Reciprocity Theorems in the Linear Thermoelasticity of Solids with Microstructure Having a Symmetric Stress Tensor

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Abstract

In this paper some reciprocity theorems of the linear thermoelasticity of an anisotropic and nonhomogeneous solid with microstructure having a symmetric stress tensor (LTSMSST) are proved.

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Key words: admissible process,heat conduction tensor, convolution.

1 Introduction

The notation, format, basic definitions and governing equations of the LTSMSST and alternative formulation of a mixed problem in B of the LTSMSST in [1] will completely used here.

The proofs of the reciprocical theorems in the different linear dynamic theories of continuum mechanics are based either on the use of Laplace transform [2] or an a such characterization of the mixed problems in B, where the initial conditions are incorporated in the field equations [3], [4].

Reciprocity theorems in the LTSMSST was analysed in [5] by the second author of this paper. The results in [5] as well as [6] and [7] offer a new method to obtain reciprocity relations in the LTSMSST.

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2 Reciprocity Theorems in the LTSMSST

Let us introduce the vectorial functions $\mathbf{S}$, $\mathbf{U} : \overline{B} \times I \to \mathbb{R}^7$,

$$\begin{cases} \mathbf{S}(\mathbf{x}, t) &= \displaystyle\sum_{i=1}^{3} t_i(\mathbf{x}, t)\mathbf{e}'_i + \sum_{j=1}^{3} c_j(\mathbf{x}, t)\mathbf{e}'_j + \frac{\overline{h}(\mathbf{x}, t)}{T_0}\,\mathbf{e}'_7, \\ \mathbf{U}(\mathbf{x}, t) &= \displaystyle\sum_{i=1}^{3} u_i(\mathbf{x}, t)\mathbf{e}'_i + \sum_{j=1}^{3} \varphi_j(\mathbf{x}, t)\mathbf{e}'_j + \theta(\mathbf{x}, t)\,\mathbf{e}'_7, \end{cases} \tag{1}$$

where $\mathcal{B}' = \{\mathbf{e}'_1, \mathbf{e}'_2, \cdots, \mathbf{e}'_7\}$ is the canonical base in $\mathbb{R}^7$. We define the following inner products in $\mathbb{R}^7$

$$\begin{cases} \mathbf{S}(\mathbf{x}, \tau) \cdot \mathbf{U}(\mathbf{x}, p) = \mathbf{t}(\mathbf{x}, \tau) \cdot \mathbf{u}(\mathbf{x}, p) + \mathbf{c}(\mathbf{x}, \tau) \cdot \boldsymbol{\varphi}(\mathbf{x}, p) + \dfrac{\overline{h}(\mathbf{x}, \tau)}{T_0}\,\theta(\mathbf{x}, p), \\ \mathbf{F}(\mathbf{x}, \tau) \cdot \mathbf{U}(\mathbf{x}, p) = \rho(\mathbf{x})\mathbf{f}(\mathbf{x}, \tau) \cdot \mathbf{u}(\mathbf{x}, p) + \rho(\mathbf{x})\mathbf{M}(\mathbf{x}, \tau) \cdot \boldsymbol{\varphi}(\mathbf{x}, p) - \\ \qquad - \rho(\mathbf{x})W(\mathbf{x}, \tau)\,\theta(\mathbf{x}, p), \end{cases} \tag{2}$$

where

$$\mathbf{F}(\mathbf{x}, \tau) = (\rho\mathbf{f}(\mathbf{x}, \tau), \rho\mathbf{M}(\mathbf{x}, \tau), -\rho\, W(\mathbf{x}, \tau)), \quad W(\mathbf{x}, \tau) = \frac{1}{T_0}\overline{r}(\mathbf{x}, \tau) + S_0. \tag{3}$$

and

$$\mathbf{U}(\mathbf{x}, p) = (\mathbf{u}(\mathbf{x}, p), \boldsymbol{\varphi}(\mathbf{x}, p), \theta(\mathbf{x}, p)). \tag{4}$$

Making use of these vectors we introduce convolutions

$$\begin{cases} \Big(\mathbf{S} * \mathbf{U}\Big)(\mathbf{x}, t) &= \displaystyle\int_0^t \mathbf{S}(\mathbf{x}, t - \tau) \cdot \mathbf{U}(\mathbf{x}, \tau)\, d\tau, \\ \Big(\mathbf{F} * \mathbf{U}\Big)(\mathbf{x}, t) &= \displaystyle\int_0^t \mathbf{F}(\mathbf{x}, t - \tau) \cdot \mathbf{U}(\mathbf{x}, \tau)\, d\tau. \end{cases} \tag{5}$$

where the inner products in the wright hand side of (5) are given by (2). In what following the superscripts α and β take independently the values 1 and 2.

