PROJECTABLE NON-LINEAR CONNECTIONS ON SUBMANIFOLDS AND ON DISTRIBUTIONS ON MANIFOLDS

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Abstract

A general theory of projectable non-linear connections is studied in a previous paper by the authors.

The aim of this paper is to give some applications of this theory concerning submanifolds and distributions on manifolds.

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The projectable non-linear connections are defined and studied in a previous paper by the authors;. we recall them briefly in the first section.

The second and the third sections contain applications of projectable nonlinear connections on submanifolds and on geometric distributions on submanifolds respectively..

1 Projectable non-linear connections

In this section we recall briefly the general constructions and results obtained in [4].

Let $\xi' \xrightarrow{f} \xi$ be a morphism of vector bundles, where $\xi' = (E', \pi', M)$ and $\xi = (E, \pi, M)$ have the same base M. A *left* (respective *right*) *F-splitting* is a left (respectively right) splitting S of the induced morphism $(\pi')^*\xi' \xrightarrow{(\pi')^*f} (\pi')^*\xi$.

Proposition 1.1 *Let $\xi \xrightarrow{P''} \xi''$ be a epimorphism of vector bundles $\xi = (E, \pi, M)$. Consider also a non-linear connection C'' on ξ'' and a right splitting S of the epimorphism $\tau E \xrightarrow{\tau P''} (P'')^*\tau E''$ of vector bundles over the base E.*

Then there is a unique non-linear connection C on the vector bundle ξ which projects by $\tau P''$ the fibres of the horizontal bundle isomorphically onto the fibres of

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the horizontal subbundle of C''. The splitting S induces a right F-splitting S'' of the epimorphism P''.

If $\xi \xrightarrow{f} \xi''$ is an epimorphism of vector bundles, we say that a non-linear connection C on ξ is *projectable* on ξ'' if there exist a non-linear connection C'' on ξ'' and a right splitting of the induced morphism $\tau E'' \xrightarrow{(P'')^* f} (P'')^*\xi''$ which induce, according to Proposition 1.1, the non-linear connection C.

There is a vector bundle $\eta = (E, P'', E'')$, which has as vertical bundle the vector bundle $\ker \tau P'' \overset{not.}{=} \mathcal{V}'\xi$. Denote as $\xi' = \ker P''$. There are canonical isomorphisms $\mathcal{V}'\xi \cong (P'')^*\eta$ and $\mathcal{V}'\xi \cong \pi^*\xi'$. Denote as $\mathcal{V}'\xi \xrightarrow{I} \tau E$ the inclusion morphism. Notice that a right splitting S, given by Proposition 1.1, is equivalent with a left splitting T of the inclusion I.

If $\xi' = (E', \pi', M)$ is a vector subbundle of the vector bundle $\xi = (E, \pi, M)$ and we take as $\xi'' = \xi/\xi'$, we fit in the above case, thus there is a canonical injective morphism $\pi^*\xi' \xrightarrow{i'} \tau E$.

Proposition 1.2 *Let $\xi' = (E', \pi', M)$ be a vector subbundle of the vector bundle $\xi = (E, \pi, M)$, T be a splitting of the canonical injective morphism $\pi^*\xi' \xrightarrow{i'} \tau E$, and C'' be a non-linear connection on the vector bundle $\xi'' = \xi/\xi'$.*

Then a unique non-linear connection C on ξ can be induced, having the properties that it is projectable on ξ'' and its horizontal bundle is included in the kernel of the splitting T,.

Proposition 1.3 *Let $\xi' = (E', \pi', M)$ be a vector subbundle of the vector bundle $\xi = (E, \pi, M)$ and T be a left splitting of the canonical injective morphism $\pi^*\xi' \xrightarrow{i'} \tau E$. Then:*

a) A unique non-linear connection C' on ξ' is induced such that the fibres of the horizontal bundle are included in the fibres of the kernel of T, in every point of E'.

b) A canonical left F-splitting T' of the inclusion $\xi' \xrightarrow{j'} \xi$ is induced.

