# ON THE HARMONIC AND KILLING TENSOR FIELDS ON A COMPACT RIEMANNIAN MANIFOLD

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Dedicated to Prof. Udriste on his sixtieth birthday.
I wish him and his family to be blessed always by God.

### Abstract

Let $(M, g)$ be an orientable and compact Riemannian manifold. The aim of the present paper is to study the vector spaces $H^q(M, \Re)$ and $K^q(M, \Re)$ of harmonic $q$-forms and Killing tensor fields of order $q$ on a compact Riemannian manifold $M$, where $q = 2, 3, ..., n-2$ and $n = \dim M$.

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**Key words:** Compact Riemannian manifold, harmonic $q$-form, Killing tensor field of order $q$, Ricci tensor field, Riemannian tensor field, special quadratic form.

## 1 Introduction

Let $(M, g)$ be a compact Riemannian manifold of dimension $n$. We denote by $K^q(M, \Re)$ the vector space of Killing tensor fields of order $q$ and $H^q(M, \Re)$ the vector space of harmonic $q$-forms on $M$, $q = 2 \ldots, n-2$. The purpose of the present paper is to study the vector spaces $K^q(M, \Re)$ and $H^q(M, \Re)$.

The whole paper contains five sections. Each of them is analyzed as follows: The first section is the Introduction. Special tensor fields on a compact Riemannian manifolds are included in the second section. The Killing tensor fields on a Riemannian manifold are studied in the third section. The fourth section contains harmonic $q$-forms on a compact Riemannian manifold. Some topological invariants of a compact manifold are studied in the last section.

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## 2 Tensor fields on a manifold

Let $(M, g)$ be a compact Riemannian manifold of dimension $n$. We consider an atlas $(U_\alpha, \phi_\alpha)_{\alpha \in A}$ , where $(U_\alpha, \phi_\alpha)$ is a chart on $M$ with local coordinate system $\{x^1_\alpha, ..., x^n_\alpha\}$.

Let $w$ be a $q$-form on $M$, that is $w \in \Lambda^q(M)$. Therefore $w$ on the chart $(U, \phi)$ with local coordinate system $\{x^1, ..., x^n\}$ can take the form

$$w = \frac{1}{q!} w_{i_1 i_2 \dots i_q} dx^{i_1} \wedge dx^{i_2} \wedge ... \wedge dx^{i_q}, \tag{1}$$

where $1 \leq i_1 < i_2 < \cdots < i_q < n$ and $w_{i_1 \dots i_q}$ are the components of $w$ on the chart $(U, \phi)$.

The local norm of $w$ is defined by

$$|w|^2 = \frac{1}{q!} w_{i_1 \dots i_q} w^{i_1 \dots i_q}, \tag{2}$$

where

$$w^{i_1 \dots i_q} = g^{i_1 j_1} ... g^{i_q j_q} w_{i_1 \dots i_q} \ . \tag{3}$$

This $q$-form $w$, by means of (2), gives a function $|w|^2$ on $M$, that means $|w|^2 \in D^0(M)$. Hence we have

$$\frac{1}{2} \Delta \left( |w|^2 \right) = < \delta \Delta w, w > - |\nabla w|^2 . \tag{4}$$

The formula (4) by integration implies

$$\int\limits_M \left[ < \delta \Delta w, w > - |\nabla w|^2 \right] dM, \tag{5}$$

where $dM$ is the volume element of $(M, g)$.

If $w \in \Lambda^q(M, \Re)$, then $\Delta w \in \Lambda^q(M)$, where $\Delta$ is the Laplace operator acting on the vector space $\Lambda^q(M, \Re)$ as a linear operator. On the chart $(U, \phi)$ with local coordinate system we have

$$\begin{aligned} \Delta w \quad = \quad & \frac{1}{q!} \Bigg[ g^{i\,j} \nabla_j \nabla_i w_{i_1 \dots i_q} - \sum_{s=1}^{q} p^{\nu}_{i_s} w_{i_1 \dots\, i_{s-1}\, \nu\ i_{s+1} \dots\, i_q} - \\ & - \sum_{t<s}^{1,\dots,q} R^{\nu u}_{i_t\, i_s} w_{i_1 \dots\, i_{t-1}\, \nu\ i_{t+1} \dots\, i_{s-1}\, u\ i_{s+1} \dots\, i_q} \Bigg] dx^{i_1} \wedge ... \wedge dx^{i_q}. \end{aligned} \tag{6}$$

