APPLICATION OF THE DECOMPOSITION METHOD TO THE KORTWEG-DE VRIES EQUATION

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Abstract

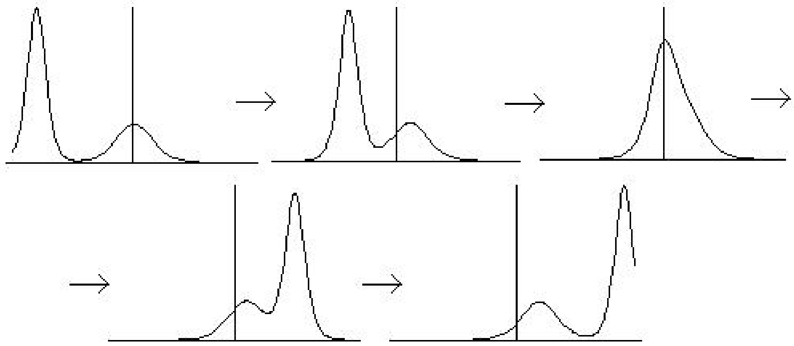
In this paper our aim is to find numerical solution of Korweg-de Vries equation using the Adomian Decomposition Method. With this method we change the non-linear problem to a mathematically tractable one with physical solution. Theoretical analysis is given and all calculations have been done and the results are discussed.

AMS Subject Classification: 65C20.
Key words: Kortweg-de Vries equation, Soliton, Adomian Decomposition Method, Adomian polynomials.

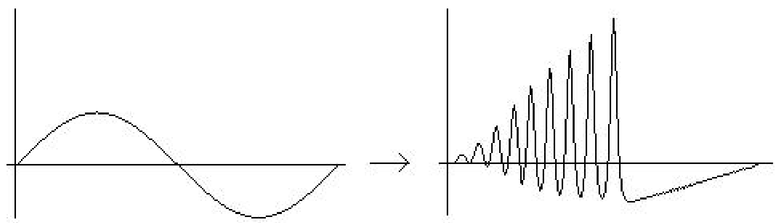
1 Introduction

The Kortweg-de Vries Equation describes the long waves over water and some wave phenomena in plasma physics. The Kortweg-de Vries (KdV) equation is the champion of model equations of nonlinear waves. It was studied by Kortweg and de Vries late in the 19th century as a water wave equation, and after a long period of sleep, revived as one of the most fundamental equation of soliton phenomena. In fact, it is from numerical experiments of this equation that Zabusky and Kruskal introduced the term "soliton". Solitons are very stable solitary waves in a solution of those equations. As the term "soliton" suggests, these solitary waves behave like "particles". When they are located mutually far apart, each of them is approximately a traveling wave with constant shape and velocity. As two such solitary waves get closer, they gradually deform and finally merge into a single wave packet; this wave packet, however, soon splits into two solitary waves with the same shape and velocity before "collision" as shown in the figure below.

Editor Gr.Tsagas *Proceedings of The Conference of Applied Differential Geometry - General Relativity and The Workshop on Global Analysis, Differential Geometry and Lie Algebras, 2000,* 79-87



The stability of solitons stems from the delicate balance of"nonlinearity" and "dispersion" in the model equations. Nonlinearity drives a solitary wave to concentrate further; dispersion is the effect to spread such a localized wave. If one of these two competing effects is lost, solitons become unstable and, eventually, cease to exist. In this respect, solitons are completely different from "linear waves" like sinusoidal waves. In fact, sinusoidal waves are rather unstable in some model equations of soliton phenomena. Computer simulations show that they soon break into a train of solitons.



The existence of certain solitary wave solutions were discovered by Kruskaland Zabusky, who first observed the emerging solitary waves by studying motion pictures of the computations. Once noted, careful computations isolated the phenomena and led to a pure mathematical solution. Also, Kenig et al [10] mentioned some global solutions for the KdV equation with unbounded data. In this paper, we will find numerical solutions using the Adomian decomposition method [3]. The advantage of this method is that the method does not take any help of linearization or any other simplifications and restrictions for handling the non-linear terms which change the physical non-linear problem to a mathematically tractable one, whose solution is not consistent with the physical solution. This method gives a computable and accurate solution of the problem for a small number of terms. In this paper we proceeded a solution using three terms.

