ON SOME CLASSES OF CONFORMALLY FLAT ALMOST HERMITIAN MANIFOLDS

Thomas Toulias and Philippos J. Xenos

Abstract

We prove some new inequalities on the curvatures of conformally flat almost Hermitian manifolds.

AMS Subject Classification: 53C15, 53C25.
Key words: Almost Hermitian manifold, sectional and bisectional curvature, confomally flat manifold.

1 PRELIMINARIES

Let $M^{2n} \equiv (M^{2n}, g, J)$ be a $2n-$dimensional $(n \geq 2)$ almost Hermitian manifold equipped with the almost Hermitian structure (g, J) and Φ the Kaehler form of M^{2n} defined by $\Phi(X,Y) = g(X, JY)$, for $X, Y \in \mathcal{X}(M^{2n})$ ($\mathcal{X}(M^{2n})$ denotes the Lie algebra of all smooth vector fields on M^{2n}). We denote by ∇, R , ρ, Q and S the Riemannian connection, the curvature tensor $(R(X,Y)Z = [\nabla_X, \nabla_Y] Z - \nabla_{[X,Y]}Z)$, the Ricci tensor, the Ricci operator $(\rho(X,Y) = g(QX,Y))$ and the scalar curvature of M^{2n}, respectively.

We denote by ρ^* the $*$Ricci tensor of M^{2n} defined by $\rho^*(x,y) = \frac{1}{2}trace\ of$ $(z \to R(x,Jy)Jz)$ for $x,y,z \in T_p(M^{2n})$, $p \in M^{2n}$. We also denote by S^* the $*$scalar curvature of M^{2n}, which is the trace of the linear endomorphism Q^* defined by $g(Q^*x,y) = \rho^*(x,y)$, for $x,y \in T_p(M^{2n})$, $p \in M^{2n}$. If M^{2n} is a Kaehler manifold, then $\rho = \rho^*$ (and therefore $S = S^*$).

On an almost Hermitian manifold M^{2n} we define the *Weyl tensor* W

$$\begin{aligned} W(X,Y)Z &= R(X,Y)Z - \tfrac{1}{2(n-1)}[g(Y,Z)QX - g(X,Z)QY + \\ &\quad + \rho(Y,Z)X - \rho(X,Z)Y] + \\ &\quad + \tfrac{S}{2(2n-1)(n-1)}[g(Y,Z)X - g(X,Z)Y], \end{aligned}$$

Editor Gr.Tsagas *Proceedings of The Conference of Applied Differential Geometry - General Relativity and The Workshop on Global Analysis, Differential Geometry and Lie Algebras, 2000,* 145-151

for all $X, Y, Z, W \in \mathcal{X}(M^{2n})$.

An almost Hermitian manifold M^{2n} is conformally flat if and only if $W(X,Y)Z \equiv 0$, or

$$\begin{aligned} R(X,Y)Z &= \frac{1}{2(n-1)}[g(Y,Z)QX - g(X,Z)QY + \rho(Y,Z)X - \rho(X,Z)Y] - \\ &-\frac{S}{2(2n-1)(n-1)}[g(Y,Z)X - g(X,Z)Y], \end{aligned} \tag{1}$$

We can define the *bisectional curvature* as

$$k(X,Y,Z,W) = \frac{R(X,Y,Z,W)}{\sqrt{\|X\|^2\|Y\|^2 - g^2(X,Y)}\sqrt{\|Z\|^2\|W\|^2 - g^2(Z,W)}},$$

for all vector fields X, Y, Z and W on M^{2n}. So, we can obtain: the *sectional curvature* $K(X,Y) = k(X,Y,X,Y)$, $\forall X, Y \in \mathcal{X}(M^{2n})$, the *holomorphic sectional curvature* $H(X) = k(X,JX,X,JX)$, $\forall X \in \mathcal{X}(M^{2n})$ and the *holomorphic bisectional curvature* $h(X,Y) = k(X,JX,Y,JY)$, $\forall X, Y \in \mathcal{X}(M^{2n})$.

The bisectional curvature [1] depends only on the plane sections p and q spanned by (X,Y) and (Z,W) respectively (i.e. $p \equiv (X,Y)$ and $q \equiv (Z,W)$), therefore we can use for the bisectional curvature the notation $k_{pq}(\equiv k(X,Y,Z,W))$.

A plane section p is called *holomorphic* if it is spanned by (X,JX) (or $p \equiv Jp$). A plane section p is called *antiholomorphic* if it is spanned by (X,Y) such that $\Phi(X,Y) = 0$, (or $p \perp Jp$).