Lemma 2.1 *If the heat conduction tensor* $\mathbf{K}$ *is symmetric and* $\mathbf{p}^{(\alpha)}$, $\mathbf{p}^{(\beta)}$ *are the solutions in* B *of the LTSMSST corresponding respectively to the external systems of data* $\mathcal{L}^{(\alpha)}$ *and* $\mathcal{L}^{(\beta)}$, *then*

$$z^{\alpha\,\beta}(\tau, p) \quad = \quad z^{\beta\,\alpha}(p, \tau), \quad (\forall)\ \tau,\, p \in\ I \times I, \tag{6}$$

where

$$\begin{aligned} z^{\alpha\,\beta}(\tau, p) = &\int_B \Big(\mathbf{F}^{(\alpha)}(\mathbf{x}, \tau) \cdot \mathbf{U}^{(\beta)}(\mathbf{x}, p) - \frac{1}{T_0}\overline{\mathbf{q}}^{(\alpha)}(\mathbf{x}, \tau) \cdot \mathbf{g}^{(\beta)}(\mathbf{x}, p)\Big) dv - \\ - &\int_B \rho(\mathbf{x})\Big(\ddot{\mathbf{u}}^{(\alpha)}(\mathbf{x}, \tau) \cdot \mathbf{u}^{(\beta)}(\mathbf{x}, p) + \mathbf{J}(\mathbf{x})[\ddot{\boldsymbol{\varphi}}^{(\alpha)}(\mathbf{x}, \tau)] \cdot \boldsymbol{\varphi}^{(\beta)}(\mathbf{x}, p)\Big) dv + \\ &+ \int_{\partial B} \mathbf{S}^{(\alpha)}(\mathbf{x}, \tau) \cdot \mathbf{U}^{(\beta)}(\mathbf{x}, p)\, da. \end{aligned} \tag{7}$$

Proof. We denote by

$$h^{\alpha\beta}(\mathbf{x},\tau,p)=\mathbf{T}^{(\alpha)}(\tau)\cdot\boldsymbol{\varepsilon}^{(\beta)}(p)+\mathbf{C}^{(\alpha)}(\tau)\cdot\boldsymbol{\kappa}^{(\beta)}(p)-\rho\,S^{(\alpha)}(\tau)\theta^{(\beta)}(p) \tag{8}$$

in which, for sake of brevity, we shall renounce at the variable $\mathbf{x}$. From the constitutive equations of the LTSMSST [1], we have

$$\begin{cases} \mathbf{T}^{(\alpha)}(\tau) = \mathbf{A}[\boldsymbol{\varepsilon}^{(\alpha)}(\tau)] + \mathbf{B}[\boldsymbol{\kappa}^{(\alpha)}(\tau)] - \theta^{(\alpha)}(\tau)\boldsymbol{\alpha}; \\ \mathbf{C}^{(\alpha)}(\tau) = \boldsymbol{\varepsilon}^{(\alpha)}(\tau)\mathbf{B} + \mathbf{H}[\boldsymbol{\kappa}^{(\alpha)}(\tau)] - \theta^{(\alpha)}(\tau)\boldsymbol{\beta}; \\ \rho S^{(\alpha)}(\tau) = \boldsymbol{\alpha}\cdot\boldsymbol{\varepsilon}^{(\alpha)}(\tau) + \boldsymbol{\beta}\cdot\boldsymbol{\kappa}^{(\alpha)}(\tau) + a\,\theta^{(\alpha)}(\tau). \end{cases} \tag{9}$$