The results in Proposition 1.3 can be improved as follows:

Proposition 1.4 *Let $\xi' = (E', \pi', M)$ be a vector subbundle of the vector bundle $\xi = (E, \pi, M)$, T be a left splitting of the canonical injective morphism $(\pi')^*\xi' \xrightarrow{i'} \tau E_{|E'}$. Then:*

a) *A unique non-linear connection C' is induced, provided that the fibres of the horizontal bundle are included in the fibres of the kernel of T, in every point of E'.*

b) *A canonical left F-splitting T' of the inclusion $\xi' \xrightarrow{j'} \xi$ is induced.*

c) *A non-linear connection C' on ξ' and a left F-splitting T' of the inclusion $\xi' \xrightarrow{j'} \xi$ define together a left splitting T of the canonical injective morphism $(\pi')^*\xi' \xrightarrow{i'} \tau E_{|E'}$.*

Proposition 1.5 *Let ξ be a vector bundle and ξ' be a vector subbundle of ξ. A non-linear connection C on ξ is projectable on a non-linear connection C'' on the vector bundle $\xi'' = \xi/\xi'$ iff the vector subbundle $\pi^*\xi'$ of $V\xi$ is parallel according to the linear Berwald connection ∇ on $V\xi$, associated with C, i.e. $\nabla_X A \in S(\pi^*\xi')$, $(\forall) A \in S(\pi^*\xi')$, $X \in \mathcal{X}(VE)$.*

2 Application to submanifolds

Let M' be a submanifold of the differentiable manifold M and $\varphi : M' \to M$ be the inclusion. The comorphism of the cotangent bundles $\tau^*M \overset{(\varphi,\varphi^*)}{\longrightarrow} \tau^*M'$ induces an epimorphism of vector bundles over the base M', $\tau^*_{M'}M = \tau^*M_{|M'} \overset{\varphi^*_{|M'}}{\longrightarrow} \tau^*M'$. In the sequel we denote as $\tau^*_{M'}M = \tau^*M_{|M'}$, as $\Pi_{M'} = \varphi^*_{|M'}$, $\nu^*_{M'}M = \ker \varphi^*_1$; $\nu^*_{M'}M$ is a vector subbundle of $\tau^*_{M'}M$ and we call it as the *conormal bundle* of the submanifold M'. There is a short exact sequence of vector bundles over the base M':

$$0 \to \nu^*_{M'}M \overset{I_{M'}}{\to} \tau^*_{M'}M \overset{\Pi_{M'}}{\to} \tau^*M' \to 0$$

which we can apply the general construction.

If C is a non-linear connection on τ^*M , then it induces a non-linear connection on the vector bundle $\tau^*_{M'}M$, which we call the *restriction* to M' of C.

Let C be a non-linear on τ^*M. We say that a submanifold M' of M is *adapted* to the connection C if its restriction to M' is projectable on the vector bundle τ^*M'. We also say that C is *projectable* along M'.

Proposition. 2.1 *A non-linear connection C on τ^*M which is projectable along a submanifold M' induces a non-linear connection C'' on τ^*M'.*

Proposition. 2.2 *Let C'' be a non-linear connection on τ^*M' and T be a right splitting of the epimorphism $\tau(T^*_{M'}M) \overset{(\Pi_{M'})_*}{\longrightarrow} \Pi^*_{M'}\tau(T^*M')$ on the base manifold $T^*_{M'}M$.*

Then there is a unique non-linear connection C on the vector bundle $\tau^*_{M'}M$ which projects on C'' and the splitting T induces a right $\tau^*_{M'}M{-}F$-splitting of the epimorphism $(\Pi_{M'})_*$.

Proposition. 2.3 *Let C'' be a non-linear connection on the vector bundle τ^*M' and S be a left splitting of the canonical injective morphism $p^*(\nu^*_{M'}M) \overset{i'}{\to} \tau(T^*_{M'}M)$, where $\tau^*_{M'}M = (T^*_{M'}M, p, M')$.*

Then there is a unique non-linear connection C on the vector bundle $\tau^*_{M'}M$, which projects on τ^*M' and which has the horizontal subbundle included in the kernel of the spliting S.

