UNIFICATION OF THE LIE ALGEBRAS $u(p,q)$

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Abstract

Let $u(p,q)$ be the known Lie algebras where $p+q=n$, $p,\ q=0,1,\ldots,\ n$. The aim of the present paper is to prove that these can be unified by means of Santilli's ideas.

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1 Introduction

We consider the vector space $\mathbf{C}^n$, on which we can consider different inner products. By using these inner products we obtain the Lie algebra $u(p,q)$ where $p+q=n$ and $p,q=0,1,\ldots,n$. It is an open problem to study the isomorphic classes of this Lie algebras ([ ]). The aim of the present paper is to study some of these Lie algebra by using Santilli's theory. The whole paper contains five sections.

The first section is the introduction. The description of the Lie algebra $u(p,q)$ is given in the second section. If also contain the Lie algebras $u^d(p,q)$, $\hat{u}(p,q)$ and $\hat{u}^d(p,q)$.

The study of $u(p,q)$, where $p+q=2$ and other cases are studied in the third section.

The fourth section contains the Lie algebra $u(p,q)$, where $p+q=3$ and other Lie algebras with the same condition Finally, the Lie algebra $u(p,q)$, where $p+q=4$ and the other Lie algebras with the same condition are studied in the last section.

2 A description of the Lie algebra $u(p,q)$

Let $\mathbf{C}^n$ be the complex Cartesian space, with coordinates $(z_1,z_2,\ldots,z_n)$ of a vector space over $\mathbf{C}$ of dimension n. If $z\in\mathbf{C}^n$, then $z=(z_1,z_2,\ldots,z_n)$. We construct the following sets:

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$$(I) \quad GL(n,C) = \{Agl(n,C)/Det(A)0\}, \tag{2.1}$$

$$(II) \quad SL(n,C) = \{AGL(n,C)/Det(A) = 1\}, \tag{2.2}$$

$$(III) \quad U(p,q) = \{AGL(n,C)/A \text{ preserves the hermitian product :}$$

$$(z,\bar{z}) = -z_1\bar{z}_1 - z_2\bar{z}_2 - \cdots - z_p\bar{z}_p + z_{p+1}\bar{z}_{p+1} + \cdots + z_{p+q}\bar{z}_{p+q}\}. \tag{2.3}$$

We put

$$U(n) = U(n,0) = U(0,n), \tag{2.4}$$

$$\begin{aligned} (IV)\ SU(p,q) &= U(p,q)SL(n,C), \\ SU(n) &= U(n)SL(n,C), \end{aligned} \tag{2.5}$$

$$(V)\ \ S(U_p \times U_q) = \left\{ \begin{pmatrix} \mathrm{A}_1 & 0 \\ 0 & \mathrm{A}_2 \end{pmatrix} / A_1 \in U(p), A_2 \in U(q), detA_1 \cdot detA_2 = 1 \right\}. \tag{2.6}$$

The above sets are Lie groups. The Lie algebras of $U(p,q)$ and $SU(p,q)$ have the form:

$$U(p,q) = \left\{ \begin{pmatrix} A_1 & A_2 \\ {}^{\mathrm{t}}\bar{A}_2 & A_3 \end{pmatrix} / \begin{array}{c} A_1,\, A_3 \text{ skew Hermitian of order } p \text{ and } q \\ \text{respectively and } A_2 \text{ is an arbitrary } q \times p \text{ matrix} \end{array} \right\}, \tag{2.7}$$

$$SU(p,q) = \left\{ \begin{pmatrix} A_1 & A_2 \\ {}^{t}\bar{A}_2 & A_3 \end{pmatrix} / \begin{array}{c} A_1,\, A_3 \text{ skew Hermitian, of order } p \text{ and } q \\ \text{respectively, } Tr\,A_1 \ + \ TrA_3 = 0 \text{ , } A_2 \text{ arbitrary} \end{array} \right\}, \tag{2.8}$$

Let g be a Lie algebra over a field $\mathbf{F}$ of characteristic zero. We assume that $\dim g = n$.