By performing the inner product in $(9)_1$ and $(9)_2$ with the vectors $\boldsymbol{\varepsilon}^{(\beta)}(p)$ and $\boldsymbol{\kappa}^{(\beta)}(p)$, respectively, and by multiplying $(9)_3$ with the real function $\theta^{(\beta)}(p)$ and then by adding the obtained results, we obtain for $h^{\alpha\beta}(\mathbf{x},\tau,p)$ the following expression

$$\begin{aligned} h^{\alpha\beta}(\mathbf{x},\tau,p) = {} & \mathbf{A}[\boldsymbol{\varepsilon}^{(\alpha)}(\tau)]\cdot\boldsymbol{\varepsilon}^{(\beta)}(p) + \mathbf{B}[\boldsymbol{\kappa}^{(\alpha)}(\tau)]\cdot\boldsymbol{\varepsilon}^{(\beta)}(p) - \theta^{(\alpha)}(\tau)\boldsymbol{\alpha}\cdot\boldsymbol{\varepsilon}^{(\beta)}(p)+ \\ & +\boldsymbol{\varepsilon}^{(\alpha)}(\tau)\mathbf{B}\cdot\boldsymbol{\kappa}^{(\beta)}(p) + \mathbf{H}[\boldsymbol{\kappa}^{(\alpha)}(\tau)]\cdot\boldsymbol{\kappa}^{(\beta)}(p) - \theta^{(\alpha)}(\tau)\boldsymbol{\beta}\cdot\boldsymbol{\kappa}^{(\beta)}(p)- \\ & -\boldsymbol{\alpha}\cdot\boldsymbol{\varepsilon}^{(\alpha)}(\tau)\theta^{(\beta)}(p) - \boldsymbol{\beta}\cdot\boldsymbol{\kappa}^{(\alpha)}(\tau)\theta^{(\beta)}(p) - a\,\theta^{(\alpha)}(\tau)\,\theta^{(\beta)}(p). \end{aligned} \tag{10}$$

In a similar way, for the expression of $h^{\beta\alpha}(\mathbf{x},p,\tau)$, we get

$$\begin{aligned} h^{\beta\alpha}(\mathbf{x},p,\tau) = {} & \mathbf{A}[\boldsymbol{\varepsilon}^{(\beta)}(p)]\cdot\boldsymbol{\varepsilon}^{(\alpha)}(\tau) + \mathbf{B}[\boldsymbol{\kappa}^{(\beta)}(p)]\cdot\boldsymbol{\varepsilon}^{(\alpha)}(\tau) - \theta^{(\beta)}(p)\boldsymbol{\alpha}\cdot\boldsymbol{\varepsilon}^{(\alpha)}(\tau)+ \\ & +\boldsymbol{\varepsilon}^{(\beta)}(\tau)\mathbf{B}\cdot\boldsymbol{\kappa}^{(\alpha)}(\tau) + \mathbf{H}[\boldsymbol{\kappa}^{(\beta)}(p)]\cdot\boldsymbol{\kappa}^{(\alpha)}(\tau) - \theta^{(\beta)}(p)\boldsymbol{\beta}\cdot\boldsymbol{\kappa}^{(\alpha)}(\tau)- \\ & -\boldsymbol{\alpha}\cdot\boldsymbol{\varepsilon}^{(\beta)}(p)\theta^{(\alpha)}(\tau) - \boldsymbol{\beta}\cdot\boldsymbol{\kappa}^{(\beta)}(p)\theta^{(\alpha)}(\tau) - a\,\theta^{(\beta)}(p)\,\theta^{(\alpha)}(\tau). \end{aligned} \tag{11}$$

By analysing expressions in (10) and (11), and taking into account the symmetries of the elastic moduli, we deduce

$$h^{\alpha\beta}(\mathbf{x},\tau,p) = h^{\beta\alpha}(\mathbf{x},p,\tau), \qquad (\forall)\ \mathbf{x}\in\overline{B}, \quad (\forall)\ \tau,\,p\in I. \tag{12}$$

We shall process the expression of $h^{\alpha\beta}(\mathbf{x},\tau,p)$ in (8) following to give another expression of it. In this sense, by taking into account the symmetry of the stress tensor $\mathbf{T}$ as well as the expression of $\boldsymbol{\varepsilon}^{(\beta)}(\mathbf{x},p)$, we obtain

$$\mathbf{T}^{(\alpha)}(\mathbf{x},\tau)\cdot\boldsymbol{\varepsilon}^{(\beta)}(\mathbf{x},p)=\mathbf{T}^{(\alpha)}(\mathbf{x},\tau)\cdot\Big(\nabla\mathbf{u}^{(\beta)}(\mathbf{x},p)\Big)^T. \tag{13}$$

