SOME REMARKS ON THE QUADRATIC q-OSCILLATOR AND QUANTUM CHAOS

A. Jannussis

Abstract

In the present paper we investigated the quantum chaos according to a paper of Arik and Karaka, where they have founded, by using a generalized quadratic oscillator, more different kinds of quadratic q - oscillator with a solvable spectrum. For the case where the real parameter satisfy a specific relation we obtain also a $SU_q(2)$ deformed algebra. From eq. (14) which describes chaos for special values of the parameter q and γ, we obtain a closed periodic solution for the harmonic oscillator. Finally we study the quatric commutation relation $A^2\left(A^+\right)^2 - Q\left(A^+\right)^2A^2 = 1$ with positive and for the case of the harmonic oscillator we obtain a successive spectrum.

AMS Subject Classification: 33E99.
Key words: oscillator, bosonization, periodic solution, perturbation.

A few years ago Arik and Karaka [1] have formulated some known algebras in forms of generalized oscillators. The above authors, except of the $SU(2)$ Lie-algebra and the $SU_q(2)$ deformed Lie-algebra founded by using a generalized quadratic oscillator, formed more different kinds of quadratic q-oscillators with a solvable spectrum. All the above cases are special cases of the generalized commutation relation

$$\alpha(AA^+)^2 + \beta A^+AAA^+ + \gamma\left(A^+A\right)^2 + \delta\left(AA^+\right) + \varepsilon A^+A + J = 0, \tag{1}$$

which characterizes the quadratic oscillator.. The parameters α, β,..., J are real numbers.

By using the bosonization method [2] for the annihilation and cration operators A, A^+, i.e.:

$$A = f(\hat{n}+1)a,\ A^+ = a^+f(\hat{n}+1), \tag{2}$$

where a, a^+, $a^+a = \hat{n}$ are the usual Boson operators and $f(\hat{n}+1)$ is the structure function, we obtain:

$$AA^+ = (\hat{n}+1)\,f^2(\hat{n}+1) = L_{\hat{n}+1},\ A^+A = \hat{n}f^2\left(\hat{n}\right) = L_{\hat{n}}. \tag{3}$$

Editor Gr.Tsagas *Proceedings of The Conference of Applied Differential Geometry - General Relativity and The Workshop on Global Analysis, Differential Geometry and Lie Algebras, 2000,* 68-75

The commutation relation (1) takes the form

$$\alpha\, L_{n+1}^2 + \beta L_n L_{n+1} + \gamma L_n^2 + \delta L_{n+1} + \varepsilon L_n + J = 0. \tag{4}$$

The above difference equation coincides exactly with the difference equation (18) of [1].

From eq. (4) we have:

$$L_{n+1} = \frac{1}{2\alpha}\left[-(\beta L_n + \delta) \pm \sqrt{(\beta^2 - 4\alpha\gamma)L_n^2 + 2(\beta\delta - 2\alpha\epsilon)L_n + \delta^2 - 4\alpha J}\right]. \tag{5}$$

In the following we put

$$(\beta^2 - 4\alpha\gamma)L_n^2 + 2(\beta\delta - 2\alpha\epsilon)L_n + \delta^2 - 4\alpha J = (kL_n + \beta\mu)^2 \tag{6}$$

and we obtain

$$\kappa^2 = \beta^2 - 4\alpha\gamma,\ \kappa\mu = \beta\delta - 2\alpha\varepsilon,\ \mu^2 = \delta^2 - 4\alpha J, \tag{7}$$

with the condition

$$\left(\beta^2 - 4\alpha\gamma\right)\left(\delta^2 - 4\alpha J\right) = (\beta\delta - 2\alpha\varepsilon)^2, \tag{8}$$

or

$$J = \frac{\varepsilon(\beta\delta - \alpha\varepsilon) - \gamma\delta^2}{\beta^2 - 4\alpha\gamma} \quad , \quad \beta^2 - 4\alpha\gamma \neq 0. \tag{9}$$

Then, the eq. (5) yields

$$L_{n+1} = \frac{1}{2\alpha}\left(\pm\kappa - \beta\right) L_n + \frac{1}{2\alpha}\left(\pm\mu - \delta\right). \tag{10}$$

For

$$q\ =\ \frac{1}{2\alpha}\left(\pm\kappa - \beta\right),\ \nu = \frac{1}{2\alpha}\left(\pm\mu - \delta\right) \tag{11}$$

and for suitable values for the different parameters $\alpha,\ \beta, ...\varepsilon$, the solution of (10) is the following

$$L_n = \nu\frac{q^n - 1}{q - 1} \tag{12}$$

and forms the deformed $SU_q(2)$ Lie-algebra with the parameters $(\alpha,\ \beta,\ldots,\ \varepsilon)$.