For sectional, holomorphic sectional and holomorphic bisectional curvatures we have respectively:

$K_p \equiv k_{pp}$, if $p \equiv (X,Y)$,

$H_p \equiv H(X) \equiv k_{pp}$, if $p \equiv (X,JX)$,

$H_{pq} = k_{pq}$, if $p \equiv (X,JX)$ and $q \equiv (Y,JY)$ where p,q are holomorphic plane sections.

If $H(x)$ is constant $f(p)$ $\forall x \in T_p(M^{2n})$ at each point p of M^{2n}, then M^{2n} is said to be of *pointwise constant holomorphic sectional curvature*. Further, if f is constant whole on M^{2n}, then M^{2n} is said to be of *constant holomorphic sectional curvature*.

On an almost Hermitian manifold M^{2n} ($2n \geq 4$), we consider an oriented plane p of the tangent space $T_m(M^{2n})$ at a point $m \in M^{2n}$. We define the *holomorphic deviation* δ_p $(0 \leq \delta_p \leq \pi)$ of p as follows

$$\cos\delta_p = \frac{g(JX,Y)}{\sqrt{\|X\|^2\|Y\|^2 - g^2(X,Y)}},$$

where $\{X,Y\}$ is a basis of p.

On a Riemannian manifold M^n of dimension n we define the *Ricci curvature* as

$$r(X) = \sum_{i=1}^{n} K(X, e_i)$$

where X is a unit vector and $\{e_1, ..., e_n\}$ is an orthonormal basis. So, we have

$$r(X) = \frac{\rho(X,X)}{\|X\|^2}, \quad \forall X \in T_p(M^n), \quad \forall p \in M^n.$$

Through this paper, we assume that all manifolds are connected and smooth and further that all quantities on manifolds are smooth, unless otherwise specified.

2 ON CURVATURES OF CONFORMALLY FLAT ALMOST HERMITIAN MANIFOLDS

One can state the following question: *If one of the curvature tensors of a manifold is absolutely less than or equal to a real constant what can we conclude about the values of the other curvature tensors?*

We try to reply in this question for $CF - manifolds$, in this section.

Proposition 1 *Let M^{2n} be a $CF - manifold$. If $|H_p| \leq f$ for every plane section p of M^{2n}, then we have*

$$|S| \leq 2n(2n-1)f, \qquad |S^*| \leq 2nf, \qquad |r + r \circ J| \leq 2(2n-1)f,$$

$$|k_{pq} + k_{JpJq}| \leq 4\frac{7n-3}{n-1}f, \quad |H_{pq}| \leq 2\frac{7n-3}{n-1}f, \quad |K_p + K_{Jp}| \leq 2\frac{7n-3}{n-1}f,$$

$$|k_{pJp}| \leq \frac{f}{n-1}\cos\delta_p[n(\cos\delta_p + 4) - 2] \leq \frac{5n-2}{n-1}f.$$

Proof. The definition of the holomorphic sectional curvature and (1) yield

$$\rho(X,X) + \rho(JX,JX) = [2(n-1)H(X) + \frac{S}{2n-1}]\,\|X\|^2 \tag{2}$$

for every vector field X on M^{2n}. From the last relation we obtain

$$|\rho(X,X) + \rho(JX,JX)| \leq [2(n-1)f + \frac{|S|}{2n-1}]\,\|X\|^2\,. \tag{3}$$

If $X \in (e_i)_{i=1}^{2n}$ and summing the results we have

$$|S| \leq 2n(2n-1)f. \tag{4}$$

Using (1) we have for arbitrary vector fields X, Y on M^{2n}

$$\begin{aligned} \rho^*(X,Y) &= \sum_{i=1}^{2n} g(R(X,Je_i)JY,e_i) \\ &= \frac{1}{2(n-1)}[g(QX,Y) + g(QJY,JX)] - \frac{S}{2(n-1)(2n-1)}g(X,Y). \end{aligned}$$

This relation can be written in the following form

$$\rho^*(X,Y) = \frac{1}{2(n-1)}[\rho(X,Y)+\rho(JX,JY)] - \frac{S}{2(n-1)(2n-1)}g(X,Y). \tag{5}$$