Similarly, we have

$$\mathbf{C}^{(\alpha)}(\mathbf{x},\tau)\cdot\boldsymbol{\kappa}^{(\beta)}(\mathbf{x},p)=\mathbf{C}^{(\alpha)}(\mathbf{x},\tau)\cdot\Big(\nabla\boldsymbol{\varphi}^{(\beta)}(\mathbf{x},p)\Big)^T. \tag{14}$$

By applying to the expressions in the wright hand side of (13) and (14) the property:

$$\mathbf{f}\cdot(\nabla\mathbf{g})^T \quad = \quad \nabla\cdot(\mathbf{f}[\mathbf{g}]) \quad - \quad (\nabla\cdot\mathbf{f}^T)\cdot\mathbf{g}, \tag{15}$$

where $\mathbf{f}$ is a two–order tensorial function and $\mathbf{g}$ is a vectorial one, and again renouncing at the writing of the argument $\mathbf{x}$, we find:

$$\mathbf{T}^{(\alpha)}(\tau)\cdot\Big(\nabla\mathbf{u}^{(\beta)}(p)\Big)^T = \nabla\cdot\Big(\mathbf{T}^{(\alpha)}(\tau)[\mathbf{u}^{(\beta)}(p)]\Big) - \Big(\nabla\cdot\mathbf{T}^{(\alpha)}(\tau)^T\Big)\cdot\mathbf{u}^{(\beta)}(p); \tag{16}$$

$$\mathbf{C}^{(\alpha)}(\tau)\cdot\Big(\nabla\boldsymbol{\varphi}^{(\beta)}(p)\Big)^T = \nabla\cdot\Big(\mathbf{C}^{(\alpha)}(\tau)[\boldsymbol{\varphi}^{(\beta)}(p)]\Big) - \Big(\nabla\cdot\mathbf{C}^{(\alpha)}(\tau)^T\Big)\cdot\boldsymbol{\varphi}^{(\beta)}(p). \tag{17}$$

The introduction of (16) and (17) respectively in (13) and (14) leads to another expressions of the left hand sides of relations (13) and (14). By replacing these last expressions in (8) and by using the equation

$$S + \frac{1}{\rho\, T_0}\nabla\cdot\overline{\mathbf{q}} = W, \tag{18}$$

we find a better expression for $h^{\alpha\beta}(\mathbf{x},\tau,p)$

$$\begin{aligned} h^{\alpha\beta}(\mathbf{x},\tau,p) \quad = \quad & \nabla\cdot\Big(\mathbf{T}^{(\alpha)}(\tau)[\mathbf{u}^{(\beta)}(p)]\Big) \quad - \quad \Big(\nabla\cdot\mathbf{T}^{(\alpha)}(\tau)^T\Big)\cdot\mathbf{u}^{(\beta)}(p) + \\ & + \nabla\cdot\Big(\mathbf{C}^{(\alpha)}(\tau)[\boldsymbol{\varphi}^{(\beta)}(p)]\Big) \quad - \quad \Big(\nabla\cdot\mathbf{C}^{(\alpha)}(\tau)^T\Big)\cdot\boldsymbol{\varphi}^{(\beta)}(p) - \\ & - \rho\Big(- \frac{1}{\rho\, T_0}\nabla\cdot\overline{\mathbf{q}}^{(\alpha)}(\tau) \quad + \quad W^{(\alpha)}(\tau)\Big)\theta^{(\beta)}(p). \end{aligned} \tag{19}$$

We can obtain a similar expression for $h^{\beta\alpha}(\mathbf{x},p,\tau)$ if we start with (11) and we take into account (13) – (17) in which α and β, on a side, and τ and p, on the other side, are interchanging

$$\begin{aligned} h^{\beta\alpha}(\mathbf{x},p,\tau) \quad = \quad & \nabla\cdot\Big(\mathbf{T}^{(\beta)}(p)[\mathbf{u}^{(\alpha)}(\tau)]\Big) \quad - \quad \Big(\nabla\cdot\mathbf{T}^{(\beta)}(p)^T\Big)\cdot\mathbf{u}^{(\alpha)}(\tau) + \\ & + \nabla\cdot\Big(\mathbf{C}^{(\beta)}(p)[\boldsymbol{\varphi}^{(\alpha)}(\tau)]\Big) \quad - \quad \Big(\nabla\cdot\mathbf{C}^{(\beta)}(p)^T\Big)\cdot\boldsymbol{\varphi}^{(\alpha)}(\tau) - \\ & - \rho\Big(- \frac{1}{\rho\, T_0}\nabla\cdot\overline{\mathbf{q}}^{(\beta)}(p) \quad + \quad W^{(\beta)}(p)\Big)\theta^{(\alpha)}(\tau). \end{aligned} \tag{20}$$