Hence the components of $\Delta w$ on $(U, \phi)$ have the form

$$
\begin{aligned}
(\Delta w)_{i_1\dots i_q} \quad = \quad & \frac{1}{q!}\left[-g^{i\,j}\nabla_i\nabla_j w_{i_1\dots i_q}+\sum_{s=1}^{q} p^{v}_{i_s} w_{i_1\dots\, i_{s-1}\,\nu\ i_s\dots\, i_q}+\right. \\
& \left. +\sum_{t<s}^{1,\dots,q} R^{u\nu}_{i_t\, i_s} w_{i_1\dots\, i_{t-1}\ u\nu\, i_{t+1}\dots\, i_{s-1}\ \nu\ i_{s+1}\dots\, i_q}\right]. \qquad (7)
\end{aligned}
$$

We obtain the inner product of the q-forms $w$ and $\Delta w$, which takes the form

$$< \Delta w, w >=< \delta\nabla w, w > +\frac{1}{(\phi-1)!}F_q\,(w)\,, \qquad (8)$$

where $F_q\,(w)$ is the following quadrate form

$$F_q\,(w) = p_{i\,j}w^{i\,i_2\dots i_q}w^{j}_{i_2\dots i_q} + \frac{q-1}{2}R_{i\,j\,k\,l}w^{i\,j\,i_3\dots i_q}w^{k\,l}_{i_3\dots i_q} \qquad (9)$$

It can be easily proved the following relation

$$\frac{1}{2}\Delta\left(|w|^2\right) =< \Delta w, w > -\left|\nabla w\right|^2 - \frac{1}{(q-1)!}F_q\,(w,w)\,. \qquad (10)$$

The integration of (10) implies

$$\int\limits_M \left[< \Delta w, w > -\left|\nabla w\right|^2 - \frac{1}{(q-1)!}F_q\,(w,w)\right]dM. \qquad (11)$$

It can be easily obtained the relation

$$< \Delta w-(q+1)\,\delta\nabla w, w >= -q < \delta\nabla w, w > +\frac{1}{(q-1)!}F_q\,(w,w)\,. \qquad (12)$$

The equality (12) by means of (4) becomes

$$\frac{1}{2}\Delta\left(|w|^2\right) = -\left|\nabla w\right|^2 + \frac{1}{q!}F_q\,(w,w) - \frac{1}{q!} < \Delta w-(q+1)\,\delta\nabla w, w >, \qquad (13)$$

which by integration implies

$$\int\limits_M \left[< \Delta w-(q+1)\,\delta\nabla w, w > +q\left|\nabla w\right|^2 - \frac{1}{(q-1)!}F_q\,(w,w)\right]dM = 0. \qquad (14)$$

We use a new expression of the quadrate form $F_q\,(w,w)$, which can be written

$$F_q\,(w,w) = B_{i_1 i_2\dots i_q\,,\,j_1 j_2\dots j_q}w^{i_1 i_2\dots i_q}w^{j_1 j_2\dots j_q}, \qquad (15)$$

where

$$B_{i_1 i_2 \ldots i_q\,,\, j_1 j_2 \ldots j_q} = \left(p_{i_1 j_1} g_{i_2 j_2} + \frac{q-1}{2} R_{i_1 i_2 j_1 j_2}\right) g_{i_3 j_3} \ldots g_{i_q j_q}\,, \tag{16}$$

where the indices satisfy the inequalities

$$1 \preceq i_1, i_2, \ldots, i_q \preceq n \qquad 1 \preceq j_1, j_2, \ldots, j_q \preceq n. \tag{17}$$