The commutation relation

$$AA^+ = 1 + qA^+A - \gamma\left(A^+A\right)^2,\ \gamma > 0 \tag{13}$$

is of interest, as it leads to the difference equation

$$L_{n+1} = 1 + qL_n - \gamma L_n^2 \tag{14}$$

which, according to [1] coincides with the difference equation of Verhultz [3] for $q = r$ and $\gamma = \frac{r}{L}$ and describing the population growth. The difference equation (14) can not be solved in forms of elementary functions and it has chaotic behavior for large r.

First we will study the equation (14) for special values of the parameters q and γ and we obtain a closed periodic solution. Second we can apply the perturbation method for small values of q for the calculation of the eigenvalues of the corresponding harmonic oscillator.

For the first case we put

$$L_n = S_n + \sigma. \tag{15}$$

Substituting (15) in (14) we obtain

$$S_{n+1} = (q - 2\sigma\gamma)\, S_n - \gamma S_n^2, \tag{16}$$

$$\gamma\sigma^2 - (q-1)\sigma - 1 = 0, \tag{17}$$

$$\sigma = \frac{(q-1) \pm \Delta}{2\gamma}, \text{ with } \Delta^2 = (q-1)^2 + 4\gamma \tag{18}$$

and eq. (16) takes the form

$$S_{n+1} = (1 \mp \Delta)\, S_n - \gamma S_n^2. \tag{19}$$

For $S_n = \dfrac{(1 \mp \Delta)}{\gamma} T_n$ the above equations yield

$$T_{n+1} = (1 \mp \Delta) T_n\, (1 - T_n) \tag{20}$$

and for $\Delta = \mp 3$, the relation (18) is written

$$\gamma = \frac{9}{4} - \frac{(q-1)^2}{4} > 0. \tag{21}$$

The logistic map (20) for $1 \mp \Delta = 4$ takes the form

$$T_{n+1} = 4T_n(1 - T_n) \tag{22}$$

with the solution [4]:

$$T_n = \frac{1}{2}\left(1 - \cos 2^n \cos^{-1}(1 - 2T_0)\right), \tag{23}$$

$$S_n = \frac{2}{\gamma}\left(1 - \cos 2^n \cos^{-1}(1 - 2T_0)\right), \tag{24}$$

$$L_n = \frac{q+2}{2\gamma} + \frac{2}{\gamma}\left(1 - \cos 2^n \cos^{-1}(1 - 2T_0)\right) \tag{25}$$

with the initial condition $L_0 = 0$ we obtain

$$2T_0 = -\frac{q+2}{4\gamma} \tag{26}$$

and the solution (25) takes the form

$$L_n = \frac{q+2}{2\gamma} + \frac{2}{\gamma}\left(1 - \cos 2^n \cos^{-1}\left(1 + \frac{q+2}{4}\right)\right). \tag{27}$$

The corresponding annihilation and cration operators A, A^+ are

$$A = \sqrt{\left[\frac{q+2}{2\gamma} + \frac{2}{\gamma}\left(1 - \cos 2^{\hat{n}+1} \cos^{-1}\left(1 + \frac{q+2}{4}\right)\right)\right]\frac{1}{\hat{n}+1}}a, \tag{(28)}$$

or

$$A^p = a^p\sqrt{\left[\frac{q+2}{2\gamma} + \frac{2}{\gamma}\left(1 - \cos 2^{\hat{n}+1} \cos^{-1}\left(1 + \frac{q+2}{4}\right)\right)\right]\frac{1}{\hat{n}+1}}. \tag{29}$$

According to Jannussis et all [5] for the q-deformed oscillators exists a scale factor between the operators A, A^+ and x, p i.e.