On g there is the Lie bracket $[\cdot,\cdot]$, that is

$$[\cdot,\cdot] : g \times g \to g, [\cdot,\cdot] : (x,y) \to [x,y], \tag{2.9}$$

having the following two properties :

$$(I)\ [x,y] = -[y,x], \ (II)\ [x,[y,z]] + [z,[x,y]] + [y,[z,x]] = 0. \tag{2.10}$$

Now, we consider on g another internal law determined by the element $T \in g$, denoted by $[\cdot,\cdot]_T$ and defined by:

$$[X,Y]_T = XTY - YTX, \ (\forall)X,Y \in g. \tag{2.11}$$

It can be easily proved that $[x,y]_T$ has the same properties as the Lie bracket (2.10). Then g with $[\cdot,\cdot]_T$ is the *Lie-Santili algebra*.

Let g and a Lie algebra over a Field of characteristic zero. The dimension of g is n. From g and using en element $\hat{I}$ we can construct the following set $\hat{g}$:

$$\hat{g} = \left\{ \hat{x} = x\hat{I}/x \in g \right\}.$$

It is possible $\hat{I}$ does not belong into g, however the product $x\hat{I}$ is defined. The set $\hat{g}$ with the usual Lie bracket, becomes a Lie algebra.

Now we can define $u^d(p,q)$, $\hat{u}(p,q)$ and $\hat{u}^d(p,q)$ as follows:

$$u^d(p,q) = u(p,q)T\hat{u}(p,q) = u(p,q)T_{p,q},\ \hat{u}^d(p,q) = \hat{u}(p,q)\hat{I}_{p,q}, \tag{2.12}$$

where

$$\begin{aligned} T_{\bar{p},q} &= \hat{I}_{\bar{p},q} = \left(\begin{array}{cccccc} & -1 & & & & \\ p\left\{\right. & & \ddots & & & \\ & & & -1 & & \\ & & & & 1 & \\ q\left\{\right. & & \ddots & & & \ddots \\ & & & & & 1 \end{array}\right), \\ T_{p,\bar{q}} &= \left(\begin{array}{cccccc} & b_1^2 & & & & \\ p\left\{\right. & & \ddots & & & \\ & & & b_p^2 & & \\ & & & -b_{p+1}^2 & & \\ q\left\{\right. & & \ddots & & \ddots & \\ & & & & & -b_{p+q}^2 \end{array}\right), \end{aligned} \tag{2.13}$$

which are isodual Lie algebra.

3 The Lie algebras $u(p,q)$, $p+q=2$

Now, we consider the Lie algebras

$$u(p,q),\ p+q=2. \tag{3.1}$$

Then (3.1) gives the following cases:

$$u(2,0), u(1,1), u(0,2). \tag{3.2}$$

From (3.2) we obtain the following Lie algebra :

$$\begin{aligned} &u^d(2,0),\ \hat{u}(2,0),\ \hat{u}^d(2,0),\ u^d(1,1),\ \hat{u}(1,1),\ \hat{u}^d(1,1), \\ &u^d(0,2),\ \hat{u}(0,2),\ \hat{u}^d(0,2). \end{aligned} \tag{3.3}$$

Now we prove the following result:

Theorem 3.1 *The following relations hold:*

$$\hat{u}\,(2,0) \cong u(2,0),\ \hat{u}\,(1,1) \cong u\,(1,1)\,,\ \hat{u}^{d}\,(2,0) \cong u\,(2,0)\,,\ \hat{u}^{d}\,(1,1) \cong u^{d}\,(1,1)\,. \quad (3.4)$$

Proof. We prove only that

$$\hat{u}\,(2,0) \cong u^{d}\,(1,1)\,, \quad (3.5)$$

the proves of the others following in the same way.

We know that if $A \in u\,(2,0)$ then

$${}^{t}A + \bar{A} = 0 \text{ or } A +^{t} \left(\bar{A}\right) = 0. \quad (3.6)$$

From (3.6) we conclude that $u(2,0)$ has the form:

$$u\,(2,0) = \left[A = \begin{pmatrix} -i\lambda & \mu + i\nu \\ -\mu + i\nu & i\tau \end{pmatrix} / \lambda, \mu, \nu, \tau \in I\!R\right]. \quad (3.7)$$

Now using Santili's theory we have:

$$\begin{aligned} u\,(1,1) &= u\,(2,0)\,T_{1,\bar{1}} = \begin{pmatrix} i\lambda & \mu + i\nu \\ -\mu + i\nu & i\tau \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \\ & \begin{pmatrix} i\lambda & -(\mu + i\nu) \\ -\mu + i\nu & -i\tau \end{pmatrix}. \end{aligned} \quad (3.8)$$