Proposition. 2.4 *Let S be a left splitting of the canonical injective morphism $p^*(\nu^*_{M'}M) \overset{i'}{\to} \tau(T^*_{M'}M)$, where $\tau^*_{M'}M = (T^*_{M'}M, p, M')$. Then:*

a) *There is a unique non-linear connection C' on $\nu^*_{M'}M$ which enjoys the property that the fibers of its horizontal bundle are included in the fibers of* $\ker S$.

b) *A left $\tau^*_{M'}M$-F-splitting S'_0 and a left $\nu^*_{M'}M$-F-splitting S' of the inclusion $\nu^*_{M'}M \overset{I_{M'}}{\to} \tau^*_{M'}M$ are induced.*

Proposition. 2.5 *A non-linear connection C on $\tau^*_{M'}M$ is projectable on the non-linear connection C'' on τ^*M' iff the vector subbundle $p^*\nu^*_{M'}M$ of $V\tau^*_{M'}M$ is parallel with respect the linear Berwald connection ∇ on $V\tau^*_{M'}M$, associated with the non-linear connection C.*

3 Application to distributions on differentiable manifolds

We suppose now that $\mathcal{D}$ is a regular distribution on the manifold M, i.e. a vector subbundle $\mathcal{D} \stackrel{\varphi}{\to} \tau M$. We consider the cotangent bundle $\tau^* M = (T^* M, p, M)$, the dual bundle $\mathcal{D}^* = (D^*, \pi^*_{\mathcal{D}}, M)$ of the distribution and the induced epimorphism $\tau^* M \stackrel{\varphi^*}{\longrightarrow} \mathcal{D}^*$. We denote as $\nu^*\mathcal{D} = \ker \varphi^*$, which is a vector subbundle of $\tau^* M$, called the *conormal bundle* of the distribution $\mathcal{D}$ on M. There is a short exact sequence of vector bundles over the base M:

$$0 \to \nu^*\mathcal{D} \stackrel{I_{\mathcal{D}}}{\to} \tau^* M \stackrel{\Pi_{\mathcal{D}}}{\to} \mathcal{D}^* \to 0$$

which we can apply the general construction. We say that a non-linear connection C on $\tau^* M$ is *projectable* along the distribution $\mathcal{D}$ on M, if C is projectable on the vector bundle $\mathcal{D}^*$.

Proposition 3.1 *A projectable non-linear connection C on $\tau^* M$ along the distribution $\mathcal{D}$ induces a non-linear connection C'' on $\mathcal{D}^*$.*

Proposition. 3.2 *Let C'' be a non-linear connection on $\mathcal{D}^*$ and T be a right splitting of the epimorphism $\tau(T^*M) \stackrel{(\Pi_{\mathcal{D}})_*}{\longrightarrow} \Pi^*_{\mathcal{D}}\tau(D^*)$ on the base T^*M.*

Then there is a unique non-linear connection C on $\tau^* M$ which projects on C'', and T induces a right $\tau^* M - F$-splitting T'', of the epimorphism $(\Pi_{\mathcal{D}})_*$.

Proposition 3.3 *Let C'' be a non-linear connection on the vector bundle $\mathcal{D}^*$ and S be a left splitting of the canonical injective morphism $p^* (\nu^*\mathcal{D}) \stackrel{i'}{\to} \tau(T^*M)$, where $\tau^* M = (T^* M, p, M)$.*

Then there exists only one non-linear connection C on the vector bundle $\tau^* M$, projectable on $\mathcal{D}^*$, which has the horizontal bundle included in the kernel of the splitting S.

Proposition 3.4 *Let S be a left splitting of the canonical injective morphism $\pi^* (\nu^*\mathcal{D}) \stackrel{i'}{\to} \tau(T^*M)$, where $\tau^* M = (T^* M, \pi, M)$. Then:*

a) *A unique non-linear connection C' on $\nu^*\mathcal{D}$ is induced having the property that the fibres of its horizontal bundle are included in the fibres of the vector bundle* $\ker S$ *in every point of the base $\tau^* M$.*

b) *A canonical left $\tau^* M$-F-splitting S'_0 and a left $\nu^*\mathcal{D}$-F-splitting S' of the inclusion $\nu^*\mathcal{D} \stackrel{I_{M'}}{\to} \tau^* M$ are induced, too.*

Remark A Hamilton structure on M gives rise to a (pseudo)metric on $\tau T^* M$ (see [3]). If there is $(\pi^* (\nu^*\mathcal{D}))^{\perp}$ according to this metric, then Proposition 1.2 applies.

Proposition 3.5 *A non-linear connection C on $\tau^* M$ is projectable on a non-linear connection C'' on the vector bundle $\mathcal{D}^*$ iff the vector subbundle $p^*\mathcal{D}^*$ of $V\tau^* M$ is parallel with respect the linear Berwald connection ∇ on $V\tau^* M$, associated with C.*

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