{1a}{}^{2c}{}_{1b} &= \delta_{1b}N^{(1)c}_{\ \ a} - \delta_{1a}N^{(1)c}_{\ \ b} = \partial_{1b}N^{(1)c}_{\ \ a} - \partial_{1a}N^{(1)c}_{\ \ b} \\ \bar{K}_{1a}{}^{3c}{}_{1b} &= \delta_{1b}N^{(2)c}_{\ \ a} - \delta_{1a}N^{(2)c}_{\ \ b} = (\partial_{1b} - N^{(1)d}_{\ \ b}\partial_{2d})N^{(2)c}_{\ \ a} - (a/b). \end{aligned}$$

In (2.6) formulae (1.14) were used.

Using (1.15), (2.5) takes the form:

(7) $$[\delta_{1a}, \delta_{1b}] = K_{1a}{}^{2c}{}_{1b}\delta_{2c} + K_{1a}{}^{3c}{}_{1b}\delta_{3c},$$

where

(8) $$\begin{aligned} K_{1a}{}^{2c}{}_{1b} &= \bar{K}_{1a}{}^{2c}{}_{1b} \\ K_{1a}{}^{3c}{}_{1b} &= \bar{K}_{1a}{}^{3c}{}_{1b} + \bar{K}_{1a}{}^{2d}{}_{1b}M^{(1)c}_{\ \ d}. \end{aligned}$$

Proposition 2.3. *The distribution T_{V_2} is integrable.*

Proof. From (1.10) and (1.15) it follows

$$[\delta_{2a}, \delta_{2b}] = K_{2a}{}^{3c}{}_{2b}\delta_{3c} = (\delta_{2b}N^{(1)c}_{\ \ a} - \delta_{2a}N^{(1)c}_{\ \ b})\delta_{3c} = 0,$$

i.e.

(9) $$[\delta_{2a}, \delta_{2b}] = 0.$$

Proposition 2.4. *The distribution T_{V_3} is integrable.*

Proof.

(10) $$[\delta_{3a}, \delta_{3b}] = 0.$$

Proposition 2.5. *The following formulae are valid:*

$$[\delta_{0a},\delta_{1b}] = K_{0a}{}^{1c}{}_{1b}\delta_{1c} + K_{0a}{}^{2c}{}_{1b}\delta_{2c} + K_{0a}{}^{3c}{}_{1b}\delta_{3c}, \tag{11}$$

where

$$\begin{aligned} K_{0a}{}^{1c}{}_{1b} &= \bar{K}_{0a}{}^{1c}{}_{1b}, \\ K_{0a}{}^{2c}{}_{1b} &= \bar{K}_{0a}{}^{2c}{}_{1b} + \bar{K}_{0a}{}^{1d}{}_{1b}M^{(1)c}{}_{d}, \\ K_{0a}{}^{3c}{}_{1b} &= \bar{K}_{0a}{}^{3c}{}_{1b} + \bar{K}_{0a}{}^{2d}{}_{1b}M^{(1)c}{}_{d} + \bar{K}_{0a}{}^{1d}{}_{1b}M^{(2)c}{}_{d}, \end{aligned} \tag{12}$$

and

$$\begin{aligned} \bar{K}_{0a}{}^{1c}{}_{1b} &= \delta_{1b}N^{(1)c}{}_{a} = \partial_{1b}N^{(1)c}{}_{a} \\ \bar{K}_{0a}{}^{2c}{}_{1b} &= \delta_{1b}N^{(2)c}{}_{a} - \delta_{0a}N^{(1)c}{}_{b} \\ \bar{K}_{0a}{}^{3c}{}_{1b} &= \delta_{1b}N^{(3)c}{}_{a} - \delta_{0a}N^{(2)c}{}_{b}. \end{aligned} \tag{13}$$

