Logarithm of Differential Forms and Regualarization of Volume Form

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Abstract

Fractional order differential forms (and vector fields) are introduced by using logarithm of derivatives. Considering such forms on a Hilbert space H equipped with a Schatten class operator G, whose ζ-function $\zeta(g,s) = tr(G^s)$ is holomorphic at $s = 0$, we can define the regularized volume form on H. It is also shown there is a 2-cocycle obstruction to the global definition of such forms, and show regularized volume form exists on a loopspace ΩM if the string class of ΩM vanishes (cf.[4], [11]).

AMS Subject Classification: 35R15, 33C55,58B30.
Key words: Fractional order differential forms and vector fields, regularized volume, regularized infinite dimensional indefinite integral operator

1 Introduction

In our previous study of regularization of the Dirac operator on a Hilbert space H , new proper values which do not come from finite dimensional Dirac operator appeared. The proper spinors belonging to these proper values are expressed as the infinite products of trigonometric functions ([6]). To clarify background of this phenomenon, we have defined regularized infinite product $: \prod x_n :$, $\sum x_n e_n \in H^-(finite)$, where $H^-(finite)$ is a modification of H and $\{e_n\}$ is a special complete ortho-normal basis of H. $: \prod x_n :$ is linear in each variable, but we can not compute $\lim_{N\to\infty} \frac{\partial^N}{\partial x_1 \cdots \partial x_n} : \prod x_n :$. But it is shown in [7], if we use fractional derivatives(cf.[12]. In this paper, physical applications of fractional calculus are also discussed), we can define an infinite order derivation $\frac{\partial^\infty}{\prod \partial x_n}$ such that

$$\frac{\partial^\infty}{\prod \partial x_n} : \prod x_n := 1.$$

Editor Gr.Tsagas *Proceedings of The Conference of Applied Differential Geometry - General Relativity and The Workshop on Global Analysis, Differential Geometry and Lie Algebras, 2001,* 11-32

Related regularized infinite dimensional indefinite integral operator was also defined.

Motivated these definitions, we try to construct fractional order differential forms and use them to the regularization of the volume form of H. For this purpose, we first study logarithm of derivation, which is the generating operator of the semi-group of fractional derivatives. Logarithm of derivation

$$\log(\frac{\partial}{\partial x_n}) = \lim_{h\to 0} h^{-1}(\frac{\partial^h}{\partial x_n^h} - I),$$

is not a pseudodifferential operator and maps single valued functions to many valued functions. But taking $w_n = \log x_n,\ n = 1, 2, \ldots$, as variables, we can proceed the calculus of logarithm of derivation in algebraic manner. For this purpose, we first consider one-variable case, and denote the power series algebra of $w = \log x$ by $\mathfrak{F}(w) = \mathfrak{F}$. We also define another product $f\sharp g$ by

$$f\sharp g = \frac{d}{dx}\int_0^x f(x-t)g(t)dt,$$

when f and g are functions of $x = \mathrm{e}^w$. Then we can define $\sharp$-product for the power serieses of w, and we have

$$\mathrm{e}^{\sharp tw} = \frac{\mathrm{e}^{-\gamma t}}{\Gamma(1+t)}x^t, \tag{1}$$

$$\mathrm{e}^{\sharp f} = \sum \frac{f^{\sharp n}}{n!}, \quad f^{\sharp n} = \overbrace{f\sharp\cdots\sharp f}^{n}. \tag{2}$$

We regard $\mathbb{F}$ to be an operator algebra acting on $\mathfrak{F}$ by the $\sharp$-product. Then, since $\lim_{t\to -n} \mathrm{e}^{\sharp t(w+\gamma)}\sharp f = \frac{d^n f}{dx^n}$, we may consider $\mathrm{e}^{\sharp -n(w+\gamma)} = \frac{d^n}{dx^n}$ as the element of $\mathbb{F}$. By using $\mathbb{F}$ and $\sharp$-product, we define fractional derivation and logarithm of derivation by

$$\frac{d^a}{dx^a}f = \mathrm{e}^{\sharp -a(w+\gamma)}\sharp f, \tag{3}$$

$$\log(\frac{d}{dx})f = -(w+\gamma)\sharp f. \tag{4}$$

The algebra $\mathfrak{F} = \mathfrak{F}(w_1, w_2, \ldots)$ is defined similar to $\mathfrak{F}(w)$. While in the definition of $\mathbb{F} = \mathbb{F}(w_1, w_2, \ldots)$, we modify $\sharp$-product as follows;

$$w_n\sharp w_m = w_n \cdot w_m + \frac{m-n}{|m-n|}\frac{\pi i}{2}, \quad n \neq m. \tag{5}$$