**Proposition 1** *The tensor field* $B = \left(B_{i_1 i_2 \ldots i_q\,,\, j_1 j_2 \ldots j_q}\right)$ *on a Riemannian manifold* $(M, g)$ *is symmetric with respect to* $(i_1 i_2, j_1 j_2)$ *and with respect of any of two indices* $(i_\nu, j_\nu)$ $\nu = 3, 4, \ldots q$ *and as well as with respect to* $(i_1, i_2, \ldots i_q)$ *and* $(j_1, j_2, \ldots, j_q)$*. Therefore the quadratic form (15) is symmetric with respect to* $(i_1, i_2, \ldots i_q)$ *and* $(j_1, j_2\,, \ldots, \,, j_q)$*.*

*Proof.* It is known from the properties of the Ricci tensor field $p$, the curvature tensor field $R$ and the metric tensor field $g$, we have the following relations

$$p_{i_1 j_1} = p_{j_1 i_1}\ ,\ \ p_{i_2 j_2} = p_{j_2 i_2}\ ,\ \ R_{i_1 i_2 j_1 j_2} = R_{j_1 j_2 i_1 i_2}, \tag{18}$$

$$g_{i_1 j_1} = g_{j_1 i_1}\ ,\ g_{i_2 j_2} = g_{j_2 i_2}\ ,\ g_{i_3 j_3} = g_{j_3 i_3}\ , \ldots,\ g_{i_q j_q} = g_{j_q i_q}. \tag{19}$$

From (7) and (8) we conclude the first part of the proposition. The other part, that means the symmetric property of the tensor field with respect to the pair

$$(i_1, i_2, \ldots i_q)\ ,\ (j_1, j_2, \ldots, j_q) \tag{20}$$

is a consequence of the same relations (18) and (19). □

Now, we intrude a new quadratic form $\overset{\wedge}{F}_q\ (w, w)$ on the vector space $\Lambda^q\,(M, \Re)$ as follow

$$\overset{\wedge}{F}_q\ (w, w) = B'_{(i_1 \ldots i_q)(j_1 \ldots j_q)} w^{i_1 \ldots i_q} w^{j_1 \ldots j_q}, \tag{21}$$

with the condition that the symbol $(i_1, i_2, \ldots i_q)$ means

$$1 \preceq i_1 < i_2 < \ldots < i_q \preceq n.$$

For this reason we form the tensor field

$$B'_{(i_1 \ldots i_q)(j_1 \ldots j_q)} = B_{k_1 \ldots k_q l_1 \ldots l_q} \delta^{k_1 \ldots k_q}_{i_1 \ldots i_q} \delta^{l_1 \ldots l_q}_{j_1 \ldots j_q}\,, \tag{22}$$

where

$$\delta^{k_1 \ldots k_q}_{i_1 \ldots i_q} \quad \text{and} \quad \delta^{l_1 \ldots l_q}_{j_1 \ldots j_q} \tag{23}$$

are the Kronecker's generalized symbols which are given by the formulas

$$\delta^{k_1 \ldots k_q}_{i_1 \ldots i_q} = \begin{vmatrix} \delta^{k_1}_{i_1} \ldots\ldots \delta^{k_q}_{i_1} \\ \ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots \\ \delta^{k_1}_{i_q} \ldots\ldots \delta^{k_q}_{i_q} \end{vmatrix}, \qquad \delta^{l_1 \ldots l_q}_{j_1 \ldots j_q} = \begin{vmatrix} \delta^{l_1}_{j_1} \ldots\ldots \delta^{l_q}_{j_1} \\ \ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots \\ \delta^{l_1}_{j_q} \ldots\ldots \delta^{l_q}_{j_q} \end{vmatrix}. \tag{24}$$

Now, we can prove the following proposition

**Proposition 2** *The tensor field* $B' = \left(B'_{(i_1...i_q)(j_1...j_q)}\right)$ *is symmetric with respect to* $\{(i_1...i_q)\ ,\ (j_1...,j_q)\}$.