Proposition 2.6. *For* $[\delta_{0a},\delta_{2b}]$ *we have*

$$[\delta_{0a},\delta_{2b}] = K_{0a}{}^{2c}{}_{2b}\delta_{2c} + K_{0a}{}^{3c}{}_{2b}\delta_{3a}, \tag{14}$$

where

$$\begin{aligned} K_{0a}{}^{2c}{}_{2b} &= \bar{K}_{0a}{}^{2c}{}_{2b} = \partial_{2b}N^{(2)c}{}_{a}, \\ K_{0a}{}^{3c}{}_{2b} &= \bar{K}_{0a}{}^{3c}{}_{2b} + \bar{K}_{0a}{}^{2d}{}_{2b}M^{(1)c}{}_{d}, \\ \bar{K}_{0a}{}^{2c}{}_{2b} &= \delta_{2b}N^{(2)c}{}_{a} = \partial_{2b}N^{(2)c}{}_{a}, \\ \bar{K}_{0a}{}^{3c}{}_{2b} &= \delta_{2b}N^{(3)c}{}_{a} - \delta_{0a}N^{(1)c}{}_{b} = \\ &(\partial_{2b} - N^{(1)d}{}_{b}\partial_{3d})N^{(3)c}{}_{a} - (\partial_{0a} - N^{(1)d}{}_{a}\partial_{1d})N^{(1)c}{}_{b}. \end{aligned} \tag{15}$$

Proposition 2.7. *For* $[\delta_{0a},\delta_{3b}]$ *we get*

$$[\delta_{0a},\delta_{3b}] = K_{0a}{}^{3c}{}_{3b}\delta_{3c}, \tag{16}$$

where

$$K_{0a}{}^{3c}{}_{3b} = \delta_{3b}N^{(3)c}{}_{a}. \tag{17}$$

Proposition 2.8. *For* $[\delta_{1a},\delta_{2b}]$ *we have*

$$[\delta_{1a},\delta_{2b}] = K_{1a}{}^{3c}{}_{2b}\delta_{3c}, \tag{18}$$

where

$$K_{1a}{}^{3c}{}_{2b} = \delta_{2b}N^{(2)c}{}_{a} - \delta_{1a}N^{(1)c}{}_{b} = \partial_{2b}N^{(2)c}{}_{a} - \partial_{1a}N^{(1)c}{}_{a}. \tag{19}$$

Proposition 2.9. *For* $[\delta_{1a},\delta_{3b}]$ *we obtain*

$$[\delta_{1a},\delta_{3b}] = 0. \tag{20}$$

Proposition 2.10. *For* $[\delta_{2a},\delta_{3b}]$ *we get*

$$[\delta_{2a},\delta_{3b}] = 0. \tag{21}$$

3 Covariant derivatives in $T(E)$

Let $\nabla : T(E)\times T(E)\to T(E)$ be a linear connection, such that

$$\nabla : (X,Y)\to \nabla_X Y\in T(E),\ \forall X,Y\in T(E).$$

The operator ∇ is called a generalized connection. It is called d-connection if $\nabla_X Y$ is in T_H, T_{V_1}, T_{V_2}, T_{V_3} if Y is in T_H, T_{V_1}, T_{V_2}, T_{V_3} respectively. For the space Osc^kM it has been studied by R. Miron and Gh. Atanasiu in [15], [16].

Definition 3.1. *The generalized connection* ∇ *on* $T(E)$ *is defined by*

(1) $$\nabla_{\delta_{\alpha a}}\delta_{\beta b}=\Gamma^{\ \gamma c}_{\beta b\ \ \alpha a}\delta_{\gamma c}.$$

In (3.1) the summation is going over γ and c.
If Y is any vector field in $T(E)$ and

$$Y=Y^{\beta b}\delta_{\beta b},$$

then

$$\begin{aligned}\nabla_{\delta_{\alpha a}}Y &= \nabla_{\delta_{\alpha a}}(Y^{\beta b}\delta_{\beta b})=(\delta_{\alpha a}Y^{\beta b})\delta_{\beta b}+\Gamma^{\ \gamma c}_{\beta b\ \ \alpha a}Y^{\beta b}\delta_{\gamma c}=\\ &(\delta_{\alpha a}Y^{\beta b}+\Gamma^{\ \beta b}_{\gamma c\ \ \alpha a}Y^{\gamma c})\delta_{\beta b}.\end{aligned}$$

Now we define the generalized covariant derivative of vector field Y in the form

(2) $$Y^{\beta b}{}_{|\alpha a}=\delta_{\alpha a}Y^{\beta b}+\Gamma^{\ \beta b}_{\gamma c\ \ \alpha a}Y^{\gamma c}.$$

We have

(3) $$\nabla_{\delta_{\alpha a}}Y=(Y^{\beta b}{}_{|\alpha a})\delta_{\beta b}.$$

Theorem 3.1. *With respect to (1.1)* $Y^{\beta b}{}_{|\alpha a}$ *will be a d-tensor field, i.e.*