Let $(e_i)_{i=1}^{2n} = \{e_1,...,e_n,Je_1,...,Je_n\}$ be an orthonormal frame. If $X = Y = e_i$ the last mentioned relation, summing with respect to i, yields

$$S^* = \frac{1}{2n-1}S. \tag{6}$$

This relation and (3) yield

$$|S^*| \le 2nf. \tag{7}$$

From (2) and (3) we have

$$|r + r\circ J| \le 2(2n-1)f. \tag{8}$$

From the definition of the Ricci curvature we have for arbitrary vector fields X and Y on M^{2n}

$$2\rho(X,Y) = r(X+Y)\left\|X+Y\right\|^2 - r(X)\left\|X\right\|^2 - r(Y)\left\|Y\right\|^2. \tag{9}$$

The last relation implies

$$\begin{aligned} 2[\rho(X,Y)+\rho(JX,JY)] &= [r(X+Y)+r(JX+JY)]\left\|X+Y\right\|^2 - \\ &\quad -[r(X)+r(JX)]\left\|X\right\|^2 - [r(Y)+r(JY)]\left\|Y\right\|^2. \end{aligned}$$

The above equation because of (4) yields

$$|\rho(X,Y)+\rho(JX,JY)| \le 2(2n-1)f[\left\|X\right\|^2 + \left\|Y\right\|^2 + g(X,Y)], \tag{10}$$

for all vector fields X, Y on M^{2n}. Using (1), (3) and (6) we obtain

$$\begin{aligned} &|R(X,Y,Z,W) + R(JX,JY,JZ,JW)| \\ \le\ & \frac{2n-1}{n-1}f\{|g(X,Z)|\ [\left\|Y\right\|^2 + \left\|W\right\|^2 + g(Y,W)] + \\ &+|g(Y,Z)|\ [\left\|X\right\|^2 + \left\|W\right\|^2 + g(X,W)] + \\ &+|g(Y,W)|\ [\left\|X\right\|^2 + \left\|Z\right\|^2 + g(X,Z)] + \\ &+|g(X,W)|\ [\left\|Y\right\|^2 + \left\|Z\right\|^2 + g(Y,Z)]\} + \\ &+\frac{2n}{n-1}f[|g(X,Z)g(Y,W)| + |g(Y,Z)g(X,W)|]. \end{aligned} \tag{11}$$

Since

$$[g(X,Y)]^2 \le \left\|X\right\|^2\left\|Y\right\|^2, \quad \forall X,Y \in \mathcal{X}(M^{2n}), \tag{12}$$

we have $\forall X,Y,Z,W \in \mathcal{X}(M^{2n})$

$$|R(X,Y,Z,W) + R(JX,JY,JZ,JW)| \le 4\frac{7n-3}{n-1}f, \tag{13}$$

or

$$|k_{pq} + k_{JpJq}| \leq 4\frac{7n-3}{n-1}f. \tag{14}$$

For arbitrary holomorphic plane sections p, q the above relation implies

$$|H_{pq}| \leq 2\frac{7n-3}{n-1}f. \tag{15}$$

From (7) we obtain

$$|R(X,Y,X,Y) + R(JX,JY,JX,JY)| \leq 2\frac{7n-3}{n-1}f, \tag{16}$$

for orthonormal vector fields X, Y, Z, W. The last relation yields

$$|K_p + K_{Jp}| \leq 2\frac{7n-3}{n-1}f. \tag{17}$$

Finally, from (7) we obtain

$$\begin{aligned} |R(X,Y,JX,JY)| &\leq \frac{2n-1}{n-1}f\,|\Phi(X,Y)|\,[\|X\|^2 + \|Y\|^2] + \\ &+ \frac{n}{n-1}\Phi^2(X,Y). \end{aligned}$$

This relation yields the last relation of the proposition. □

Considering some special classes of the almost Hermitian manifolds of the proposition 1, we obtain better results. So, we have the following propositions:

Proposition 2 *Let M^{2n} be a $CF-manifold$ with non-negative Ricci curvature. If $|H_p| \leq f$ for every plane section p of M^{2n}, then we have*

$$|k_{pq}| \leq 4\frac{7n-3}{n-1}f, \quad |K_p| \leq 2\frac{7n-3}{n-1}f,$$

$$|k_{pJp}| \leq \frac{f}{2(n-1)}\,|\cos\delta_p|\,[n(\cos\delta_p + 4) - 2] \leq \frac{5n-2}{2(n-1)}f.$$