*Proof.* According to the proposition 1 we have

$$B_{k_1...k_q,\,l_1...l_q} = B_{l_1...l_q,\,k_1...k_q}. \tag{25}$$

From the properties of Kronecker's tensor field we conclude that

$$\delta^i_j = \begin{cases} 1 & \text{if} \quad i = j \\ 0 & \text{if} \quad i \neq j \end{cases}, \tag{26}$$

$$\delta^{l_1...l_q}_{j_1...j_q} = \delta^{k_1...k_q}_{i_1...i_q}. \tag{27}$$

From (25), (26) and (27) we have

$$B'_{(i_1...i_q)(j_1...j_q)} = B_{(j_1...j_q)(i_1...i_q)} \tag{28}$$

□

Now, under the introduction of the tensor field $B$, the quadratic form $\overset{\wedge}{F_q}(w,w)$ becomes

$$\overset{\wedge}{F_q}(w,w) = \frac{1}{(q-1)!} B_{(j_1...j_q)(i_1...i_q)} w^{i_1...i_q}\, w^{j_1...j_q}, \tag{29}$$

where the indices satisfy the inequalities

$$1 \leq i_1 < i_2 < ... < i_q \leq n, \quad 1 \leq j_1 < j_2 < ... < j_q \leq n. \tag{30}$$

The relation between the quadratic forms $F_q(w,w)$ and $\overset{\wedge}{F_q}(w,w)$ is the following

$$\overset{\wedge}{F_q}(w,w) = \frac{1}{(q-1)!} F_q(w,w). \tag{31}$$

Now, we can prove the following theorem

**Theorem 3** *Let* $(M,g)$ *be a compact Riemannian manifold of dimension* $n$, $n \succeq 3$. *The nullity of the quadratic form* $\overset{\wedge}{F_q}(w,w)$ *is a global property.*

*Proof.* Let $(U,\phi)$ be a chart on $M$ with local coordinate system $\left(x^1,...,x^n\right)$. We assume that the nullity of $\overset{\wedge}{F_q}(w,w)$ on $(U,\phi)$ is $\kappa$, that is

$$nullity\ \overset{\wedge}{F_q}(w,w) = \kappa. \tag{32}$$

If $(V,\upsilon)$ is another chart on $M$ with local coordinate system $\left(y^1,...,y^n\right)$ such that $U \cap V \neq 0$, then the components $\left\{w'^{l_1...l_q}\right\}$ of $w$ on $V$ are connected with the components $\left\{w^{i_1...i_q}\right\}$ of $w$ on $U$ with the relations

$$w^{\,i_1 \dots i_q} = \frac{\partial x^{i_1}}{\partial y^{l_1}} \dots \frac{\partial x^{i_q}}{\partial y^{l_q}} \acute{w}^{l_1 \dots l_q}. \tag{33}$$

The relations (33) are valid on $U \cap V$ and the local coordinates $\left(x^1, ..., x^n\right)$ and $\left(y^1, ..., y^n\right)$ are connected by the relations

$$x^1 = x^1\left(y^1, ..., y^n\right) ... x^n = x^n\left(y^1, ..., y^n\right). \tag{34}$$

The quadratic form $\overset{\wedge}{F}_q\,(w, w)$ on the local coordinate system $\left(y^1, ..., y^n\right)$ and by meaning of (33) and (34) takes the expression

$$\overset{\wedge}{F}_q\,(\acute{w}, \acute{w}) = P \cdot Q, \tag{35}$$

where

$$P = \acute{w}^{\,l_1 \dots l_q} \frac{\partial x^{i_1}}{\partial y^{l_1}} \dots \frac{\partial x^{i_q}}{\partial y^{l_q}}, \quad Q = \frac{\partial x^{j_1}}{\partial y^{m_1}} \dots \frac{\partial x^{j_q}}{\partial y^{m_q}} \acute{w}^{m_1 \dots m_q}. \tag{36}$$

The change of the local coordinate system $\left(x^1, ..., x^n\right)$ on the chart $(U, \phi)$ to the $\left(y^1, ..., y^n\right)$ on $(V, \upsilon)$ brings a change into the base

$$\left\{dy^{i_1} \wedge dy^{i_2} \wedge ... \wedge dy^{i_q}\right\} \tag{37}$$

of the vector space $\Lambda^q\left(U \cap V,\ \Re\right)$, which is determined by the relation

$$dx^{i_1} \wedge dx^{i_2} \wedge ... \wedge dx^{i_q} = \frac{\partial x^{i_1}}{\partial y^{l_1}} \frac{\partial x^{i_2}}{\partial y^{l_2}} \dots \frac{\partial x^{i_q}}{\partial y^{l_q}} dy^{l_1} \wedge dy^{l_2} \wedge ... \wedge dy^{l_q}. \tag{38}$$