(4) $$Y^{\beta b'}{}_{|\alpha a'}=\frac{\partial x^a}{\partial x^{a'}}\frac{\partial x^{b'}}{\partial x^b}Y^{\beta b}{}_{|\alpha a}$$

if all $\Gamma^{\ \beta b}_{\gamma c\ \ \alpha a}$ *are transformed as d-tensors, i.e.*

(5) $$\Gamma^{\ \beta b}_{\gamma c\ \ \alpha a}\frac{\partial x^a}{\partial x^{a'}}\frac{\partial x^{b'}}{\partial x^b}=\Gamma^{\ \beta b'}_{\gamma c'\ \ \alpha a'}\frac{\partial x^{c'}}{\partial x^c}$$

except $\Gamma^{\ \beta b}_{\beta c\ \ 0a}$ *(no summation over* β*,* $\beta=0,1,2,3$*) which have the following transformation law*

(6) $$\Gamma^{\ \beta b}_{\beta c\ \ 0a}\frac{\partial x^a}{\partial x^{a'}}\frac{\partial x^{b'}}{\partial x^b}=\Gamma^{\ \beta b'}_{\beta c'\ \ 0a'}\frac{\partial x^{c'}}{\partial x^c}+\frac{\partial x^a}{\partial x^{a'}}\frac{\partial^2 x^{b'}}{\partial x^a\partial x^c},$$

for $\beta=0,1,2,3$.

Proof. Starting from (3.4) and using the tensor character of $\delta_{\alpha a'}$ and $Y^{\beta b'}$ we get

$$Y^{\beta b'}{}_{|\alpha a'} = \frac{\partial x^a}{\partial x^{a'}}\left[\frac{\partial x^{b'}}{\partial x^b}\delta_{\alpha a}Y^{\beta b} + Y^{\beta c}\delta_{\alpha a}\frac{\partial x^{b'}}{\partial x^c}\right] + \Gamma_{\gamma c'}{}^{\beta b'}{}_{\alpha a'}Y^{\gamma c}\frac{\partial x^{c'}}{\partial x^c} =$$

$$\frac{\partial x^a}{\partial x^{a'}}\frac{\partial x^{b'}}{\partial x^b}[\delta_{\alpha a}Y^{\beta b} + \Gamma_{\gamma c}{}^{\beta b}{}_{\alpha a}Y^{\gamma c}].$$

From the above follows:

(7) $$\Gamma_{\gamma c}{}^{\beta b}{}_{\alpha a}\frac{\partial x^a}{\partial x^{a'}}\frac{\partial x^{b'}}{\partial x^b} = \Gamma_{\gamma c'}{}^{\beta b'}{}_{\alpha a'}\frac{\partial x^{c'}}{\partial x^c} + \delta^{\beta}_{\gamma}\frac{\partial x^a}{\partial x^{a'}}\delta_{\alpha a}\frac{\partial x^{b'}}{\partial x^c}.$$

If $\beta \neq \gamma$ the last term in (3.7) vanishes.
If $\alpha \neq 0$, then $\delta_{\alpha a}\frac{\partial x^{b'}}{\partial x^c} = 0$ (see (1.10) and (1.1)(a)).
If $\alpha = 0$, then

$$\delta_{0a}\frac{\partial x^{b'}}{\partial x^c} = \frac{\partial^2 x^{b'}}{\partial x^a \partial x^c},$$

which proves (3.6).

4 The torsion tensor of generalized connection

The torsion tensor $T(X,Y)$ is defined as usual by:

(1) $$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y].$$

If X and Y expressed in the basis B have the form

(2) $$X = X^{\alpha a}\delta_{\alpha a},\ Y = Y^{\beta b}\delta_{\beta b},$$

then using linearity of ∇ and (3.1) we get

(3) $$\nabla_X Y = \nabla_{X^{\alpha a}\delta_{\alpha a}}(Y^{\beta b}\delta_{\beta b}) = X^{\alpha a}(\delta_{\alpha a}Y^{\beta b})\delta_{\beta b} + X^{\alpha a}Y^{\beta b}\Gamma_{\beta b}{}^{\gamma c}{}_{\alpha a}\delta_{\gamma c}.$$

Further we have

(4) $$[X,Y] = [X^{\alpha a}\delta_{\alpha a}, Y^{\beta b}\delta_{\beta b}] = X^{\alpha a}(\delta_{\alpha a}Y^{\beta b})\delta_{\beta b} - Y^{\beta b}(\delta_{\beta b}X^{\alpha a})\delta_{\alpha a} + X^{\alpha a}Y^{\beta b}[\delta_{\alpha a}, \delta_{\beta b}].$$