The Lie algebra $u\,(2,0)$ can be obtained as follows:

$$\begin{aligned} u\,(0,2) &= u\,(2,0)\,T_{2,0} = u\,(2,0)\,T_{0,\bar{2}} = \\ \begin{pmatrix} i\lambda & \mu + i\nu \\ -\mu + i\nu & i\tau \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} &= \begin{pmatrix} -i\lambda & -(\mu + i\nu) \\ \mu - i\nu & -i\tau \end{pmatrix}, \end{aligned} \quad (3.9)$$

which implies

$$u\,(0,2) = -u\,(2,0)\,. \quad (3.10)$$

Now we write in detail the mentioned cases (3.3).

$$\begin{aligned} u^{d}\,(2,0) &= u\,(2,0)\,T^{d}_{2,0} = u\,(2,0)\,T^{d}_{0,\bar{2}} = \\ \begin{pmatrix} i\lambda & \mu + i\nu \\ -\mu + i\nu & i\tau \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} &= -u\,(2,0)\,, \end{aligned} \quad (3.11)$$

$$\begin{aligned} u^{d}\,(1,1) &= u\,(2,0)\,T^{d}_{\bar{1},1} = u\,(2,0)\,T^{d}_{1,\bar{1}} = \\ \begin{pmatrix} i\lambda & \mu + i\nu \\ -\mu + i\nu & i\tau \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} &= -u\,(1,1)\,, \end{aligned} \quad (3.12)$$

$$\begin{aligned} \hat{u}\,(2,0) &= u\,(2,0)\,T_{2,0} = \begin{pmatrix} i\lambda & \mu + i\nu \\ -\mu + i\nu & i\tau \end{pmatrix} \begin{pmatrix} b_1^2 & 0 \\ 0 & b_2^2 \end{pmatrix} = \\ & \begin{pmatrix} i\lambda b_1^2 & (\mu + i\nu)\, b_2^2 \\ (-\mu + i\nu)\, b_1^2 & -i\tau b_2^2 \end{pmatrix}, \end{aligned} \quad (3.13)$$

$$\begin{aligned} \hat{u}\,(1,1) &= u\,(2,0)\,T_{1,\bar{1}} = \begin{pmatrix} i\lambda & \mu+i\nu \\ -\mu+i\nu & i\tau \end{pmatrix} \begin{pmatrix} b_1^2 & 0 \\ 0 & -b_2^2 \end{pmatrix} = \\ & \begin{pmatrix} i\lambda b_1^2 & -\left(\mu+i\nu\right) b_2^2 \\ \left(-\mu+i\nu\right) b_1^2 & -i\tau b_2^2 \end{pmatrix}, \end{aligned} \tag{3.14}$$

$$\begin{aligned} \hat{u}^d\,(2,0) &= u\,(2,0)\,T_{0,\bar{2}} = \begin{pmatrix} i\lambda & \mu+i\nu \\ -\mu+i\nu & i\tau \end{pmatrix} \begin{pmatrix} -b_1^2 & 0 \\ 0 & -b_2^2 \end{pmatrix} = \\ & \begin{pmatrix} -i\lambda b_1^2 & -\left(\mu+i\nu\right) b_2^2 \\ \left(\mu-i\nu\right) b_1^2 & -i\tau b_2^2 \end{pmatrix}, \end{aligned} \tag{3.15}$$

$$\begin{aligned} \hat{u}^d\,(1,1) &= u\,(2,0)\,T_{\bar{1},1} = \begin{pmatrix} i\lambda & \mu+i\nu \\ -\mu+i\nu & i\tau \end{pmatrix} \begin{pmatrix} -b_1^2 & 0 \\ 0 & b_2^2 \end{pmatrix} = \\ & \begin{pmatrix} -i\lambda b_1^2 & \left(\mu+i\nu\right) b_2^2 \\ \left(\mu-i\nu\right) b_1^2 & i\tau b_2^2 \end{pmatrix}, \end{aligned} \tag{3.16}$$

We consider the mapping $f : u\,(2,0) \to \hat{u}\,(2,0)$,

$$f : A = \begin{pmatrix} i\lambda & \mu+i\nu \\ -\mu+i\nu & i\tau \end{pmatrix} \to f(A) = \begin{pmatrix} i\lambda b_1^2 & \left(\mu+i\nu\right) b_2^2 \\ \left(-\mu+i\nu\right) b_1^2 & i\tau b_2^2 \end{pmatrix} = \hat{A}. \tag{3.17}$$

and

$$B = \begin{pmatrix} i\lambda' & \mu'+i\nu' \\ -\mu'+i\nu' & i\tau' \end{pmatrix} \to f(B) = \begin{pmatrix} i\lambda' b_1^2 & \left(\mu'+i\nu'\right) b_2^2 \\ \left(-\mu'+i\nu'\right) b_1^2 & i\tau' b_2^2 \end{pmatrix} = \hat{B}. \tag{3.18}$$