This expression implies

$$rank\left(\frac{\partial x^{i_1}}{\partial y^{l_1}} \frac{\partial x^{i_2}}{\partial y^{l_2}} \dots \frac{\partial x^{i_q}}{\partial y^{l_q}}\right) = \left(\begin{array}{c} n \\ q \end{array}\right). \tag{39}$$

From (35) by means of (39) implies that the nullity of the quadratic form $\overset{\wedge}{F}_q\,(w, w)$ is constant and equal $\kappa$ on the whole manifold. □

## 3 Killing tensor fields

Let $(M, g)$ be a compact Riemannian manifold of dimension $n$. Let $T$ be an antisymmetric tensor field of type $(0, q)$. It is known that for $T$ we can associate an exterior $q$-form $w$. This exterior $q$-form $w$ is called *Killing* if it satisfies the relation

$$(q + 1)\,\nabla w = dw. \tag{40}$$

The relation (40) implies that the tensor field $\nabla w$ is antisymmetric and also

$$\delta w = 0. \tag{41}$$

The tensor field $T$ is called *Killing* if the associated exterior $q$-form $w$ is Killing..

Let $(U,\phi)$ be a chart on $(M,g)$ with local coordinate system $\left(x^1,...,x^n\right)$. Let $\left\{w_{i_1...i_q}\right\}$ be the components of $w$ on $(U,\phi)$. If $w$ is a Killing, then $\left\{w_{i_1...i_q}\right\}$ satisfy the relations

$$g^{ji}\nabla_j\nabla_i w_{i_1...i_q} + \frac{1}{q}\sum_{s=1}^{n} p^{v}_{i_s} w_{i_1...i_{s-1}\,v\,i_{s+1}...i_q} +$$

$$\frac{1}{q}\sum_{t<s}^{1...q} R^{vu}_{i_t\,i_s} w_{i_1...i_{t-1}\,v\,i_{t+1}...i_{s-1}\,u\,i_{s+1}...i_q} = 0 \tag{42}$$

and

$$g^{i\,j}\nabla_j w_{i_1 i_2...i_q} = 0 \tag{43}$$

on the chart $(U,\phi)$.

If $w$ is a Killing exterior $q$-form, the formula (11) takes the form

$$\int_M \left[ q\ \left|\nabla w\right|^2 - \frac{1}{(q-1)!}F_q\left(w,w\right)\right] dM. \tag{44}$$

Now, we can prove the following theorem

**Theorem 4** *Let $(M,g)$ be a compact orientable Riemannian manifold of dimension $n$. We assume that the quadratic form $\overset{\wedge}{F_q}\left(w,w\right)$ is semi-negative on the whole manifold $M$. We also assume that on one chart of $M$ the nullity of $\overset{\wedge}{F_q}\left(w,w\right)$ is equal to the number of linearly independent parallel exterior forms of order $q$.*

*Then the dimension of $K_q\left(M,\Re\right)$ of Killing exterior $q$-forms on $M$ is given by*

$$\dim\, K_q\left(M,\Re\right) = \textit{nulity}\ \ \overset{\wedge}{F_q}\left(w,w\right) \tag{45}$$

*Proof.* Let $w$ be a Killing exterior form of order $q$. Under the assumption, we have that the quadratic form $\overset{\wedge}{F_q}\left(w,w\right)$ is semi-definite on the whole manifold. From the formula (44) we obtain

$$\nabla w = 0 \qquad \overset{\wedge}{F_q}\left(w,w\right) = 0. \tag{46}$$

It is known that if $f \in D^o\left(M\right)$ and $\Delta f \succeq 0$ or $\Delta f \preceq 0$, then $f = c$.

From the relation (8), the equation $\Delta w = (q+1)\ \delta\,\nabla w$ and the above remark we conclude that

$$\left|w\right|^2 = c, \quad c = \text{constant}. \tag{47}$$

We assume that on the chart $(U,\phi)$ of the manifold $M$ with local coordinate system $\left(x^1...x^n\right)$ the nullity of the quadratic form $\overset{\wedge}{F_q}\left(w,w\right)$ is equal to $\kappa$ and there are $\kappa$

linearly independent different than zero exterior $q$-forms, $w^{(\beta)}_{(i_1 i_2 \dots i_q)}$ $\beta = 1, 2, ..., \kappa$, which satisfy the relations (46). These exterior $q$-forms are parallel and also Killing.