The substitution of (4.3) and (4.4) into (4.1) gives

Theorem 4.1. *The torsion tensor of generalized connection has the form*

(5) $$T(X,Y) = X^{\alpha a}Y^{\beta b}[(\Gamma_{\beta b}{}^{\gamma c}{}_{\alpha a} - \Gamma_{\alpha a}{}^{\gamma c}{}_{\beta b})\delta_{\gamma c} - [\delta_{\alpha a}, \delta_{\beta b}]] = T_{\beta b}{}^{\gamma c}{}_{\alpha a}X^{\alpha a}Y^{\beta b}\delta_{\gamma c},$$

where

(6) $$T_{\beta b}{}^{\gamma c}{}_{\alpha a} = \Gamma_{\beta b}{}^{\gamma c}{}_{\alpha a} - \Gamma_{\alpha a}{}^{\gamma c}{}_{\beta b} - K_{\alpha a}{}^{\gamma c}{}_{\beta b},$$

(7) $$[\delta_{\alpha a}, \delta_{\beta b}] = K_{\alpha a}{}^{\gamma c}{}_{\beta b}\delta_{\gamma c}.$$

From (2.1)-(2.21) it follows

$$X^{\alpha a}Y^{\beta b}[\delta_{\alpha a},\delta_{\beta b}] = \tag{8}$$

$$\begin{aligned}
X^{0a}Y^{0b}(K_{0a}{}^{1c}{}_{0b}\delta_{1c}+K_{0a}{}^{2c}{}_{0b}\delta_{2c}+K_{0a}{}^{3c}{}_{0b}\delta_{3c})+\\
(X^{0a}Y^{1b}-X^{1b}Y^{0a})(K_{0a}{}^{1c}{}_{1b}\delta_{1c}+K_{0a}{}^{2c}{}_{1b}\delta_{2c}+K_{0a}{}^{3c}{}_{1b}\delta_{3c})+\\
(X^{0a}Y^{2b}-X^{2b}Y^{0a})(K_{0a}{}^{2c}{}_{2b}\delta_{2c}+K_{0a}{}^{3c}{}_{2b}\delta_{3c})+\\
(X^{0a}Y^{3b}-X^{3b}Y^{0a})K_{0a}{}^{3c}{}_{3b}\delta_{3c}+\\
X^{1a}Y^{1b}(K_{1a}{}^{2c}{}_{1b}\delta_{2c}+K_{1a}{}^{3c}{}_{1b}\delta_{3c})+\\
(X^{1a}Y^{2b}-X^{2b}Y^{1a})K_{1a}{}^{3c}{}_{2b}\delta_{3c}.
\end{aligned}$$

As in $T_{\beta b}{}^{\gamma c}{}_{\alpha a}$ α,β,γ take values from $\{0,1,2,3\}$, so there are 4^3 different types of torsion coefficients. From (4.6) and (4.8) it is clear that 4^3-19 of them are the difference of connection coefficients, but 19 of them according to (4.8) have the additional term $-K_{\alpha a}{}^{\gamma c}{}_{\beta b}$. If it is supposed that T_H, T_{V_1}, T_{V_2}, T_{V_3} are integrable distributions, then all K's beside $X^{0a}Y^{0b}$ and $X^{1a}Y^{1b}$ are equal to zero.