It can be easily proved that

$$f\left(\alpha A+\beta B\right) = \alpha f\left(A\right) + \beta f\left(B\right),\ \alpha,\beta \in \mathbf{C}, \tag{3.19}$$

$$f([A,B]) = [f(A),f(B)] = [\hat{A},\hat{B}] = \widehat{[A,B]}. \tag{3.20}$$

We prove that :
I) $f(A+B) = f(A)+f(B)$ and
II) $f(cA) = cf(A)$.
$f(A+B) = f\left[\begin{pmatrix} i\lambda & \mu+i\nu \\ -\mu+i\nu & i\tau \end{pmatrix} + \begin{pmatrix} i\lambda' & \mu' + i\nu' \\ -\mu' + \text{iv'} & i\tau' \end{pmatrix}\right] =$
$f\begin{pmatrix} i(\lambda + \lambda') & \mu + \mu' + i(\nu + \nu') \\ -(\mu + \mu') + i(\nu+\nu') & i(\tau + \tau') \end{pmatrix}$. Thus

$$f\left(A+B\right) = \begin{pmatrix} i\left(\lambda + \lambda'\right) b_1^2 & [\mu + \mu' + i(\nu+\nu')]b_2^2 \\ [-(\mu + \mu') + i(\nu+\nu')]b_1^2 & i(\tau+\tau')b_2^2 \end{pmatrix} \tag{1}$$

$$f(A)+f(B) = \begin{pmatrix} i\lambda b_1^2 & (\mu + i\nu)b_2^2 \\ (-\mu + i\nu)b_1^2 & i\tau b_2^2 \end{pmatrix} +$$

$\begin{pmatrix} i\lambda' b_1^2 & (\mu' + i\nu')b_2^2 \\ (-\mu' + i\nu')b_1^2 & i\tau' b_2^2 \end{pmatrix}$, thus

$$f(A)+f(B)=\begin{pmatrix} i\,(\lambda+\lambda')\,b_1^2 & ((\mu+\mu') + i\,(\nu+\nu'))\,b_2^2 \\ (-\,(\mu+\mu')+i\,(\nu+\nu'))\,b_1^2 & i\,(\tau+\tau')\,b_2^2 \end{pmatrix}. \tag{2}$$

From the relations (1) and (2) it follows that $f(A+B)=f(A)+f(B)$.
$f(cA)=f\left[c\begin{pmatrix} i\lambda & \mu+i\nu \\ -\mu+i\nu & i\tau \end{pmatrix}\right]=f\begin{pmatrix} i\lambda c & (\mu+i\nu)\,c \\ (-\mu+i\nu)\,c & i\tau c \end{pmatrix}=$
$\begin{pmatrix} i\lambda c b_1^2 & c\,(\mu+i\nu)\,b_2^2 \\ c\,(-\mu+i\nu)\,b_1^2 & ic\tau b_2^2 \end{pmatrix}=c\begin{pmatrix} i\lambda b_1^2 & (\mu+i\nu)\,b_2^2 \\ (-\mu+i\nu)\,b_1^2 & i\tau b_2^2 \end{pmatrix}=cf(A).$
$f\left[\begin{pmatrix} i\lambda & \mu+i\nu \\ -\mu+i\nu & i\tau \end{pmatrix}\begin{pmatrix} i\lambda' & \mu'+i\nu' \\ -\mu'+i\nu' & i\tau' \end{pmatrix}-\right.$
$\left.\begin{pmatrix} i\lambda' & \mu'+i\nu' \\ -\mu'+i\nu' & i\tau' \end{pmatrix}\begin{pmatrix} i\lambda & \mu+i\nu \\ -\mu+i\nu & i\tau \end{pmatrix}\right]=$
$\left[\begin{pmatrix} -\lambda\lambda' + (\mu+i\nu)(-\mu'+i\nu') & i\lambda\,(\mu'+i\nu') + i\tau'(\mu+i\nu) \\ (-\mu+i\nu)i\lambda' + i\tau(-\mu'+i\nu') & (-\mu+i\nu)(\mu'+i\nu') - \tau\tau' \end{pmatrix}-\right.$
$\left.\begin{pmatrix} -\lambda\lambda' + (\mu'+i\nu')(-\mu+i\nu) & i\lambda'\,(\mu+i\nu) + i\tau(\mu'+i\nu') \\ (-\mu'+i\nu')i\lambda' + i\tau\,(-\mu+i\nu) & (-\mu'+i\nu')(\mu+i\nu) - \tau\tau' \end{pmatrix}\right]=$
$f\begin{pmatrix} 2i(\mu\nu'-\mu'\nu) & (\tau-\lambda)(\nu'-i\mu') + (\lambda'-\tau')(\nu-i\mu) \\ (\lambda-\tau)(\nu'-\mathrm{i}\mu') + (\nu'-\lambda')(\nu+i\mu) & 2i(\mu'\nu-\mu\nu') \end{pmatrix}=$
$f\begin{pmatrix} -ip & x\bar{z}+y\bar{z}_1 \\ -xz-yz_1 & ip \end{pmatrix}=\begin{pmatrix} -ipb_1^2 & (x\bar{z}+y\bar{z}_1)b_2^2 \\ (-xz-yz_1)b_1^2 & ipb_2^2 \end{pmatrix}$, .where $p=2\,(\mu'\nu-\mu\nu')$, $x=\lambda'-\tau'$, $y=\tau-\lambda$, $z=\nu+i\mu$, $z_1=\mu'+i\mu'$.
Thus