The whole paper contains five sections. Each of them is analyzed as follows. The first section is the introduction.

The formulation of the problem is studied in the second section. The theoretic approach is given in the third section. The determination of the Adomian's Special Polynomials is studied in the fourth section. The fifth section includes the results, the diagrams and the discussions.

2 Formulation of the Problem

The Kortweg-de Vries equation in one dimension has the following form:

$$\frac{\partial u}{\partial t} - u\frac{\partial u}{\partial x} - \frac{\partial^3 u}{\partial x^3} = 0, \tag{1}$$

where the first term is the linear, the second is the non-linear and the third is the highest order term.

If we define [13]

$$L_t u = \frac{\partial u}{\partial t} = Ru, \;\; L_x u = \frac{\partial^3 u}{\partial x^3} = Lu, \;\; Nu = u\frac{\partial u}{\partial x}, \tag{2}$$

where Nu represents the non-linear term, Lu the highest order term, and Ru is the rest of the equation, then equation (1) takes the form

$$Ru - Nu - Lu = 0.$$

The boundary conditions are defined as follows:

$$u(t,0) = u(t,1) = 0 \;\text{ for }\; t \succeq 0 \tag{3}$$

and the initial condition as a sinusoidal initial value:

$$u(0,x) = \sin \pi x. \tag{4}$$

3 Theoretic approach

We solve equation (1) for $L_t u$ and $L_x u$ separately and we get

$$L_t u = L_x u + Nu, \tag{5}$$

$$L_x u = L_t u - Nu. \tag{6}$$

Let L_t^{-1} and L_x^{-1} be the inverse operators of $L_t u$ and $L_x u$ respectively, given by the form:

$$L_t^{-1} = \int (\cdot)\, dt \;\text{ and }\; L_x^{-1} = \int\int\int (\cdot)\, dxdxdx. \tag{7}$$

Then operating both sides of equations (5) and (6) with the inverse operators (7), we obtain.

$$u = \phi_0 + L_t^{-1}\left(\frac{\partial^3 u}{\partial x^3} + u\frac{\partial u}{\partial x}\right), \tag{8}$$

$$u = \psi_0 + L_x^{-1}\left(\frac{\partial u}{\partial t} - u\frac{\partial u}{\partial x}\right), \tag{9}$$

where ϕ_0 and ψ_0 are the solutions of the equations

$$\frac{\partial u}{\partial t} = 0 \quad \text{and} \quad \frac{\partial^3 u}{\partial x^3} = 0 \tag{10}$$

respectively. The equations (10) can be solved subjected to the corresponding initial condition (4) and boundary conditions (3) and we obtain:

$$\phi_0 = 0 \ \text{ and } \ \psi_0 = \sin \pi x. \tag{11}$$

Now, adding (8) and (9) and dividing by 2, we get the following form:

$$\begin{aligned} u &= \frac{1}{2}\left[(\phi_0 + \psi_0) + L_t^{-1}\left(\frac{\partial^3 u}{\partial x^3} + u\frac{\partial u}{\partial x}\right) + L_x^{-1}\left(\frac{\partial u}{\partial t} - u\frac{\partial u}{\partial x}\right)\right] \\ &= \frac{\sin \pi x}{2} + \frac{1}{2}\left[L_t^{-1}\left(\frac{\partial^3 u}{\partial x^3} + u\frac{\partial u}{\partial x}\right) + L_x^{-1}\left(\frac{\partial u}{\partial t} - u\frac{\partial u}{\partial x}\right)\right], \end{aligned} \tag{12}$$

where

$$u_0 = \frac{1}{2}(\phi_0 + \psi_0) = \frac{\sin \pi x}{2}. \tag{13}$$

After that, we write the parametrized form of (12) which is:

$$u = u_0 + \lambda\frac{1}{2}\left[L_t^{-1}\left(\frac{\partial^3 u}{\partial x^3} + u\frac{\partial u}{\partial x}\right) + L_x^{-1}\left(\frac{\partial u}{\partial t} - u\frac{\partial u}{\partial x}\right)\right] \tag{14}$$

and the parametrized decomposition forms of u and Nu as

$$u = \sum_{n=0}^{\infty} \lambda^n u_n, \tag{15}$$

$$Nu = \frac{\partial u}{\partial x} = \sum_{n=0}^{\infty} \lambda^n A_n, \tag{16}$$

where An are the Adomian's special polynomials [1,2] to be determined later. Here the parameter λ looks like a perturbation parameter; but actually is not a perturbation parameter; it is used only for grouping the terms.

Now substitution of (15) and (16) into (14) gives

$$\sum_{n=0}^{\infty}\lambda^n u_n = u_0 + \lambda\frac{1}{2}\left[\begin{array}{c} L_t^{-1}\left(\frac{\partial^3\sum\limits_{n=0}^{\infty}\lambda^n u_n}{\partial x^3} + \sum\limits_{n=0}^{\infty}\lambda^n A_n\right) + \\ L_x^{-1}\left(\frac{\partial\sum\limits_{n=0}^{\infty}\lambda^n u_n}{\partial t} - \sum\limits_{n=0}^{\infty}\lambda^n A_n\right)\end{array}\right] \tag{17}$$

If we compare like-power terms of λ from both sides of equation (17), and taking under consideration that parameter λ is being proved [5,6] that has the unique value $\lambda = 1$, we get

$$\begin{aligned}
u_0 &= \sin\pi x \\
u_1 &= \frac{1}{2}\left[L_t^{-1}\left(\frac{\partial^3 u_0}{\partial x^3} + A_0\right) + L_x^{-1}\left(\frac{\partial u_0}{\partial t} - A_0\right)\right] \\
u_2 &= \frac{1}{2}\left[L_t^{-1}\left(\frac{\partial^3 u_1}{\partial x^3} + A_1\right) + L_x^{-1}\left(\frac{\partial u_1}{\partial t} - A_1\right)\right] \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
u_{n+1} &= \frac{1}{2}\left[L_t^{-1}\left(\frac{\partial^3 u_n}{\partial x^3} + A_n\right) + L_x^{-1}\left(\frac{\partial u_n}{\partial t} - A_n\right)\right], \quad n = 0, 1, 2, ..., n
\end{aligned} \tag{18}$$

Next, we proceed to determine Adomian's special polynomials A_n.

4 Determination of Adomian's Special Polynomials

The A_n polynomials are defined in such a way that each A_n depends only on $u_0, u_1, ..., u_n$ for $n = 0, 1, 2, ..., n$, i.e., $A_0 = A(u_0)$, $A_1 = A_1(u_0, u_1)$, $A_2 = A_2(u_0, u_1, u_2)$, etc. In order to do this we substitute (15) into (16) and we have

$$\begin{aligned}
Nu &= u\frac{\partial u}{\partial x} = \left(u_0 + \lambda u_1 + \lambda^2 u_2 + \lambda^3 u_3 + ...\right)\left(\frac{\partial u_0}{\partial x} + \lambda\frac{\partial u_1}{\partial x} + \lambda^2\frac{\partial u_2}{\partial x} + \lambda^3\frac{\partial u_3}{\partial x} + ...\right) \\
&= u_0\frac{\partial u_0}{\partial x} + \lambda\left(u_0\frac{\partial u_1}{\partial x} + u_1\frac{\partial u_0}{\partial x}\right) + \lambda^2\left(u_0\frac{\partial u_2}{\partial x} + u_1\frac{\partial u_1}{\partial x} + u_2\frac{\partial u_0}{\partial x}\right) + \\
&\quad + \lambda^3\left(u_0\frac{\partial u_3}{\partial x} + u_1\frac{\partial u_2}{\partial x} + u_2\frac{\partial u_1}{\partial x} + u_3\frac{\partial u_0}{\partial x}\right) + \lambda^4(...).
\end{aligned} \tag{19}$$