5 The curvature theory of ∇

The curvature tensor for the generalized connection ∇ is defined as usual

$$R(X,Y)Z=\nabla_X\nabla_YZ-\nabla_Y\nabla_XZ-\nabla_{[X,Y]}Z. \tag{1}$$

If the notations (4.2) and $Z=Z^{\gamma c}\delta_{\gamma c}$ are used, then

$$\begin{aligned}
&\nabla_X\nabla_YZ=\nabla_{X^{\alpha a}}\delta_{\alpha a}\nabla_{Y^{\beta b}}\delta_{\beta b}Z^{\gamma c}\delta_{\gamma c}=\\
&\nabla_{X^{\alpha a}}\delta_{\alpha a}[Y^{\beta b}(\delta_{\beta b}Z^{\gamma c})\delta_{\gamma c}+Y^{\beta b}Z^{\gamma c}\Gamma_{\gamma c}{}^{\delta d}{}_{\beta b}\delta_{\delta d}]=\\
&X^{\alpha a}(\delta_{\alpha a}Y^{\beta b})(\delta_{\beta b}Z^{\gamma c})\delta_{\gamma c}+X^{\alpha a}Y^{\beta b}(\delta_{\alpha a}\delta_{\beta b}Z^{\gamma c})\delta_{\gamma c}+\\
&X^{\alpha a}Y^{\beta b}(\delta_{\beta b}Z^{\gamma c})\Gamma_{\gamma c}{}^{\delta d}{}_{\alpha a}\delta_{\delta d}+X^{\alpha a}(\delta_{\alpha a}Y^{\beta b})Z^{\gamma c}\Gamma_{\gamma c}{}^{\delta d}{}_{\beta b}\delta_{\delta d}+\\
&X^{\alpha a}Y^{\beta b}(\delta_{\alpha a}Z^{\gamma c})\Gamma_{\gamma c}{}^{\delta d}{}_{\beta b}\delta_{\delta d}+X^{\alpha a}Y^{\beta b}Z^{\gamma c}(\delta_{\alpha a}\Gamma_{\gamma c}{}^{\delta d}{}_{\beta b})\delta_{\delta d}+\\
&X^{\alpha a}Y^{\beta b}Z^{\gamma c}\Gamma_{\gamma c}{}^{\varepsilon e}{}_{\beta b}\Gamma_{\varepsilon e}{}^{\delta d}{}_{\alpha a}\delta_{\delta d}.
\end{aligned} \tag{2}$$

From (4.4) and (4.7) it follows

$$\begin{aligned}
\nabla_{[XY]}Z\ =\ &X^{\alpha a}(\delta_{\alpha a}Y^{\beta b})(\delta_{\beta b}Z^{\gamma c})\delta_{\gamma c}+X^{\alpha a}(\delta_{\alpha a}Y^{\beta b})Z^{\gamma c}\Gamma_{\gamma c}{}^{\delta d}{}_{\beta b}\delta_{\delta d}\\
&-Y^{\beta b}(\delta_{\beta b}X^{\alpha a})(\delta_{\alpha a}Z^{\gamma c})\delta_{\gamma c}-Y^{\beta b}(\delta_{\beta b}X^{\alpha a})Z^{\gamma c}\Gamma_{\gamma c}{}^{\delta d}{}_{\alpha a}\delta_{\delta d}\\
&X^{\alpha a}Y^{\beta b}[(\delta_{\alpha a}\delta_{\beta b}-\delta_{\beta b}\delta_{\alpha a})Z^{\gamma c}]\delta_{\gamma c}+\\
&X^{\alpha a}Y^{\beta b}Z^{\gamma c}K_{\alpha a}{}^{\varepsilon e}{}_{\beta b}\Gamma_{\gamma c}{}^{\delta d}{}_{\varepsilon e}\delta_{\delta d}.
\end{aligned} \tag{3}$$

Substituting (5.2) and (5.3) into (5.1) we obtain

Theorem 5.1. *In Osc^3M the curvature tensor for the generalized connection ∇ has the form:*

(4) $$R(X,Y)Z = R_{\gamma c}{}^{\delta d}{}_{\beta b \alpha a} Z^{\gamma c} Y^{\beta b} X^{\alpha a} \delta_{\delta d},$$

where

(5) $$R_{\gamma c}{}^{\delta d}{}_{\beta b\ \alpha a} = K_{\gamma c}{}^{\delta d}{}_{\beta b\ \alpha a} - K_{\alpha a}{}^{\varepsilon e}{}_{\beta b} \Gamma_{\gamma c}{}^{\delta d}{}_{\varepsilon e},$$

(6) $$K_{\gamma c}{}^{\delta d}{}_{\beta b\ \alpha a} = (\delta_{\alpha a} \Gamma_{\gamma c}{}^{\delta d}{}_{\beta b} + \Gamma_{\gamma c}{}^{\varepsilon e}{}_{\beta b} \Gamma_{\varepsilon e}{}^{\delta d}{}_{\alpha a}) - (\alpha a/\beta b).$$

It is clear that in (5.5) are only those $K_{\alpha a}{}^{\varepsilon e}{}_{\beta b}$ are different from zero, which appear in (4.8). As α, β, γ, δ are the elements of the set $\{0,1,2,3\}$, so there exist 4^4 types of curvature tensors.