Fractional differential forms are defined by using $\mathrm{e}^{\sharp a(w_n+\gamma)}$, together with the relation $\mathrm{e}^{\sharp 2(w_n+\gamma)} \cong 0$. Then, by using the analytic continuation of $\mathrm{e}^{\sharp\omega(s)}$, $\omega(s) = \sum \mu_n^s(w_n+\gamma)$, we define the regularized volume form of H. It is expected this construction of the volume form may give an explanation of the use of Ray-Singer determinant in the calculation of the path integral such as

$$\int e^{-\pi(x,Dx)}\mathcal{D}x = \frac{1}{\sqrt{detD}}.$$

To define fractional forms and regularized volume form on curved space, we need to define coordinate transformation of $\mathbb{F}$. If $g = e^h$, $\phi = e^u$, then by Campbell-Haussdorff formula, we get $g^t\phi^t = e^{t(h+u)+t^2CH(h,u;t)}$, if $|t|$ is small. Denoting the analytic continuation of $t(h+u)+t^2CH(h,u;t)$ along the path β which joins 0 and 1 by $h +_\beta u$, we get coordinate transformation of $\mathbb{F}$. In general, the action $h+_\beta$ is not associative. The obstruction to the associativity of this action is expressed as a 2-cocycle.

The outline of the paper is as follows; In section 2, we review regularized infinite product and its relation to the fractional calculus according to [7](see also [6]). Logarithm of derivation for 1-variable functions is discussed according to [1] and [2], in section 3. Then the algebras $\mathfrak{F}$ and $\mathbb{F}$ of several variables are introduced in section 4, and fractional differential forms are defined in section 5 together with the definition of fractional vector fields and the pairing of fractional differential forms and vector fields. Regularized volume form on H and its relation to the regularized infinite product are also discussed in this section. Coordinate transformation of $\mathbb{F}$ and fractional forms are discussed in section 6 by using the idea in [3].

By using these results and results in [4], we discuss fractional differential forms on a mapping space $Map(X,M)$ in section 7. To define fractional differential forms on $Map(X,M)$, the first string class $s^1(\tau)$ of the tangent bundle τ of $Map(X,M)$ needs to vanish. Also there exists a 2-dimensional obstruction class which is defined under the assumption *there are no crossing of spectres of* $D + A_U(x)$*.* Here the Sobolev structure of $Map(X,M)$ is fixed by D and A_U is a connection of τ with respect to D ([4]). This assumption on the spectres of $D + A_U(x)$ is satisfied if $X = S^1$, that is $Map(X,M)$ is a loopspace. But otherwise, it seems too strong, and we seek for another condition.

If D is positive, and if the virtual dimension $\nu = \zeta(D + A_U, 0)$, of $Map(X,M)$ can be choose to be a constant integer, there are no further obstruction to construct regularized volume form of $Map(X,M)$. If such selection can not be possible, then the regularized volume form may be many-valued unless $Map(X,M)$ is simply connected. On the other hand, obstruction to the construction of regularized volume form must exists and relate to the second string class $s^2(\tau)$ of τ if we use Dirac type operator as the Sobolev structure (cf.[4], see also [5], [8], [9]). Especially, we show regularized volume form exists on a loopspace ΩM if the first and second string classes of ΩM vanishes (cf.[11]).

2 Regularized infinite product and fractional calculus

Let $\{H, G\}$ be a pair of a Hilbert space H and a non-degenerate Hermitian Schatten class operator G acting on H, such that the $\zeta(G, s) = trG^s$ allows analytic continuation to $s = 0$ and holomorphic at $s = 0$. We note such pairing is closely related to Connes' non-commutative geometry from the spectral point of view ([10]) but more concrete. For simple, we assume positivity of G and arrange the proper values $\{\mu_n : Ge_n = \mu_n e_n\}$ of G as follows; $\mu_1 \geq \mu_2 \geq \cdots > 0$. Here $\{e_n\}$ are the proper vectors of G which span H. We assume $\{e_n\}$ is an ortho-normal system and fix the complete ortho-normal basis to be $\{e_n\}$. The coordinate $x = \sum x_n e_n \in H$ is fixed to be $(x_1, x_2, \ldots)$.

$\{H, G\}$ has the following numerical invariants

1. The regularized dimension $\nu = \zeta(G, 0)$ of H.
2. The location of the first pole d of $\zeta(G, s)$.
3. The Ray-Singer determinant $detG = e^{\zeta'(G,0)}$ of G.