Let $w^{(\kappa+1)}$ be another parallel exterior $q$-form different from zero, which satisfies the relation

$$\overset{\wedge}{F}_q \left( w^{(\kappa+1)}, w^{(\kappa+1)} \right) = 0. \tag{48}$$

We shall prove that the exterior $q$-form $w^{(\kappa+1)}$ can be written in a unique manner as a linear combination of $w^{(\beta)}_{(i_1 i_2 \dots i_q)}$ $\beta = 1, 2, ..., \kappa$ with constant coefficients on the chart $(U, \phi)$.

We assume that

$$w^{(\kappa+1)}_{(i_1 i_2 \dots i_q)} = \sum_{\beta=1}^{\kappa} \phi_\beta \, w^{(\beta)}_{(i_1 i_2 \dots i_q)}, \tag{49}$$

where $\phi_\beta$ are functions of $\left(x^1, x^2, ..., x^n\right)$. If we apply the operator of covariant differentiation $\nabla_j$ on the relation (49), we obtain for every $j$ the system

$$\sum_{\beta=1}^{\kappa} \partial_j \phi^{(\beta)}_{\beta \, (i_1 i_2 \dots i_q)} = 0, \tag{50}$$

which has a number of $\begin{pmatrix} n \\ q \end{pmatrix}$ equations with $\kappa$ unknown

$$\partial_j \phi_1, \partial_j \phi_2, ..., \partial_j \phi_\kappa \, . \tag{51}$$

From our assumption we know that the degree of the matrix

$$\left\{ w^{(\beta)}_{(i_1 i_2 \dots i_q)} \right\} \tag{52}$$

of the coefficient of the unknown is equal to $\kappa$

Therefore for every point of $U$, there exists at least one determinant of order $\kappa$ different than zero, the referred above matrix. The corresponding homogeneous system, which has as matrix of the coefficients the determinant which is different than zero. Hence, we obtain

$$\partial_j \phi_\beta = 0 \quad (\beta = 1, 2, ...\kappa) \quad \text{for every} \;\; j = 1, 2, \dots, n, \tag{53}$$

which imply

$$\phi_\beta = c_\beta \quad \beta = 1, 2, ...\kappa \tag{54}$$

Hence, we have

$$w^{(\kappa+1)}_{(i_1 i_2 \dots i_q)} = \sum_{\beta=1}^{\kappa} c_\beta \, w^{(\beta)}_{(i_1 i_2 \dots i_q)}. \tag{55}$$

On the other hand, by means of the formulas of changing the local coordinate system we easily conclude that the linear connection (55) with constant coefficients is valid on the whole manifold.

Finally, since every exterior $q$-form $w^{(\beta)}$ satisfies the relation

$$\left| w^{(\beta)} \right| = c_\beta \ , \ (c_\beta \neq 0) \tag{56}$$

and taking under consideration the theorem 3, we conclude that the exterior $q$-form $w^{(\beta)}$, $\beta = 1, 2, ...\kappa$ are the only non-zero linearly independent Killing exterior $q$-form on the whole manifold. □

# 4 Harmonic $q$-forms

Let $w$ be an exterior $q$-form. This is called *harmonic* if it satisfies the relations

$$dw = 0, \quad \delta w = 0, \tag{57}$$

or equivalently

$$\Delta w = 0. \tag{58}$$

Let $(U, \phi)$ be a chart on the manifold with local coordinate system $\left(x^1 ... x^n\right)$. Let $\left\{ w_{i_1 \ldots i_q} \right\}$ be the components of $w$ on $U$. If $w$ is harmonic, then its components satisfy the conditions

$$g^{ji} \nabla_j \nabla_i w_{i_1 \ldots i_q} - \sum_{s=1}^{q} R^{v}_{i_s} w_{i_1 \ldots i_{s-1}\, v\, i_{s+1} \ldots i_q} - \sum_{t<s}^{1 \ldots q} R^{vu}_{i_t\, i_s} w_{i_1 \ldots i_{t-1}\, v\, i_{t+1} \ldots i_{s-1}\, u\, i_{s+1} \ldots i_q} = 0. \tag{59}$$