From (19) we conclude that the Adomian Polynomials have the following form:

$$\begin{aligned}
A_0 &= u_0\frac{\partial u_0}{\partial x}, \\
A_1 &= u_0\frac{\partial u_1}{\partial x} + u_1\frac{\partial u_0}{\partial x},
\end{aligned}$$

$$A_2 = u_0\frac{\partial u_2}{\partial x} + u_1\frac{\partial u_1}{\partial x} + u_2\frac{\partial u_0}{\partial x}, \tag{20}$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

Hence, the polynomial A_0 has the following form:

$$A_0 = u_0\frac{\partial u_0}{\partial x} = \frac{1}{4}\pi\cos\pi x\sin\pi x. \tag{21}$$

Using (13) and A_0 from (20) into the expression of u_1 in (18) and then performing the integrations with respect to t and x respectively, we have

$$u_1 = \frac{x^2}{64} - \frac{1}{4}\pi^3\cos\pi x - \frac{\cos 2\pi x}{128\pi^2} + \frac{1}{8}\pi t\cos\pi x\sin\pi x,$$

then from (13) and (22) we have the two terms solution

$$u = u_0 + u_1 \tag{22}$$

given by the following form

$$\frac{1}{2}\sin\pi x + \frac{x^2}{64} - \frac{1}{4}\pi^3\cos\pi x - \frac{\cos 2\pi x}{128\pi^2} + \frac{1}{8}\pi t\cos\pi x\sin\pi x. \tag{23}$$

Now, if we suggest as a solution of u an approximation of three terms then using calculations of u_1 from (22) and A_1 from (20) into the expression of u_2 in (18) and then performing the integrations with respect to t and x respectively, we have the solution

$$u = u_0 + u_1 + u_2$$

wich is given by the following form:

$$\begin{aligned}&\frac{5}{8}\sin\pi x + \frac{x^2}{64} - \frac{1}{4}\pi^3\cos\pi x - \frac{\cos 2\pi x}{256\pi^2} + \frac{1}{8}\pi t\cos\pi x\sin\pi x +\\ &\frac{9t\cos\pi x}{1024\pi} + \frac{x\cos\pi x}{64\pi^3} + \frac{1}{256}\pi t x^2\cos\pi x - - \frac{5}{32}\pi^4 t^2\cos 2\pi x - \frac{35t\cos 3\pi x}{9216\pi} -\\ &\frac{23\sin\pi x}{1024\pi^4} - \frac{1}{256}\pi^2 t^2\sin\pi x - \frac{1}{16}\pi^6 t^2\sin\pi x + \frac{1}{128}tx\sin\pi x + \frac{x^2\sin\pi x}{256\pi^2} -\\ &\frac{5}{128}\pi t\sin 2\pi x - \frac{\sin 3\pi x}{9216\pi^4} + \frac{3}{256}\pi^2 t^2\sin 3\pi x.\end{aligned}$$

The complete computer program written in Mathematica 3.0 in order to get numerical results is:

```
uo=(1/2)Sin [Pi x]
xuo=D[uo,x]
tuo=D[uo,t]
truo=D[uo,x,x,x]
amiden=Expand[uo xuo]
```

```
a=Integrate[tuo-amiden,x,x,x], b=Integrate[truo+amiden,t]
u1=Expand[(a/2)+(b/2)]
u=Expand[uo+u1]
xu1=D[u1,x], tru1=D[u1,x,x,x], tu1=D[u1,t]
aena=Expand[uo xu1+u1 xuo]
c=Integrate[tu1-aena,x,x,x], d=Integrate[tru1+aena,t]
u2=Expand[(c/2)+(d/2)]
u=Expand[u+u2]
u/.{x->.75, t->.01}
```