By using the motion equations of the LTSMSST [1], the properties of the nabla operator ∇, and the result (2), for $h^{\alpha\beta}(\mathbf{x},\tau,p)$ we finaly get the expression

$$\begin{aligned} h^{\alpha\beta}(\mathbf{x},\tau,p) \; = \; & \nabla\cdot\Big(\mathbf{T}^{(\alpha)}(\tau)[\mathbf{u}^{(\beta)}(p)] \quad + \mathbf{C}^{(\alpha)}(\tau)[\boldsymbol{\varphi}^{(\beta)}(p)] + \frac{1}{T_0}\overline{\mathbf{q}}^{(\alpha)}(\tau)\,\theta^{(\beta)}(p)\Big) + \\ & + \mathbf{F}^{(\alpha)}(\tau)\cdot\mathbf{U}^{(\beta)}(p) \quad - \frac{1}{T_0}\overline{\mathbf{q}}^{(\alpha)}(\tau)\cdot\mathbf{g}^{(\beta)}(p) - \\ & - \rho\Big(\ \ddot{\mathbf{u}}^{(\alpha)}(\tau)\cdot\mathbf{u}^{(\beta)}(p) + \mathbf{J}[\ddot{\boldsymbol{\varphi}}^{(\alpha)}(\tau)]\cdot\boldsymbol{\varphi}^{(\beta)}(p)\Big). \end{aligned} \tag{21}$$

The expression of $h^{\beta\alpha}(\mathbf{x},p,\tau)$, similar those in (21), is

$$\begin{aligned} h^{\beta\alpha}(\mathbf{x},p,\tau) \;=\; \nabla\cdot\Big(\mathbf{T}^{(\beta)}(p)[\mathbf{u}^{(\alpha)}(\tau)] \;&+ \mathbf{C}^{(\beta)}(p)[\boldsymbol{\varphi}^{(\alpha)}(\tau)]+\frac{1}{T_0}\overline{\mathbf{q}}^{(\beta)}(p)\,\theta^{(\alpha)}(\tau)\Big)+\\ +\mathbf{F}^{(\beta)}(p)\cdot\mathbf{U}^{(\alpha)}(\tau) \;&-\; \frac{1}{T_0}\overline{\mathbf{q}}^{(\beta)}(p)\cdot\mathbf{g}^{(\alpha)}(\tau)-\\ -\rho\Big(\ddot{\mathbf{u}}^{(\beta)}(p)\cdot\mathbf{u}^{(\alpha)}(\tau) &+ \mathbf{J}[\ddot{\boldsymbol{\varphi}}^{(\beta)}(p)]\cdot\boldsymbol{\varphi}^{(\alpha)}(\tau)\Big). \end{aligned} \tag{22}$$

In identity (12), in which the met functions have the expressions (21) and (22), we integrate on B simultaneous with the use of divergence theorem written as the case stands as:

$$\int_B \nabla\cdot(\mathbf{f}[\mathbf{g}])\,dv \;=\; \int_{\partial B}\mathbf{f}^T[\mathbf{n}]\cdot\mathbf{g}\,da; \tag{23}$$

$$\int_B \nabla\cdot(h\,\mathbf{g})\,dv \;=\; \int_{\partial B}\mathbf{g}[\mathbf{n}]\,h\,da, \tag{24}$$

where $\mathbf{n}$ is the outward unit normal vector to the boundary ∂B of the body B, and h is a real function. By taking into account the Cauchy type relations, the Fourier–Stokes one [1], the symmetry of the heat conduction tensor $\mathbf{K}$, equations (1) and (2), we obtain (6) and the Lemma is proved. □

By using this Lemma, we will prove some reciprocity theorems of the LTSMSST. In all reciprocity theorems two admissible processes [1] of the LTSMSST corresponding to two different external systems of data are considered. As special cases, from the proved reciprocity relations of LTSMSST we can deduce those of the linear elasticity [4][**61.1**] as well as those in linear thermoelasticity [5][**21.3**].