$$x = \frac{1}{2}\sqrt{\frac{\hbar(q+1)}{m\omega}}\,(A + A^+) \;\;,\;\; p = -\frac{i}{2}\sqrt{\hbar m\omega(q+1)}\,(A - A^+) \tag{30}$$

and the Hamiltonian of the harmonic oscillator takes the form

$$H = \frac{p^2}{2m} + \frac{m}{2}\omega^2 x^2 = \frac{\hbar\omega}{4}(q+1)\,(AA^+ + A^+A), \tag{31}$$

or

$$\begin{aligned} H &= \frac{\hbar\omega}{4}(1+q)\left[\frac{q+2}{2\gamma} + \frac{2}{\gamma}\left(1 - 2^{\hat{n}+1}\cos^{-1}\left(1 + \frac{q+2}{4}\right)\right) + \right. \\ &\left. \frac{q+2}{2\gamma}\left(1 - 2^{\hat{n}}\cos^{-1}\left(1 + \frac{q+2}{4}\right)\right)\right]. \end{aligned} \tag{32}$$

The energy eigenvalues are the following

$$\begin{aligned} E_n &= \frac{\hbar\omega(q+1)}{9-(q-1)^2}\left[\frac{q+2}{2} + 2\left(1 - \cos 2^{n+1}\cos^{-1}\left(1 + \frac{q+2}{4}\right)\right) + \right. \\ &\left. \frac{q+2}{2} + 2(1 - \cos 2^n \cos^{-1}\left(1 + \frac{q+2}{4}\right)\right] \end{aligned} \tag{33}$$

with the restriction

$$9 > (q-1)^2\,. \tag{34}$$

For $n = 0$, we obtain the ground state, i.e.e

$$E_0 = \frac{\hbar\omega}{2}\frac{q+1}{9-(q-1)^2}\left[q + 2 + 4\,(1 - \cos 2)\cos^{-1}\left(1 + \frac{q+2}{4}\right)\right] \tag{35}$$

and satisfy the generalized uncertainty principle [5,6)]

$$(\Delta x)\ \Delta p) \geq \frac{\hbar'}{2}, \tag{36}$$

where

$$\hbar' = \hbar \frac{q+1}{9-(q-1)^2}\left[q+2+4\,(1-\cos 2)\cos^{-1}\left(1+\frac{q+2}{4}\right)\right]. \tag{37}$$

For the second case we consider the equation (14) and we put $-\gamma = \lambda$ i.e. eq. (14) yields

$$L_{n+1} = 1 + qL_n + \lambda L_n^2. \tag{38}$$

By using the perturbation method for small values of λ the solution has the form

$$L_n = L_n^{(0)} + \lambda L_n^{(1)} + \lambda^2 L_n^{(2)} + \cdots + \lambda^\kappa L_n^{(k)} + \cdots. \tag{39}$$

Substituting (39) in (38) we obtain

$$L_{n+1}^{(0)}+\lambda L_{n+1}^{(1)}+\lambda^2 L_{n+1}^{(2)}+\lambda^3 L_n^{(3)}+\cdots \equiv 1+q\left(L_n^{(0)} + \lambda L_n^{(1)} + \lambda^2 L_n^{(2)} + \lambda^3 L_n^{(3)} + \cdots\right) +$$

$$+\lambda\left[(L_n^{(0)})^2 + \lambda\, 2L_n^{(0)}L_n^{(1)} + \lambda^2\left(2L_n^{(0)}L_n^{(2)} + \left(L_n^{(0)}\right)^2\right)\right]. \tag{40}$$

From the above relation we have the following recursion equations:

$$L_{n+1}^{(0)} - qL_n^{(0)} = 1 \text{ with } \mathrm{L}_0^{(0)} = 0, \tag{41}$$

$$L_{n+1}^{(1)} - qL_n^{(1)} = \left(L_n^{(0)}\right)^2, \tag{42}$$

$$L_{n+1}^{(2)} - qL_n^{(2)} = 2L_n^{(0)}L_n^{(1)}, \tag{43}$$

$$L_{n+1}^{(3)} - qL_n^{(3)} = 2L_n^{(0)}L_n^{(2)} + \left(L_n^{(0)}\right)^2. \tag{44}$$