5 Results, Diagrams and Discussion

Here are the results for a small time period $t > 0$, with time values $t = 0.01$, 0.05, 0.1, 0.15, 0.2, 0.25 and for space values $x = -0.75$, -0.5, -0.25, 0.25, 0.5, 0.75

x = - 0.75	**Dec. Anal.**		**x = 0.25**	**Dec. Anal.**
t	**3-terms**		**t**	**3-terms**
0,01	-0,3731		0,01	0,3846
0,05	-0,0492		0,05	0,0666
0,1	0,5466		0,1	-0,5217
0,15	1,3546		0,15	-1,3223
0,2	2,3747		0,2	-2,3351
0,25	3,607		0,25	-3,5599
x = - 0.50			**x = 0.50**	
t			**t**	
0,01	-0,6129		0,01	0,6247
0,05	-0,4317		0,05	0,5168
0,1	0,1344		0,1	0,1794
0,15	1,0779		0,15	-0,3832
0,2	2,3987		0,2	-1,1709
0,25	4,0967		0,25	-2,1836
x = - 0.25			**x = 0.75**	
t			**t**	
0,01	-0,4922		0,01	0,5003
0,05	-0,6123		0,05	0,6145
0,1	-0,5716		0,1	0,5665
0,15	-0,3186		0,15	0,3063
0,2	0,1464		0,2	-0,1661
0,25	0,8237		0,25	-0,8506

Until now nonlinear partial differential equations have defied analytical solution for a century and analytical approaches have been replaced by numerical methods which discretize the problem and lead to severe problems of computational time on supercomputer. So, this global methodology has made it possible to solve nonlinear, partial differential equations without a need for linearization or assumptions of "weak" nonlinearity, "small" fluctuations, and to avoid discretized methods which lead to the massive computational requirements in solving such equations and as a result of these, to seek continuous, verifiable, analytic solutions without the massive printouts and restrictive assumptions which necessarily change the physical problem into a mathematical tractable and different problem not yielding the same solution.

References

[1] Adomian G., *Nonlinear Stochastic Operator Equations*, Academic Press, 1986.

[2] Adomian G., *Nonlinear Stochastic Systems Theory and Applications to Physics*, Kluwer Academic Publishers, 1989.

[3] Adomian G., *Application of the Decomposition Method to the Navier-Stokes Equations*, J. Math. Anal. Appl., 119 (1986) 340-360.

[4] Adomian G., Appld. Math. Lett., 6, No. 5 (1993) 35-36.

[5] Cherruault Y., Kybernetes, 18, No2 (1989) 31-39.

[6] Cherruault Y., Math. Comp. Modelling, 16, No.2 (1992) 85-93.

[7] Rach R., *A Convenient Computational Form of the Adomian Polynomials*, J. Math. Anal. Appl., 102 (1984) 415-419.

[8] Adomian G. and Malakian, *Existence of the Inverse of a Linear Stochastic Operator*, J. Math. Anal. Appl., 114 (1986) 55-56.

[9] Adomian G. and Rach R., *Inversion of Nonlinear Stochastic Operators*, J. Math. Anal. Appl., 91 (1983) 39-46.

[10] Kenig C. E., Ponce G. and Vega L. *Global solutions for the KdV equation with unbounded data*, J. Diff. Eqns, (1997) 339-364.

[11] Gardner C. S., Greene J. M., Kruskal M. D., and Miura R. M., *Method for Solving the Kortweg-de Vries Equation*, Phys. Rev. Lett. 19 (1967) 1095-1097.

[12] Helal M. A., *A Chebyshev-spectral method for solving Kortweg de Vries equation with hydrodynamical applications*, J. Appl. Math., 40 (1989) 267-287.

[13] Mamaloukas C., *An Approximate Solution of Burgers Equation using Adomian's Decomposition Method*, (in this volume).

[14] Adomian G. and Malakian, *Self-correcting approximate solutions by the iterative method for nonlinear Stochastic differential Equations*, J. Math. Anal. Appl., 76 (1980) 309-327.

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