Theorem 2.1. *If the heat connduction tensor* $\mathbf{K}$ *is symmetric, and* $\mathbf{p}^{(\alpha)}$, $\alpha=1,2$, *are the solutions of the mixed problems in* B *of the LTSMSST corresponding to the external systems of data* $\mathcal{L}^{(\alpha)}$, *then the following reciprocity relation holds*

$$\begin{aligned} &\int_B\Big(\mathbf{F}^{(1)}*\mathbf{U}^{(2)}\Big)(\mathbf{x},t)\,dv + \int_{\partial B}\Big(\mathbf{S}^{(1)}*\mathbf{U}^{(2)}\Big)(\mathbf{x},t)\,da+\\ &+\int_B\rho(\mathbf{x})\Big(\mathbf{u}_0^{(1)}(\mathbf{x})\cdot\mathbf{v}^{(2)}(\mathbf{x},t) + \mathbf{v}_0^{(1)}(\mathbf{x})\cdot\mathbf{u}^{(2)}(\mathbf{x},t)\Big)\,dv+\\ &+\int_B\rho(\mathbf{x})\Big(\mathbf{J}(\mathbf{x})[\boldsymbol{\varphi}_0^{(1)}(\mathbf{x})]\cdot\boldsymbol{\nu}^{(2)}(\mathbf{x},t) + \mathbf{J}(\mathbf{x})[\boldsymbol{\nu}_0^{(1)}(\mathbf{x})]\cdot\boldsymbol{\varphi}^{(2)}(\mathbf{x},t)\Big)\,dv =\\ &=\int_B\Big(\mathbf{F}^{(2)}*\mathbf{U}^{(1)}\Big)(\mathbf{x},t)\,dv + \int_{\partial B}\Big(\mathbf{S}^{(2)}*\mathbf{U}^{(1)}\Big)(\mathbf{x},t)\,da+\\ &+\int_B\rho(\mathbf{x})\Big(\mathbf{u}_0^{(2)}(\mathbf{x})\cdot\mathbf{v}^{(1)}(\mathbf{x},t) + \mathbf{v}_0^{(2)}(\mathbf{x})\cdot\mathbf{u}^{(1)}(\mathbf{x},t)\Big)\,dv+\\ &+\int_B\rho(\mathbf{x})\Big(\mathbf{J}(\mathbf{x})[\boldsymbol{\varphi}_0^{(2)}(\mathbf{x})]\cdot\boldsymbol{\nu}^{(1)}(\mathbf{x},t) + \mathbf{J}(\mathbf{x})[\boldsymbol{\nu}_0^{(2)}(\mathbf{x})]\cdot\boldsymbol{\varphi}^{(1)}(\mathbf{x},t)\Big)\,dv. \end{aligned} \tag{25}$$

Proof. In identity (6) of Lemma 2.1 we take $\alpha=1$, $\beta=2$, $p=t-\tau$ and then we integrate into respect of τ from 0 to t. In those integrals in which the second derivative into respect of τ appear, we twice integrate by parts and simultaneously we use both the initial conditions of the considered mixed problems [1] and the definition of convolution. We obtain:

$$\begin{aligned}\int_0^t \ddot{\mathbf{u}}^{(1)}(\mathbf{x},\tau)\cdot\mathbf{u}^{(2)}(\mathbf{x},t-\tau)\,d\tau &= \left(\mathbf{v}^{(1)} * \mathbf{v}^{(2)}\right)(\mathbf{x},t)+\\ &+\mathbf{v}^{(1)}(\mathbf{x},t)\cdot\mathbf{u}_0^{(2)}(\mathbf{x})-\mathbf{v}_0^{(1)}(\mathbf{x})\cdot\mathbf{u}^{(2)}(\mathbf{x},t);\end{aligned}\tag{26}$$

