GEODESICS IN THE HIGHER ORDER FINSLER GEOMETRY

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Abstract

The geometry of order $k \geq 1$ is presented in the author's monograph [4]. Here, in Thessaloniki at Prof. Tsagas's Workshop , we would to pointed out the geometrical theory of geodesics, from the point of view of variational calculus, in the higer order Finsler spaces.

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1 Introduction

After eighty years from the discovery of Finsler spaces, (Paul Finsler 1918), they are known and appreciated for the theoretical reason, Engineering, Theoretical Physics, Optimal control, Biology etc. This fact was expressed by author, in the "Joint Summer Research Conference on Finsler Geometry" organized by S.S.Chern at University of Seattle (1995, [2]), as follows:

"After three quarters of a century of existence Finsler Geometry constitutes an imposing edifice, which numerous scientists tray to understand, but without noticing the immense scientific labor set at its foundations."

In the same time I have emphases the importance of the extension of the notion of Finsler space to higher order.

Sixty-five years ago A. Kawaguchi [6] and J.L.Synge gave first definition. They have considered the integral of action $I(c)$ of the square of fundamental function F of the space and imposed the following condition: $I(c)$ does not depend on the parameterization of the curve c. Thus, the Zermelo conditions for F hold. But, for $k > 1$, they lead to condition: the fundamental tensor g_{ij} of the space is degenerate. So the Kawaguchi-Synge theory of higher order Finsler spaces is geometrically nonconvenient.

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In the paper [8], together with Sorin Sabau we define Finsler space of order $k, F^{(k)n}$, as a natural extension of the classical notion of Finsler space.

Therefore, in the present lecture we give: the notion of Finsler space of order$\geq$ 1, geodesics in $F^{(k)n}$, variational problem, geodesics in canonical parameterization, higher order energy and its law of conservation.

2 Preliminaries. Finsler space of order k

The concept of higher order Finsler space is a natural extension of the classical one. Namely, a Finsler space of order $k > 1$ is a pair $F^{(k)n} = (M, F)$ determined by a real C^∞-manifold of dimension n and a function:

$$F : Osc^k M \to R$$

having the following properties:

a) F is of C^∞-class on $\widetilde{E} = Osc^k M \setminus \{0\}$ and continuous on the null section;

b) F is positive;

c) the Hessian with the elements:

$$g_{ij} = \frac{1}{2}\frac{\partial^2 F^2}{\partial y^{(k)i}\partial y^{(k)j}} \tag{1.1}$$

is positively defined on $\widetilde{E}$.

Clearly, here $(Osc^k M, \pi^k, M)$ is the k-osculator bundle of the manifold M and $(x^i, y^{(1)i}, \ldots, y^{(k)i})$ are the canonical local coordinates of the points $u = (x, y^{(1)}, \ldots, y^{(k)}) \in E = Osc^k M$, $(i, j, h, \ldots = 1, \ldots, n)$. g_{ij} from (1.1) is a $d-tensor$ field on $\widetilde{E}$ and is called the fundamental tensor field of the space $F^{(k)n}$. Of course, it follows:

$$rank\,||g_{ij}|| = n \text{ on } \widetilde{E} \tag{1.2}$$

and the tensors g_{ij} are $0-\hom ogeneous$ on the fibre of $\widetilde{E}$. Therefore we have:

$$\pounds_{\overset{k}{\Gamma}} g_{ij} = 0, \tag{1.3}$$

where $\pounds_{\overset{k}{\Gamma}}$ is the operator of Lie derivative with respect to the Liouville vector field $\overset{k}{\Gamma}$:

$$\overset{k}{\Gamma} = y^{(1)i}\frac{\partial}{\partial y^{(1)i}} + \ldots + k y^{(k)i}\frac{\partial}{\partial y^{(k)i}}. \tag{1.4}$$

In the case $k = 1$ the previous definition gives the classical notion of Finsler space.

In the general case $k \geq 1$ the existence of space $F^{(k)n}$ on the paracompact manifold M is proved in the book [4].

3 Geodesics in spaces $F^{(k)n}$

Let $c : t \in [0,1] \to (x^i(t)) \in U \subset M$ be a smooth parametrized curve and $\widetilde{c} : t \in [0,1] \to (x^i(t), \frac{dx^i}{dt}(t), \ldots, \frac{1}{k!}\frac{d^k x^i}{dt^k}(t)) \in \pi^{-1}(U) \subset \widetilde{E}$ its extension of order k. The length of c, $l(c)$ in the space $F^{(k)n}$ is defined by:

$$l(c) = \int_0^1 F(x(t), \frac{dx(t)}{dt}, \ldots, \frac{1}{k!}\frac{d^k x(t)}{dt^k})dt. \tag{2.1}$$

It is known fact that $l(c)$ essentially depends on the parameterization of the curve c. However, the variational problem involving the functional $l(c)$ can be studied. We shall present it omitted proofs.