The integral formula (14), if $w$ is a harmonic form and by means of (59), takes the form

$$\int_M \left[ |\nabla w|^2 + \frac{1}{(q-1)!} F_q(w, w) \right] dM = 0. \tag{60}$$

The following results are known. If $F_q(w, w)$ is semi-definite on the whole manifold, then every harmonic form is parallel. If $F_q(w, w) > 0$, then there exists no harmonic form and the $q$ Betti number is zero, then

$$b_q(M) = 0. \tag{61}$$

Now, we can prove the following theorem

**Theorem 5** *Let $(M, g)$ be an orientable and compact Riemannian manifold of dimension $n$. We assume that the quadratic form $\overset{\wedge}{F_q}(w, w)$ is semi-positive on the whole manifold $M$ and the same time the nullity of $F_q(w, w)$ is equal with the number* *of linearity independent parallel exterior q-forms, then q Betti number of M is equal to the nullity of* $\overset{\wedge}{F_q}(w,w)$*, that is* $b_q(M) = \text{ nullity } \left(\overset{\wedge}{F_q}(w,w)\right)$.

*Proof.* From the assumption the $\overset{\wedge}{F_q}(w,w)$ is semi-positive on the whole manifold $M$, we conclude, by the meaning of the formula of (60), the relations

$$\overset{\wedge}{F_q}(w,w) = 0 \ , \ \nabla w = 0. \tag{62}$$

From the equation (10), using the known result, if $f \in D^o(M)$ and $\Delta f \succeq 0$ implies $f =$ constant, we conclude that

$$|w|^2 = c.$$

If we use the same method as in the theorem 3 we conclude that there are $\kappa$ linearly independent parallel exterior $q$-forms, which are the only harmonic $q$-forms on the manifold. Therefore we have the equality

$$\dim\ H^q(M,\Re) = b_q = \kappa.$$

□

## 5 Determination of the Betti numbers

Let $M$ be an orientable and compact manifold. It is known that the $q$-Betti number $b_q$ is a topological invariant. One problem of algebraic topology is to determine $b_q(M)$, $q = 1, 2, ..., n-1$, where $n = \dim M$. In some cases we use Riemannian metrics on $M$ to determine $b_q$, $q = 1, 2, ..., n-1$.This method connects Algebraic Topology and Differential Geometry.

Let $H(M)$ be the set of all Riemannian metrics on $M$. It is known that $H(M)$ is a Banach space of infinite dimension. Let $g \in H(M)$ be a Riemannian metric on $M$. If $w \in \Lambda^q(M)$, then we can form

$$\overset{\wedge}{F_q}(w,w). \tag{63}$$

If

$$\overset{\wedge}{F_q}(w,w) \succeq 0 \quad \text{and} \qquad \nabla w = 0, \tag{64}$$

then $w$ is parallel and also harmonic. If there are $\kappa$ exterior $q$-forms

$$w_1,\ w_2,\ ...,\ w_\kappa$$

such that they satisfy

$$\overset{\wedge}{F_q}(w_1,w_1) \succeq 0\ \ \nabla w_1 = 0,\ \ \overset{\wedge}{F_q}(w_2,w_2) \succeq 0\ \ \nabla w_2 = 0, ....., \overset{\wedge}{F_q}(w_\kappa,w_\kappa) \succeq 0\ \ \nabla w_\kappa = 0, \tag{65}$$

then $b_q = \kappa$. Thus we have the following result.

**Proposition 6** *Let $M$ be an orientable and compact manifold of dimension $n$. We assume that there are $\kappa$ exterior $q$-forms $w_1$, $w_2$, ..., $w_\kappa$ and a Riemannian metric $g$ such that these forms are parallel with respect to the Levi-Civita connection $\nabla$ and also $\overset{\wedge}{F}_q\,(w_j, w_j) \geq 0, \;\; j = 1, ..., \kappa$. Then $b_q = \kappa$*

The above proposition permits to determine the Betti numbers on an orientable and compact Riemannian manifold.

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