$$\begin{aligned}\int_0^t \mathbf{J}(\mathbf{x})[\ddot{\boldsymbol{\varphi}}^{(1)}(\mathbf{x},\tau)]\cdot\boldsymbol{\varphi}^{(2)}(\mathbf{x},t-\tau)\,d\tau &= \left(\mathbf{J}[\boldsymbol{\nu}^{(1)}] * \boldsymbol{\nu}^{(2)}\right)(\mathbf{x},t)+\\ &+\mathbf{J}(\mathbf{x})[\boldsymbol{\nu}^{(1)}(\mathbf{x},t)]\cdot\boldsymbol{\varphi}_0^{(2)}(\mathbf{x})-\mathbf{J}(\mathbf{x})[\boldsymbol{\nu}_0^{(1)}(\mathbf{x})]\cdot\boldsymbol{\varphi}^{(2)}(\mathbf{x},t);\end{aligned}\tag{27}$$

$$\begin{aligned}\int_0^t \ddot{\mathbf{u}}^{(2)}(\mathbf{x},\tau)\cdot\mathbf{u}^{(1)}(\mathbf{x},t-\tau)\,d\tau &= \left(\mathbf{v}^{(2)} * \mathbf{v}^{(1)}\right)(\mathbf{x},t)+\\ &+\mathbf{v}^{(2)}(\mathbf{x},t)\cdot\mathbf{u}_0^{(1)}(\mathbf{x})-\mathbf{v}_0^{(2)}(\mathbf{x})\cdot\mathbf{u}^{(1)}(\mathbf{x},t);\end{aligned}\tag{28}$$

$$\begin{aligned}\int_0^t \mathbf{J}(\mathbf{x})[\ddot{\boldsymbol{\varphi}}^{(2)}(\mathbf{x},\tau)]\cdot\boldsymbol{\varphi}^{(1)}(\mathbf{x},t-\tau)\,d\tau &= \left(\mathbf{J}[\boldsymbol{\nu}^{(2)}] * \boldsymbol{\nu}^{(1)}\right)(\mathbf{x},t)+\\ &+\mathbf{J}(\mathbf{x})[\boldsymbol{\nu}^{(2)}(\mathbf{x},t)]\cdot\boldsymbol{\varphi}_0^{(1)}(\mathbf{x})-\mathbf{J}(\mathbf{x})[\boldsymbol{\nu}_0^{(2)}(\mathbf{x})]\cdot\boldsymbol{\varphi}^{(1)}(\mathbf{x},t).\end{aligned}\tag{29}$$

Based on the commutativity of convolution, the first term in the wright hand side of (26) is equal to the first term in the wright hand side of (28), that is

$$\left(\mathbf{v}^{(1)} * \mathbf{v}^{(2)}\right)(\mathbf{x},t) \quad = \quad \left(\mathbf{v}^{(2)} * \mathbf{v}^{(1)}\right)(\mathbf{x},t).\tag{30}$$

Then, from the symmetry of tensor $\mathbf{J}$ and commutativity of convolution, we have

$$\left(\mathbf{J}[\boldsymbol{\nu}^{(1)}] * \boldsymbol{\nu}^{(2)}\right)(\mathbf{x},t) \quad = \quad \left(\mathbf{J}[\boldsymbol{\nu}^{(2)}] * \boldsymbol{\nu}^{(1)}\right)(\mathbf{x},t).\tag{31}$$

The Fourier–Stokes relation, the symmetry of the heat conduction tensor [1], and the associativity of convolution lead to identity

$$\int_0^t \overline{\mathbf{q}}^{(1)}(\mathbf{x},\tau)\cdot\mathbf{g}^{(2)}(\mathbf{x},t-\tau)\,d\tau \quad = \quad \int_0^t \overline{\mathbf{q}}^{(2)}(\mathbf{x},\tau)\cdot\mathbf{g}^{(1)}(\mathbf{x},t-\tau)\,d\tau.\tag{32}$$

The integration of relation (6) into respect to τ from 0 to t, in which $\alpha = 1$, $\beta = 2$, $p = t-\tau$, and the use of identities $(26)-(32)$, leads to (25) and the theorem is proved. □