Consider the curves:

$$c_\varepsilon : t \in [0,1] \to (x^i(t) + \varepsilon V^i(t)) \in M, \tag{2.2}$$

where ε is a real number, choice such that $\operatorname{Im} c_\varepsilon \subset U, V^i(t) = V^i(x(t))$ being a regular vector field on the open set U restricted to the curve c. We assume that the curves c and c_ε have the same endpoints $c(0)$ and $c(1)$ and the same osculator spaces of order $1, \ldots, k-1$. This is:

$$V^i(0) = V^i(1), \frac{d^\alpha V^i}{dt^\alpha}(0) = \frac{d^\alpha V^i}{dt^\alpha}(1), (\alpha = 1, \ldots, k-1). \tag{2.3}$$

The length of a parameterized curve c_ε is the following:

$$l(c_\varepsilon) = \int_0^1 F(x + \varepsilon V, \frac{dx}{dt} + \varepsilon\frac{dV}{dt}, \ldots, \frac{1}{k!}(\frac{d^k x}{dt^k} + \varepsilon\frac{d^k V}{dt^k}))dt. \tag{2.1$'$}$$

A necessary condition that $l(c)$ be an extremal value for the functionals $l(c_\varepsilon)$ is as follows:

$$\frac{dl(c_\varepsilon)}{d\varepsilon} \mid_{\varepsilon=0} = 0. \tag{2.4}$$

From the previous condition we derive:

$$\frac{dl(c_\varepsilon)}{d\varepsilon} \mid_{\varepsilon=0} = \int_0^1 (\frac{\partial F}{\partial x^i}V^i + \frac{\partial F}{\partial y^{(1)i}}\frac{dV^i}{dt} + \ldots + \frac{1}{k!}\frac{\partial F}{\partial y^{(k)i}}\frac{d^k V^i}{dt^k})dt, \tag{2.4$'$}$$

where:

$$y^{(1)i} = \frac{dx^i}{dt}, \ldots, y^{(k)i} = \frac{1}{k!}\frac{d^k x^i}{dt^k}. \tag{2.5}$$

Now, setting:

$$I^1_V(F) \quad = \quad V^i\frac{\partial F}{\partial y^{(k)i}}, \tag{2.6}$$

$$I_V^2(F) = V^i \frac{\partial F}{\partial y^{(k-1)i}} + \frac{dV^i}{dt}\frac{\partial F}{\partial y^{(k)i}}, \ldots,$$
$$I_V^k(F) = V^i \frac{\partial F}{\partial y^{(1)i}} + \frac{dV^i}{dt}\frac{\partial F}{\partial y^{(2)i}} + \ldots + \frac{1}{(k-1)!}\frac{d^{k-1}V^i}{dt^{k-1}}\frac{\partial F}{\partial y^{(k)i}},$$

we remark that $I_V^1(F), \ldots, I_V^k(F)$ are the main invariants of the fundamental function F. They have the properties:

$$I_V^\alpha(F)(c(0)) = I_V^\alpha(F)(c(1)) = 0, (\alpha = 1, \ldots, k). \tag{2.6$'$}$$

Also, the Euler-Lagrange operator $\overset{\circ}{E}_i$ applied to F give us:

$$\overset{\circ}{E}_i (F) = \frac{\partial F}{\partial x^i} - \frac{d}{dt}\frac{\partial F}{\partial y^{(1)i}} + \ldots + (-1)^k \frac{1}{k!}\frac{d^k}{dt^k}\frac{\partial F}{\partial y^{(k)i}}. \tag{2.7}$$

One deduces important identities:

$$\begin{aligned} & \frac{\partial F}{\partial x^i}V^i + \frac{\partial F}{\partial y^{(1)i}}\frac{dV^i}{dt} + \ldots + \frac{1}{k!}\frac{\partial F}{\partial y^{(k)i}}\frac{d^kV^i}{dt^k} \\ = & \overset{\circ}{E}_i (F)V^i + \frac{d}{dt}I_V^k(F) - \frac{1}{2!}\frac{d^2}{dt^2}I_V^{k-1}(F) + \ldots + \\ & +(-1)^{k-1}\frac{1}{k!}\frac{d^k}{dt^k}I_V^1(F). \end{aligned} \tag{2.8}$$

Therefore, we obtain:

$$\begin{aligned} \frac{dl(c_\varepsilon)}{d\varepsilon} \mid_{\varepsilon=0} = & \int_0^1 \overset{\circ}{E}_i (F)V^i dt + \int_0^1 \frac{d}{dt}\{I_V^k(F) - \\ & -\frac{1}{2!}\frac{d}{dt}I_V^{k-1}(F) + \ldots + (-1)^{k-1}\frac{1}{k!}\frac{d^{k-1}}{dt^{k-1}}I_V^1(F)\}dt \end{aligned}$$

and using (2.6) and $(2.6)'$ it follows that:

$$\frac{dl(c_\varepsilon)}{d\varepsilon} \mid_{\varepsilon=0} = \int_0^1 \overset{\circ}{E}_i (F)V^i dt.$$