After some computations, the solution of the (41) - (44) are the following

$$L_{n+1}^{(0)} = \frac{q^{n+1}-1}{q-1}, \text{ with } L_0^{(0)} = 0, \tag{45}$$

$$L_{n+1}^{(1)} = q^{n-1}+\frac{q^n}{(q-1)^2}\left[q^2\frac{q^{n-1}-1}{q-1} - 2(n-1) + \frac{1}{q^n}\ \frac{q^{n-1}-1}{q-1}\right], \text{ with } L_0^{(1)} = 0,\ L_1^{(0)} = 1, \tag{46}$$

$$L_{n+1}^{(3)} = 2q^{n+1}\sum_{\ell=2}^{n}\frac{L_\ell^{(0)}L_\ell^{(2))}}{q^\ell},, \text{ with } L_0^{(2)} = 0, L_1^{(2)} = 0,\ L_2^{(2)} = 0, \tag{47}$$

$$L_{n+1}^{(3)} = q^{n+1}\sum_{\ell=3}^{n}\frac{2L_\ell^{(0)}L_\ell^{(2))} + (L_\ell^{(1)})^2}{q^\ell}, \text{ with } L_0^{(3)} = 0,\ \ L_1^{(3)} = 0,\ L_2^{(3)} = 0,\ L_3^{(3)} = 0. \tag{48}$$

Working in the same way we can calculate the coefficients $L_{n+1}^{(k)}$ for $k = 4, 5, \ldots$.

Based on the above results we will calculate in a forthcoming paper the first, second (and so on) order harmonic oscillator eingenvalues on in terms of the parameter λ.

In the sequel we will study the following commutation relation

$$A^2\left(A^+\right)^2 - Q\left(A^+\right)^2 A^2 = 1. \tag{49}$$

According to the relation (2) and after some computations, the commutation relation (49) takes the form

$$(n+2)(n+3)\, f^2(n+2)\, f^2(n+3) - n(n+1)\, f^2(n+1)\, f^2(n) = 1. \tag{50}$$

For

$$L_{n+1} = (n+1)\, f^2(n+1), \tag{51}$$

eq.(50) yields

$$L_{n+2}L_{n+3} - QL_nL_{n+1} = 1. \tag{52}$$

Our problem in this case is the determination of the coefficient L_n from the solution of the equation (52) with the requirement that the operators A, A^+ to be the annihilation and the creation operator correspondingly.

For the solution of the nonlinear recursion equation (52) we consider the following initial conditions: $L_0 = 0$, $L_1 = 1$ and $L_2 = c$, where the parameter $c \neq 0, \infty$ and positive.

From eq. (52) for $n = 0$ we obtain

$$L_2L_3 = cL_3 = 1 \text{ or } L_3 = \frac{1}{c}. \tag{53}$$

Also for $n = 1, 2, 3, 4, 5.......$ in a successive way we obtain the coefficients $L_4, L_5, ...$ i.e.

$$\begin{aligned}
&L_0 = 0,\\
&L_1 = 1,\\
&L_2 = c,\\
&L_3 = \frac{1}{c},\\
&L_4 = c\,(1+Qc),\\
&L_5 = \frac{1+Q}{c\,(1+Qc)},\\
&L_6 = \frac{c\,(1+Qc)\left(1+Q+Q^2c\right)}{1+Q},\\
&L_7 = \frac{(1+Q)\left(1+Q+Q^2\right)}{c\,(1+Qc)\,(1+Q+Q^2c)},\\
&L_8 = \frac{c\,(1+Qc)\left(1+Q+Q^2c\right)\left(1+Q+Q^2+Q^3c\right)}{(1+Q)\,(1+Q+Q^2)},\\
&L_9 = \frac{(1+Q)\left(1+Q+Q^2\right)\left(1+Q+Q^2+Q^3\right)}{c\,(1+Qc)\,(1+Q+Q^2c)\,(1+Q+Q^2+Q^3c)}\,,\\
&L_{10} = \frac{c\,(1+Qc)\left(1+Q+Q^2c\right)\left(1+Q+Q^2+Q^3c\right)\left(1+Q+Q^2+Q^3+Q^4c\right)}{(1+Q)\,(1+Q+Q^2)\,(1+Q+Q^2+Q^3)},
\end{aligned} \tag{54}$$

$$L_{11}=\frac{(1+Q)\left(1+Q+Q^2\right)\left(1+Q+Q^2+Q^3\right)\left(1+Q+Q^2+Q^3+Q^4\right)}{c\,(1+Qc)\,(1+Q+Q^2c)\,(1+Q+Q^2+Q^3c)\,(1+Q+Q^2+Q^3+Q^4c)},$$

....