From (4.7), (5.5) and (5.6) it follows

(7) $$\begin{aligned} R_{\gamma c}{}^{\delta d}{}_{\beta b\ \alpha a} &= -R_{\gamma c}{}^{\delta d}{}_{\alpha a\ \beta b} \\ K_{\gamma c}{}^{\delta d}{}_{\beta b\ \alpha a} &= -K_{\gamma c}{}^{\delta d}{}_{\alpha a\ \beta b} \\ K_{\alpha a}{}^{\gamma c}{}_{\beta b} &= -K_{\beta b}{}^{\gamma c}{}_{\alpha a} \end{aligned}$$

Taking into account (4.8) the explicit form of (5.5) is the following

(8) $$\begin{aligned} R_{\gamma c}{}^{\delta d}{}_{0b\ 0a} &= K_{\gamma c}{}^{\delta d}{}_{0b\ 0a} - K_{0a}{}^{1e}{}_{0b} \Gamma_{\gamma c}{}^{\delta d}{}_{1e} - K_{0a}{}^{2e}{}_{0b} \Gamma_{\gamma c}{}^{\delta d}{}_{2e} - K_{0a}{}^{3e}{}_{0b} \Gamma_{\gamma c}{}^{\delta d}{}_{3e}, \\ R_{\gamma c}{}^{\delta d}{}_{1b\ 0a} &= K_{\gamma c}{}^{\delta d}{}_{1b\ 0a} - K_{0a}{}^{1e}{}_{1b} \Gamma_{\gamma c}{}^{\delta d}{}_{1e} - K_{0a}{}^{2e}{}_{1b} \Gamma_{\gamma c}{}^{\delta d}{}_{2e} - K_{0a}{}^{3e}{}_{1b} \Gamma_{\gamma c}{}^{\delta d}{}_{3e}, \\ R_{\gamma c}{}^{\delta d}{}_{2b\ 0a} &= K_{\gamma c}{}^{\delta d}{}_{2b\ 0a} - K_{0a}{}^{2e}{}_{2b} \Gamma_{\gamma c}{}^{\delta d}{}_{2e} - K_{0a}{}^{3e}{}_{2b} \Gamma_{\gamma c}{}^{\delta d}{}_{3e}, \\ R_{\gamma c}{}^{\delta d}{}_{3b\ 0a} &= K_{\gamma c}{}^{\delta d}{}_{3b\ 0a} - K_{0a}{}^{3e}{}_{3b} \Gamma_{\gamma c}{}^{\delta d}{}_{3e}, \\ R_{\gamma c}{}^{\delta d}{}_{1b\ 1a} &= K_{\gamma c}{}^{\delta d}{}_{1b\ 1a} - K_{1a}{}^{2e}{}_{1b} \Gamma_{\gamma c}{}^{\delta d}{}_{2e} - K_{1a}{}^{3e}{}_{1b} \Gamma_{\gamma c}{}^{\delta d}{}_{3e}, \\ R_{\gamma c}{}^{\delta d}{}_{2b\ 1a} &= K_{\gamma c}{}^{\delta d}{}_{2b\ 1a} - K_{1a}{}^{3e}{}_{2b} \Gamma_{\gamma c}{}^{\delta d}{}_{3e}, \\ R_{\gamma c}{}^{\delta d}{}_{3b\ 1a} &= K_{\gamma c}{}^{\delta d}{}_{3b\ 1a}, \qquad R_{\gamma c}{}^{\delta d}{}_{2b\ 2a} = K_{\gamma c}{}^{\delta d}{}_{2b\ 2a}, \\ R_{\gamma c}{}^{\delta d}{}_{3b\ 2a} &= K_{\gamma c}{}^{\delta d}{}_{3b\ 2a}, \qquad R_{\gamma c}{}^{\delta d}{}_{3b\ 3a} = K_{\gamma c}{}^{\delta d}{}_{3b\ 3a}, . \end{aligned}$$

In the above formulae γ and δ take values from the set $\{0,1,2,3\}$.