Now, taking into account that $\int_0^1 \overset{\circ}{E}_i (F)V^i dt = 0$ for any V^i give us $\overset{\circ}{E}_i (F) = 0$ and using (2.4) we have:

Theorem 1 *In order that the length $l(c)$ of the parameterized curve be an extremal value for the functional $l(c_\varepsilon)$ it is necessary that the following Euler-Lagrange equation hold:*

$$\begin{aligned} \overset{\circ}{E}_i (F) \ : & = \frac{\partial F}{\partial x^i} - \frac{d}{dt}\frac{\partial F}{\partial y^{(1)i}} + \ldots + (-1)^k \frac{1}{k!}\frac{d^k}{dt^k}\frac{\partial F}{\partial y^{(k)i}} = 0 \\ y^{(1)i} & = \frac{dx^i}{dt}, \ldots, y^{(k)i} = \frac{1}{k!}\frac{d^kx^i}{dt^k}. \end{aligned} \tag{2.9}$$

The parameterized curves $c : [0,1] \to M$, solutions of the Euler-Lagrange equations (2.9) are called the *extremal curves* of the length $l(c)$.

4 Geodesics in the canonical parametrization

Let us consider a canonical parametrization of the curve $c : t \in [0,1] \to (x^i(t)) \in M$.

The function $t \to s(t)$:

$$s(t) = \int_{t_0}^{t} F(x(\tau), \frac{dx(\tau)}{d\tau}, \ldots, \frac{1}{k!}\frac{d^k x(\tau)}{d\tau^k})d\tau,\ t_0, t \in [0,1]$$

is monotone, because $\dfrac{ds}{dt} = F > 0$.

So, it is invertible. We can introduce a parameter s (called natural or canonical) with property:

$$F(x(s), \frac{dx}{ds}, \ldots, \frac{1}{k!}\frac{d^k x}{ds^k}) = 1. \tag{3.1}$$

Now we give:

Definition 2 The extremal curves of the lenght $l(c)$ in the canonical parametrization are called *geodesics of the space* $F^{(k)n}$.

We get an extension of a very known classical result in the geometry of Finsler spaces:

Theorem 3 *In the canonical parametrization, the geodesics of the Finsler space of order* $k, F^{(k)n}$*, are given by the equations:*

$$\overset{\circ}{E}_i (F^2) = 0,\ y^{(1)i} = \frac{dx^i}{ds}, \ldots, y^{(k)i} = \frac{1}{k!}\frac{d^k x^i}{ds^k} \tag{3.2}$$

Proof. A straightforward calculus, lead to identity:

$$\frac{1}{2}\overset{\circ}{E}_i (F^2) = F\overset{\circ}{E}_i (F) + \frac{dF}{dt}\overset{1}{E}_i (F) + \ldots + \frac{d^k F}{dt^k}\overset{k}{E}_i (F), \tag{3.3}$$

where $\overset{\beta}{E}_i, (\beta = 1, \ldots, k)$ are the Craig-Synge operators:

$$\overset{\beta}{E}_i = \sum_{\alpha=1}^{k} (-1)^\alpha \frac{1}{\alpha!} \binom{\alpha}{\alpha - \beta} \frac{d^{\alpha-\beta}}{dt^{\alpha-\beta}} \frac{\partial}{\partial y^{(\alpha)i}},\ \beta = 1, \ldots, k. \tag{3.4}$$

Taking into account (3.1), from (3.3) it follows: In the canonical parametrization, the equations $\overset{\circ}{E}_i (F) = 0$ and $\overset{\circ}{E}_i (F^2) = 0$ are the equivalent. □

In particular, for $k = 1$, (3.2) reduces to:

$$\frac{\partial F^2}{\partial x^i} - \frac{d}{ds}\frac{\partial F^2}{\partial y^i} = 0,\ y^i = \frac{dx^i}{ds}.$$

These are the classical differential equations of the geodesics, in canonical parametrization, of a Finsler spaces.