$$f\,([A,B])=\begin{pmatrix} -ipb_1^2 & (x\bar{z}+y\bar{z}_1)b_2^2 \\ (-xz-yz_1)b_1^2 & ipb_2^2 \end{pmatrix}. \tag{(3.21)}$$

On the other hand $[f(A),f(B)]=\left[\hat{A},\hat{B}\right]=\hat{A}T\hat{B}-\hat{B}T\hat{A}=A\hat{B}-B\hat{A}$.
$A\hat{B}-B\hat{A}=\begin{pmatrix} i\lambda & \mu+iv \\ -\mu+i\nu & i\tau \end{pmatrix}\begin{pmatrix} i\lambda' b_1^2 & (\mu'+i\nu')b_2^2 \\ (-\mu'+i\nu')b_1^2 & i\tau' b_2^2 \end{pmatrix}-$
$\begin{pmatrix} i\lambda' & \mu'+iv' \\ -\mu'+iv' & i\tau' \end{pmatrix}\begin{pmatrix} i\lambda b_1^2 & (\mu+i\nu)b_2^2 \\ (-\mu+i\nu)b_1^2 & i\tau b_2^2 \end{pmatrix}=$
$\begin{pmatrix} -\lambda\lambda' b_1^2 + (\mu+i\nu)\,(-\mu'+i\nu')b_1^2 & i\lambda\,(\mu'+iv')b_2^2 + i\tau'(\mu+i\nu)b_2^2 \\ [-i\lambda'(\mu+iv) + i\tau\,(-\mu'+i\nu')]b_1^2 & [(-\mu+\mathrm{i}\nu)\,(\mu'+iv')-\tau\tau']b_2^2 \end{pmatrix}-$
$\begin{pmatrix} [-\lambda\lambda' + (\mu'+i\nu')\,(-\mu+i\nu)]b_1^2 & [i\lambda\,(\mu+i\nu+i\tau'(\mu'+i\nu')]b_2^2 \\ [i\lambda'(-\mu'+i\nu') + i\tau'\,(-\mu+i\nu)]b_1^2 & [(-\mu'+i\nu')(\mu+i\nu-\tau\tau']b_2^2 \end{pmatrix}=$
$\begin{pmatrix} 2i(\nu'\mu-\nu\mu')b_1^2 & [(\lambda'-\tau')(\nu-\mathrm{i}\mu) + (\tau-\lambda)(\nu'-i\mu')]b_2^2 \\ [(\tau'-\lambda')(\nu+i\mu) + (\lambda-\tau)(\nu'+i\mu')]b_1^2 & 2i(\mu'\nu-\mu\nu')b_2^2 \end{pmatrix}=$
$\begin{pmatrix} -ipb_1^2 & (x\bar{z}+y\bar{z}_1)b_2^2 \\ -(xz+yz_1)b_1^2 & ipb_2^2 \end{pmatrix}$. Thus

$$A\hat{B}-B\hat{A}=\begin{pmatrix} -ipb_1^2 & (x\bar{z}+y\bar{z}_1)b_2^2 \\ -(xz+yz_1)b_1^2 & ipb_2^2 \end{pmatrix}. \tag{3.22}$$