6 The Bianchi identities

When $R(X,Y)Z$ is defined by (5.1) and $T(X,Y)$ by (4.1) the first Bianchi identity in the global form ([14]) is given by

$$cycl\{X,Y,Z\}[R(X,Y)Z - (\nabla_X T)(Y,Z) - T(T(X,Y),Z)] = 0.$$

Theorem 6.1. *The first Bianchi equation in Osc^3M has the form*

(1) $$\begin{aligned} & cycl\{\gamma c, \beta b, \alpha a\} R_{\gamma c}{}^{\delta d}{}_{\beta b\ \alpha a} = \\ & cycl\{\gamma c, \beta b, \alpha a\} (\bar{T}_{\gamma c}{}^{\delta d}{}_{\beta b|\alpha a} + \bar{T}_{\gamma c}{}^{\varepsilon e}{}_{\beta b} \bar{T}_{\alpha a}{}^{\delta d}{}_{\varepsilon e} + K_{\gamma c}{}^{\varepsilon e}{}_{\beta b} \Gamma_{\alpha a}{}^{\delta d}{}_{\varepsilon e}) \end{aligned}$$

where

(2) $$\bar{T}_{\gamma c}{}^{\delta d}{}_{\beta b} = \Gamma_{\gamma c}{}^{\delta d}{}_{\beta b} - \Gamma_{\beta b}{}^{\delta d}{}_{\gamma c} = T_{\gamma c}{}^{\delta d}{}_{\beta b} - K_{\gamma c}{}^{\delta d}{}_{\beta b}.$$

Proof. The proof is obtained by direct calculation using (4.6), (4.7), (5.5) and (5.6). $R_{\gamma c}{}^{\delta d}{}_{\beta b\ \alpha a}$ is tensor, so the sums of terms on the right hand side of (6.1) is also tensor.

Theorem 6.1.' *The first Bianchi identity in* Osc^3M *has the form*

(3) $$cycl\{\gamma c,\beta b,\alpha a\}(K_{\gamma c}{}^{\delta d}{}_{\beta b\ \alpha a}-\bar{T}_{\gamma c}{}^{\delta d}{}_{\beta b|\alpha a}-\bar{T}_{\gamma c}{}^{\varepsilon e}{}_{\beta b}\bar{T}_{\alpha a}{}^{\delta d}{}_{\varepsilon e})=0.$$

Proof. Using the antisymmetry of $K_{\gamma c}{}^{\varepsilon e}{}_{\beta b}$ we can write (6.1) in the form

$$cycl\{\gamma c,\beta b,\alpha a\}(R_{\gamma c}{}^{\delta d}{}_{\beta b\ \alpha a}+K_{\alpha c}{}^{\varepsilon e}{}_{\beta b}\Gamma_{\gamma c}{}^{\delta d}{}_{\varepsilon e})=$$
$$cycl\{\gamma c,\beta b,\alpha a\}(\bar{T}_{\gamma c}{}^{\delta d}{}_{\beta b|\alpha a}+\bar{T}_{\gamma c}{}^{\varepsilon e}{}_{\beta b}\bar{T}_{\alpha a}{}^{\delta d}{}_{\varepsilon e}).$$

From the above equation and (5.5) it follows (6.3).

Theorem 6.2. *In the torsion free space* Osc^3M *the first Bianchi identity has the form*

(4) $$cycl\{\gamma c,\beta b,\alpha a\}R_{\gamma c}{}^{\delta d}{}_{\beta b\ \alpha a}=$$
$$cycl\{\gamma c,\beta b,\alpha a\}[\bar{T}_{\gamma c}{}^{\delta d}{}_{\beta b|\alpha a}+\bar{T}_{\gamma c}{}^{\varepsilon e}{}_{\beta b}(\bar{T}_{\alpha a}{}^{\delta d}{}_{\varepsilon e}-\Gamma_{\alpha a}{}^{\delta d}{}_{\varepsilon e})].$$

Proof. From $T_{\gamma c}{}^{\delta d}{}_{\beta b}=0$ and (6.2) it follows

(5) $$\bar{T}_{\gamma c}{}^{\delta d}{}_{\beta b}=-K_{\gamma c}{}^{\delta d}{}_{\beta b}.$$

The substitution of (6.5) into (6.1) results (6.4).

Special cases

Proposition 6.1. *For* $\alpha=\beta=\gamma=0$ *the first Bianchi identity has the form*

(6) $$cycl\{c,b,a\}R_{0c}{}^{\delta d}{}_{0b0a}=$$
$$cycl\{c,b,a\}(\bar{T}_{0c}{}^{\delta d}{}_{0b|0a}+\bar{T}_{0c}{}^{\varepsilon e}{}_{\beta b}\bar{T}_{0a}{}^{\delta d}{}_{\varepsilon e}+K_{0c}{}^{\varepsilon e}{}_{0b}\Gamma_{0a}{}^{\delta d}{}_{\varepsilon e}),$$

where

(7) $$\bar{T}_{0c}{}^{0d}{}_{0b}=\Gamma_{0c}{}^{0d}{}_{0b}-\Gamma_{0b}{}^{0d}{}_{0c}=T_{0c}{}^{0d}{}_{0b},$$

(8) $$\bar{T}_{0c}{}^{\delta d}{}_{0b}=\Gamma_{0c}{}^{\delta d}{}_{0b}-\Gamma_{0b}{}^{\delta d}{}_{0c}=T_{0c}{}^{\delta d}{}_{0b}-K_{0c}{}^{\delta d}{}_{0b},\ \delta=1,2,3.$$

Proof. (6.7) and (6.8) are the consequences of (2.3) and (4.7).