Theorem 2.2. *If the heat connduction tensor* $\mathbf{K}$ *is symmetric, and* $\mathbf{p}^{(1)}$, $\mathbf{p}^{(2)}$, *are the solutions of the mixed problems in* B *of the LTSMSST corresponding to the external systems of data* $\mathcal{L}^{(1)}$, $\mathcal{L}^{(2)}$, *respectively, then the following reciprocity relation holds*

$$\begin{aligned}&\int_B \left(\widetilde{\mathbf{F}}^{(1)} * \mathbf{U}^{(2)}\right)(\mathbf{x},t)\,dv \quad + \quad \int_{\partial B}\left(\mathbf{S}^{(1)} * \mathbf{U}^{(2)}\right)(\mathbf{x},t)\,da =\\ &= \int_B \left(\widetilde{\mathbf{F}}^{(2)} * \mathbf{U}^{(1)}\right)(\mathbf{x},t)\,dv \quad + \quad \int_{\partial B}\left(\mathbf{S}^{(2)} * \mathbf{U}^{(1)}\right)(\mathbf{x},t)\,da,\end{aligned}\tag{33}$$

where

$$\widetilde{\mathbf{F}}^{(\alpha)} = \rho\Big(i * \mathbf{f}^{(\alpha)} + \mathbf{u}_0^{(\alpha)} + t\,\mathbf{v}_0^{(\alpha)},\ i * \mathbf{M}^{(\alpha)} + \mathbf{J}[\boldsymbol{\varphi}_0^{(\alpha)} + t\,\boldsymbol{\nu}_0^{(\alpha)}],\ -i * W^{(\alpha)}\Big). \tag{34}$$

Proof. In reciprocity relation (25) we perform convolution with the identity function i and we take into account that the following equalities hold

$$\begin{aligned} & i * \Big(\mathbf{u}_0^{(\alpha)}(\mathbf{x}) \cdot \mathbf{v}^{(\beta)}(\mathbf{x}, t) + \mathbf{v}_0^{(\alpha)}(\mathbf{x}) \cdot \mathbf{u}^{(\beta)}(\mathbf{x}, \mathbf{t})\Big) = \\ & = -t\,\mathbf{u}_0^{(\alpha)}(\mathbf{x}) \cdot \mathbf{u}_0^{(\beta)}(\mathbf{x}) + \Big((\mathbf{u}_0^{(\alpha)} + t\,\mathbf{v}_0^{(\alpha)}) * \mathbf{u}^{(\beta)}\Big)(\mathbf{x}, t); \end{aligned} \tag{35}$$

$$\begin{aligned} & i * \Big(\mathbf{J}(\mathbf{x})[\boldsymbol{\varphi}_0^{(\alpha)}(\mathbf{x})] \cdot \boldsymbol{\nu}^{(\beta)}(\mathbf{x}, t) + \mathbf{J}(\mathbf{x})[\boldsymbol{\nu}_0^{(\alpha)}(\mathbf{x})] \cdot \boldsymbol{\varphi}^{(\beta)}(\mathbf{x}, \mathbf{t})\Big) = \\ & = -t\,\mathbf{J}(\mathbf{x})[\boldsymbol{\varphi}_0^{(\alpha)}(\mathbf{x})] \cdot \boldsymbol{\varphi}_0^{(\beta)}(\mathbf{x}) + \Big(\mathbf{J}[\boldsymbol{\varphi}_0^{(\alpha)} + t\,\boldsymbol{\nu}_0^{(\alpha)}] * \boldsymbol{\varphi}^{(\beta)}\Big)(\mathbf{x}, t). \end{aligned} \tag{36}$$

By using the comutativity of convolution, the first terms in the wright hand sides of relations (35) and (36) will be vanish by those equal with them found in the other hand side of the equality obtained by performing convolution in (25) with function i. The remain terms join to the functions introduced by (34). □

Corollary 2.1. *Let* $\mathbf{p}^{(\alpha)}$, $\alpha = 1, 2$, *be solutions of the mixed problems in* B *of the LTSMSST corresponding to the external systems of data* $\mathcal{L}^{(1)}$ *and* $\mathcal{L}^{(2)}$ *having null initial conditions. Then the reciprocity relations* (25) *and* (34) *becomes*

$$\begin{aligned} & \int_B \Big(\mathbf{F}^{(1)} * \mathbf{U}^{(2)}\Big)(\mathbf{x}, t)\, dv + \int_{\partial B} \Big(\mathbf{S}^{(1)} * \mathbf{U}^{(2)}\Big)(\mathbf{x}, t)\, da = \\ & = \int_B \Big(\mathbf{F}^{(2)} * \mathbf{U}^{(1)}\Big)(\mathbf{x}, t)\, dv + \int_{\partial B} \Big(\mathbf{S}^{(2)} * \mathbf{U}^{(1)}\Big)(\mathbf{x}, t)\, da. \end{aligned} \tag{37}$$

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