Using the relation (3.21) and (3.22) it follows that the relation (3.20) holds. It can be also proved that f is one-to-one and onto, that means bijective. Therefore f is an isomorphism. □

4 The Lie algebras $u(p,q)$, $p+q=3$

In this section we consider the Lie algebras $u(p,q)$, $p+q=3$, which gives the following cases:

$$u(3,0), u(2,1), u(1,2), u(0,3). \tag{4.1}$$

From these we obtain:

$$\begin{aligned}&\left\{u^d(3,0), u^d(2,1), u^d(1,2), u^d(0,3)\right\}, \left\{\hat{u}(3,0), \hat{u}(2,1), \hat{u}(1,2), \hat{u}(0,3)\right\},\\ &\left\{\hat{u}^d(3,0), \hat{u}^d(2,1), \hat{u}^d(1,2), \hat{u}^d(0,3)\right\}.\end{aligned} \tag{4.2}$$

We write analytically $u(3,\ 0)$ from which we can obtain (4.2).

$$u(3) = u(3,0) = \left\{ A = \begin{pmatrix} \mathrm{i}\lambda & \mu\ + i\nu & p + i\theta \\ -\mu + i\nu & i\nu_1 & p_1 + i\theta_1 \\ -p + i\theta & -\,p_1\ + i\theta_1 & i\theta_2 \end{pmatrix} /\lambda, \mu, ..., \theta_2 \in IR \right\}.$$

Now, we have the following result.

Theorem 4.1 *The following isomorphisms are valid:*

$$\begin{aligned}\hat{u}\,(3,0) &\cong u\,(3,0)\,,\ \hat{u}^d\,(3,0) \cong u^d\,(3,0)\,,\\ \hat{u}\,(2,1) &\cong u\,(2,1)\,,\ \hat{u}^d\,(2,1) \cong u^d\,(2,1)\,..\end{aligned}$$

Proof. It can be followed the same way as in the proof of Theorem 3.1.

5 The Lie algebras $u(p,q)$, $p+q=4$

We have the Lie algebras

$$u(4) = u(4,0), u(3,1), u(2,2), u(1,3), u(0,4) = -u(4). \tag{5.1}$$

From these we obtain the following cases:

$$\begin{array}{l}\left\{\hat{u}(4), u^d(4), \hat{u}^d(4)\right\},\ \left\{\hat{u}(3,1), u^d(3,1), \hat{u}^d(3,1)\right\},\\ \left\{\hat{u}(2,2), u^d(2,2), \hat{u}^d(2,2)\right\},\ \left\{\hat{u}(1,3), u^d(1,3), \hat{u}^d(1,3)\right\},\\ \left\{\hat{u}(0,4), u^d(0,4), \hat{u}^d(0,4)\right\}.\end{array} \tag{ 5.2}$$

We express analytically $u(4) = u(4,0)$ from which we can get (5.2) using the mentioned methodology..

$$\begin{aligned} u(4,0) \quad = \quad & \left\{ A = \begin{pmatrix} i\lambda & \mu\ + i\nu & p + i\theta & u + i\tau \\ -\mu\ + i\nu & i\nu_1 & p_1\ + i\theta_1 & u_1 +\ i\tau_1 \\ -p + i\theta & -p_1\ + i\theta_1 & i\theta_2 & u_2 + i\tau_2 \\ -u + i\tau & -\,u_1\ + i\tau_1 & -\,u_2\ + i\tau_2 & i\tau_3 \end{pmatrix} / \right. \\ & \lambda, \mu, ..., \tau_3 \in IR\}\,. \end{aligned} \tag{5.3}$$

The following result can be proved using the same methodology as for Theorems 3.1 and 4.1.

Theorem 5.1 *The following Lie algebras are isomorphic:*

$$\begin{array}{l} \hat{u}(4,0) \cong u(4,0),\ \hat{u}^d(4,0) \cong u^d(4,0), \\ \hat{u}(3,1) \cong u(3,1),\ \hat{u}^d(3,1) \cong u^d(3,1), \\ \hat{u}(2,2) \cong u(2,2),\ \hat{u}^d(2,2) \cong u^d(2,2). \end{array}$$

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