Proposition 6.2. *When the horizontal distribution is integrable and* $\alpha=\beta=\gamma=0$ *then the first Bianchi identity has the form:*

(9) $$cycl\{c,b,a\}K_{0c}{}^{\delta d}{}_{0b\ 0a}=cycl\{c,b,a\}(T_{0c}{}^{\delta d}{}_{0b|0a}+T_{0c}{}^{\varepsilon e}{}_{0b}\bar{T}_{0a}{}^{\delta d}{}_{\varepsilon e}).$$

Proof. When the horizontal distribution T_H is integrable, then from Proposition 2.1. it follows $K_{0a}{}^{\delta d}{}_{0b} = 0$, $\delta = 0,1,2,3$. From this equations and (5.5) in this case we have

$$R_{0c}{}^{\delta d}{}_{0b\,0a} = K_{0c}{}^{\delta d}{}_{0b\,0a}$$

and from (6.7) and (6.8) we obtain

$$\bar{T}_{0c}{}^{\delta d}{}_{0b} = T_{0c}{}^{\delta d}{}_{0b},\ \delta = 0,1,2,3.$$

Proposition 6.3. *For $\alpha = \beta = \gamma = \delta = 0$ and integrable horizontal distribution T_H, the first Bianchi identity has the form*

$$cycl\{c,b,a\}K_{0c}{}^{0d}{}_{0b\,0a} = cycl\{c,b,a\}(T_{0c}{}^{0d}{}_{0b|0a} + T_{0c}{}^{\varepsilon e}{}_{0b}T_{0a}{}^{0d}{}_{\varepsilon e}). \tag{10}$$

Proof. From (4.8) it is obvious that all $K_{0a}{}^{0d}{}_{\varepsilon e}$ $\varepsilon = 0,1,2,3$ are equal to zero and in this case $\bar{T}_{0a}{}^{0d}{}_{\varepsilon e} = T_{0a}{}^{0d}{}_{\varepsilon e}$.

Remark. If we in (6.10) drop the index 0 before indices a, b, c and e and take the summation only for $\varepsilon = 0$ we obtain the known Bianchi identity for Lagrange, Finsler or Riemann space in the form:

$$cycl\{c,b,a\}K_{c\,ba}^{\,d} = cycl\{c,b,a\}(T_{c\,b|a}^{\,d} + T_{c\,b}^{\,e}T_{a\,e}^{\,d}). \tag{11}$$

If the mentioned Lagrange, Finsler or Riemann spaces are torsion free (6.11) reduces to

$$cycl\{c,b,a\}K_{c\,ba}^{\,d} = 0.$$

Proposition 6.4. *For $\gamma = 1$, $\beta = \alpha = 0$ the first Bianchi identity has the form*

$$cycl\{1c,0b,0a\}(K_{1c}{}^{\delta d}{}_{0b\,0a} - \bar{T}_{1c}{}^{\delta d}{}_{0b|0a} - \bar{T}_{1c}{}^{\varepsilon e}{}_{0b}\bar{T}_{0a}{}^{\delta d}{}_{\varepsilon e}) = 0.$$

The above equation follows from (6.3). The other special cases can be obtained in the similar way.

Theorem 6.3. *The second Bianchi identity in Osc^3M is*

$$cycl\{\gamma c,\beta b,\alpha a\}[R_{\delta d}{}^{\varepsilon e}{}_{\gamma c\beta b|\alpha a} + R_{\delta d}{}^{\varepsilon e}{}_{\gamma c\kappa k}T_{\beta b}{}^{\kappa k}{}_{\alpha a}] = 0. \tag{12}$$

Proof. It can be obtained from the global equation

$$cycl\{Z,Y,X\}[(\nabla_X R)(Y,Z,U) + R(T(X,Y),Z)U] = 0.$$

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