Theorem 4 *The system of differential equations of geodesics of $F^{(k)n}$:*

$$\begin{aligned} \frac{\partial F^2}{\partial x^i} - \frac{d}{ds}\frac{\partial F^2}{\partial y^{(1)i}} + \ldots + (-1)^k \frac{1}{k!}\frac{d^k}{ds^k}\frac{\partial F^2}{\partial y^{(k)i}} &= 0 \\ y^{(1)i} &= \frac{dx^i}{ds}, \ldots, y^{(k)i} = \frac{1}{k!}\frac{d^k x^i}{ds^k}, \end{aligned} \tag{3.5}$$

have the following properties:

Theorem 5 *1^0 It is autoadjoint ([9]).*

2^0 It is invariant to the coordinates transformations on $F^{(k)n}$:

$$\begin{cases} \widetilde{x}^i = \widetilde{x}^i(x^1, x^2, \ldots, x^n); \det \left\| \dfrac{\partial \widetilde{x}^i}{\partial x^j} \right\| <> 0, \\ \widetilde{y}^{(1)i} = \dfrac{\partial \widetilde{x}^i}{\partial x^j} y^{(1)j}, \ldots, k\widetilde{y}^{(k)i} = \dfrac{\partial \widetilde{y}^{(k-1)i}}{\partial x^j} y^{(1)j} + \ldots + k\dfrac{\partial \widetilde{y}^{(k-1)i}}{\partial y^{(k-1)i}} y^{(k)j}. \end{cases} \tag{3.6}$$

3^0 It is invariant to the affine transformation of parameters $s \to as + b$, $a <> 0$, $a,\ b \in \mathcal{R}$.

4^0 It is of the form:

$$g_{ij}\frac{d^{2k}x^j}{ds^{2k}} + \phi_i(x, \frac{dx}{ds}, \ldots, \frac{d^{2k-1}x}{ds^{2k-1}}) = 0. \tag{3.7}$$

Proof. For the property 1° we send to the Santilli's book [9].

2^0 It is known that $\overset{\circ}{E_i}\ (F^2)$ is a *d-covector field.* Therefore $\overset{\circ}{E_i}\ (F^2) = 0$ has a geometrical meaning with respect to the coordinates transformations (3.6). But $\overset{\circ}{E_i}\ (F^2)$ is exactly (3.5).

3^0 An affine transformation $s \to as+b$, give us the transformation $(x, y^{(1)}, \ldots, y^{(k)}) \to (\widetilde{x}, \widetilde{y}^{(1)}, \ldots, \widetilde{y}^{(k)})$, where $\widetilde{y}^{(1)i} = ay^{(1)i}, \ldots, \widetilde{y}^{(k)i} = a^k y^{(k)i}$. By means of the property of the homogeneity of the fundamental function F^2 on the fibres of $Osc^k M$ it follows that $\overset{\circ}{E_i}\ (F^2)$ is transformed in $a^{2k}\ \overset{\circ}{E_i}\ (F^2)$. Therefore $\overset{\circ}{E_i}\ (F^2) = 0$ is invariant with respect to affine transformation of parameters.

4^0 Since we have:

$$\frac{d}{ds}\frac{\partial F^2}{\partial y^{(k)i}} = \Gamma \frac{\partial F^2}{\partial y^{(k)i}} + \frac{2}{k!} g_{ij} \frac{d^{k+1}x^i}{ds^{k+1}},$$

where Γ is the following operator:

$$\Gamma = y^{(1)i}\frac{\partial}{\partial x^i} + 2y^{(2)i}\frac{\partial}{\partial y^{(1)i}} + \ldots + ky^{(k)i}\frac{\partial}{\partial y^{(k-1)i}}$$

and:

$$y^{(1)i} = \frac{dx^i}{ds}, \ldots, y^{(k)i} = \frac{1}{k!}\frac{d^k x^i}{ds^k},$$

the system of differential equations (3.7) holds. $\square$

Of course, from (3.7) it follows that the system of differential equations of geodesics in the spaces $F^{(k)n}$ can be written in the form:

$$\frac{d^{2k}x^i}{ds^{2k}} + g^{ij}\phi_j(x, \frac{dx}{ds}, \ldots, \frac{d^{2k-1}x}{ds^{2k-1}}) = 0. \qquad (3.7')$$

A theorem of existence and uniqueness for geodesics can be easily formulated. An application of the previous theory is as follows.

The function F^2 in $F^{(k)n}$ is a regular Lagrangian of order k. Along a canonical parameterized curve c the energy of order k, [6] is given by:

$$\varepsilon_c^k(F^2) = I^k(F^2) - \frac{1}{2!}\frac{dI^{k-1}(F^2)}{ds} + \ldots + (-1)^{k-1}\frac{1}{k!}\frac{d^{k-1}I^1(F^2)}{ds^{k-1}} - F^2. \qquad (3.8)$$

The following formula is known, [6]:

$$\frac{d}{ds}\varepsilon_c^k(F^2) = -\overset{\circ}{E}_i (F^2)\frac{dx^i}{ds}. \qquad (3.8')$$

Consequently, the following law of conservation holds:

Theorem 6 *The energy of order $k, \varepsilon_c^k(F^2)$, of a Finsler space $F^{(k)n}$ is conserved along every geodesics of the Finsler space of order k, $F^{(k)n}$.*

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