From the solution (51) we obtain the structure function $f\,(\hat{n}+1)$ i.e.

$$f\,(\hat{n}+1)=\sqrt{\frac{L_{\hat{n}+1}}{\hat{n}+1}} \tag{55}$$

and the operators A, A^+ take the forms

$$A=\sqrt{\frac{L_{\hat{n}+1}}{\hat{n}+1}}a,\ A^+=a^+\sqrt{\frac{L_{\hat{n}+1}}{\hat{n}+1}} \tag{56}$$

and the following relations are valid:

$$A\,|n\rangle=\sqrt{L_n}\,|n-1\rangle\,,\ A^+\,|n\rangle=\sqrt{L_{n+1}}\,|n+1\rangle\,, \tag{57}$$

$$\left[A,A^+\right]=L_{\hat{n}+1}-L_{\hat{n}},\ \left\{A,A^+\right\}=L_{\hat{n}+1}+L_{\hat{n}}. \tag{58}$$

Therefore the eigenvalues of the harmonic oscillator will have the form

$$E_n=\frac{\hbar\omega}{2}\,(L_{n+1}+L_n)\,. \tag{59}$$

For $n=0$ we have the ground state

$$E_0=\frac{\hbar\omega}{2}, \tag{60}$$

which coincides exactly with the ground state of the simple harmonic oscillator. In the sequel for $n=1,2,\ldots$ we have:

$$E_1=\frac{\hbar\omega}{2}\,(c+1)\,, \tag{61}$$

$$E_2=\frac{\hbar\omega}{2}\left(\frac{1}{c}+c\right), \tag{62}$$

$$E_3=\frac{\hbar\omega}{2}\left(c\,(1+Qc)+\frac{1}{c}\right), \tag{63}$$

$$E_4=\frac{\hbar\omega}{2}\left(\frac{1+Q}{c\,(1+Qc)}+c\,(1+Qc)\right), \tag{64}$$

....

From the above relations becomes evident the need of the existence of the positive constant c , which plays the role of a scale factor and permits the calculation of the coefficients L_n.

The case $c=1$ performs physical interest inasmuch leads in closed solution, i.e.

$$L_{2n}=\frac{Q^n-1}{Q-1}=[n]\ \text{for}\ n=0,1,2,3\ldots \tag{65}$$

$$L_{2n-1} = 1 \text{ for } n = 1, 2, 3 \dots \tag{66}$$

and with the corresponding values for the case of the harmonic oscillator eq. (59) which are of the form

$$E_0 = \frac{\hbar\omega}{2}, \tag{67}$$

$$E_{2n} = E_{2n-1} = \frac{\hbar\omega}{2}\,([n]+1) \text{ for } n = 1, 2, 3 \dots . \tag{68}$$

From the above results we see that the ground state is non degenerate and all the excited states are twice degenerate, i.e. $E_1 = E_2$, $E_3 = E_4, \dots$.

Also the of physical interest is the case $Q = 1$ with the eigenvalues

$$E_0 = \frac{\hbar\omega}{2}, \tag{69}$$

$$E_{2n} = E_{2n-1} = \frac{\hbar\omega}{2}\,(n+1)\,. \tag{70}$$

From all we know from the existing literature we can mention that the study of the commutation relation (49) has been done for the first time, inasmuch we have proved the existence of the annihilation and creation operators A, A^+ and furthermore we have found the eigenvalues of the harmonic oscillator $H = \frac{p^2}{2m} + \frac{m}{2}\omega^2 x^2$. In a future paper we will study physical applications of the above results.

References

[1] .M. Arik, M. Karaca, *Quadratic q - Oscillators*, p.30, Proccedings of the VII International Conference on Symmetry methods in Physics (ICSMP - 95), Dubna 10 - 16 July 1993 (Moscow Region, Russia), Eds. : A. N. Sissakian, G. S. Bogosyan.

[2] A. Jannussis, G. Brodimas, D. Sourlas and V. Zisis, Lett. Nuovo Cim. 30, 123 (1981).

[3] R. M. May, Nature 261, 459 (1976).

[4] W. H. Steeb, J. A. Louw, *Chaos and Quantum Chaos*, World Scientific, 6 (1986).

[5] A. Jannussis, G. Brodimas, Mod. Phys. Lett. A15, 1385 (2000).

[6] R. Santilli, *Elements of Hadronic Mechanics*, Vol. I, *Mathematical Foundations* (1993), Vol. II *Theoretical Foundations* (1994): Academy of Sciences of Ukraine, Institute for Theoretical Physics, Kiev Ukraine, Second Edition (1995).

Author's address:

A. Jannussis
Department of Physics
University of Patras
26500 Patras, Greece