Proof. Since M^{2n} has non-negative Ricci curvature (4) takes the form

$$0 \leq r \leq 2(2n-1)f. \tag{18}$$

Because of the non-negativeness of the Ricci tensor and the relations (5), (8) we obtain

$$|\rho(X,Y)| \leq 2(2n-1)f[\|X\|^2 + \|Y\|^2 + g(X,Y)], \tag{19}$$

for arbitrary vector fields X and Y on M^{2n}. Using the relation $|S| \leq 2n(2n-1)f$, (1) and (**??**) we have

$$\begin{aligned}
&|R(X,Y,Z,W)| \\
\leq\ & \frac{2n-1}{n-1} f\{|g(X,Z)|\ \ [\|Y\|^2 + \|W\|^2 + g(Y,W)] + \\
&+ |g(Y,Z)|\ \ [\|X\|^2 + \|W\|^2 + g(X,W)] + \\
&+ |g(Y,W)|\ \ [\|X\|^2 + \|Z\|^2 + g(X,Z)] + \\
&+ |g(X,W)|\ \ [\|Y\|^2 + \|Z\|^2 + g(Y,Z)]\} + \\
&+ \frac{2n}{n-1} f[|g(X,Z)g(Y,W)| + |g(Y,Z)g(X,W)|],
\end{aligned}$$

for arbitrary vector fields X, Y, Z and W on M^{2n}. The above relation, using the same way of the proof of the proposition 2, leads us to the seeking results. □

Following the proof of the proposition 1 and the $J-$invariance of the Ricci tensor we can prove the following result.

Proposition 3 *Let M^{2n} be a $CF-manifold$ with $J-$invariant Ricci tensor. If $|H_p| \leq f$ for every plane section p of M^{2n}, then we have*

$$|r| \leq (2n-1)f, \quad |k_{pq}| \leq 2\frac{7n-3}{n-1} f, \quad |K_p| \leq \frac{7n-3}{n-1} f.$$

Proposition 4 *Let M^{2n} be a $CF-manifold$. If $|r| \leq f$, then we have*

$$|k_{pq}| \leq 2\frac{7n-3}{(n-1)(2n-1)} f,$$

$$|K_p| \leq \frac{7n-3}{(n-1)(2n-1)} f, \quad |H_p| \leq \frac{3n-1}{(n-1)(2n-1)} f,$$

$$|k_{pJp}| \leq \frac{f}{n-1} \cos\delta_p (\frac{n}{2n-1} \cos\delta_p + 1) \leq \frac{3n-1}{(n-1)(2n-1)} f.$$

Proof. From the definition of the holomorphic sectional curvature and the conformally flatness we obtain

$$H(X) = \frac{1}{2(n-1)} [\rho(X,X) + \rho(JX,JX) - \frac{S}{2n-1}], \tag{20}$$

for an arbitrary vector field X on M^{2n}. Because of the assumption the above relation yields

$$|H_p| \leq \frac{1}{n-1} (f + \frac{|S|}{2(2n-1)}). \tag{21}$$

From $|r| \leq f$ we have $|S| \leq 2nf$, so (10) implies the second of the seeking result.

The assumption $|r| \leq f$ using (5) yields

$$|\rho(X,Y)| \leq f[\|X\|^2 + \|Y\|^2 + g(X,Y)], \tag{22}$$

for arbitrary vector fields X and Y on M^{2n}. The above relation because of (1) and $|S| \leq 2nf$ implies

$$\begin{aligned} & |R(X,Y,Z,W)| \\ \leq\ & \frac{1}{2(n-1)} f\{|g(X,Z)|\ \ [\|Y\|^2 + \|W\|^2 + g(Y,W)] + \\ & + |g(Y,Z)|\ \ [\|X\|^2 + \|W\|^2 + g(X,W)] + \\ & + |g(Y,W)|\ \ [\|X\|^2 + \|Z\|^2 + g(X,Z)] + \\ & + |g(X,W)|\ \ [\|Y\|^2 + \|Z\|^2 + g(Y,Z)]\} + \\ & + \frac{n}{(n-1)(2n-1)} f[|g(X,Z)g(Y,W)| + |g(Y,Z)g(X,W)|], \end{aligned}$$

for all vector fields X, Y, Z and W on M^{2n}. The above inequality (as in the proposition 1) yields the seeking results. □

References

[1] Goldberg S.I. and Kobayashi S, *Holomorphic bisectional curvature*, J. Diff. Geometry 1 (1967) 225-233.

Authors' address:

Thomas Toulias and Philippos J. Xenos

Division of Mathematics, Department of Mathematics and Physics
School of Thechnology, Aristotle University of Thessaloniki
Thessaloniki, 540 06, GREECE