By using G, we introduce the Sobolev k-norm $||x||_k$ of $x \in H$ by

$$||x||_k = ||G^{-k}x||, \quad ||x|| \textit{ is the norm of } x \textit{ in } H.$$

The k-Sobolev space constructed by this norm and H is denoted by W^k. By definition, $W^k \subset W^l$ if $k > l$. We set $H^- = \cap_{k<0} W^k$. H is contained in H^-. We set $\mathbf{e} = \sum \mu_n^{d/2} e_n$. Then $\mathbf{e}$ belongs to H^-, but does not belong to H. We consider the following subspaces of H^-.

$$H^-(finite) = \{\sum x_n e_n \in H^- | \lim_{n\to\infty} \mu_n^{-d/2} x_n \textit{ exists.}\}, \tag{6}$$

$$H^-(0) = \{\sum x_n e_n \in H^- | \lim_{n\to\infty} \mu_n^{-d/2} x_n = 0\}. \tag{7}$$

By definitions, we have

$$H^-(finite) = H^-(0) \oplus \mathbf{C}\mathbf{e}. \tag{8}$$

As for topologies, we regard $H^-(0)$ to be a subspace of H^-, while we give product space topology of $H^-(0)$ and $\mathbf{C}$ to $H^-(finite)$.

By (8), $x = \sum x_n e_n \in H^-(finite)$ is uniquely written as $y + t\mathbf{e}, y = \sum y_n e_n \in H^-(0)$. If $t \neq 0$, we can write

$$\log x_n = \log(\frac{\mu_n^{-d/2} x_n}{t}) + \frac{d}{2} \log \mu_n + \log t.$$

Hence we have

$$\sum \mu_n^s \log x_n = \sum \mu_n^s \log(\frac{\mu_n^{-d/2} x_n}{t}) + \frac{d}{2}\zeta'(G, s) + \log(t\zeta(G, s)).$$

Since $(\mu_n^{-d/2} x_n)/t = 1 + \mu_n^{-d/2} y_n/t$, we define the regularized product $: prod x_n :$ of $x_1, x_2, \ldots$, as follows;

Definition. If $t \neq 0$ and $\prod(1 + \frac{\mu_n^{-d/2} y_n}{t})$ converges, we define the regularized product $: \prod x_n :$ by

$$: \prod x_n := t^{\nu} (detG)^{d/2} \prod (1 + \frac{\mu_n^{-d/2} y_n}{t}). \tag{9}$$

Theorem 1. (i). $: \prod x_n :$ *is defined if and only if* $x \notin H^-(0)$ *and* $\sum |\mu_n^{-d/2} y_n| < \infty$.

(ii). $: \prod x_n :$ *is uniquely defined if and only if* ν *is an integer.*

(iii). $: \prod x_n :$ *is linear in each* x_n.

(iv). $: \prod x_n :$ *is a positive real number if each* x_n *is a positive real number.*

By Theorem 1, $\frac{\partial}{\partial x_m} : \prod x_n :$ does not depend on x_m. Hence we may consider

$$\frac{\partial}{\partial x_m} : \prod x_n :=: \prod_{n \neq m} x_n : .$$

But we can not compute $\lim_{N \to \infty} \frac{\partial^N}{\partial x_1 \cdots \partial x_N} : \prod x_n :$. Because we have

$$\frac{\partial^N}{\partial x_1 \cdots \partial x_n} \int_0^{x_1} \cdots \int_0^{x_N} 1 dx_1 \cdots dx_n = 1,$$

$$\int_0^{x_1} \cdots \int_0^{x_N} 1 dx_1 \cdots dx_N = x_1 \cdots x_N.$$

Since $: \prod x_n :$ is the analytic continuation of $\prod x_n^{\mu_n^s}$ to $s = 0$, to calculate $\lim_{N \to \infty} \frac{\partial^N}{\partial x_1 \cdots \partial X_N} : \prod x_n :$, it is appropriate to use fractional calculus (fractional order derivation and indefinite integral).

Fractional order derivation $\frac{d^a}{dx^a}$ is defined by several ways, *e.g.* , by Fourier or Laplace transformation($\frac{d^a}{dx^a} \mathcal{L}[f(t)](x) = \mathcal{L}[(-t)^a f(t)](x)$), by Riemann Hilbert integral ($\frac{d^a}{dx^a} f = \frac{1}{\Gamma(1-a)} \int_0^x (x-t)^{-a} f'(t) dt,\ a < 1$), and by $\frac{d^a}{dx^a} f = \lim_{h \to 0} h^{-a} (\tau_h - \mathrm{I})^a f$, where $\tau_h f(x) = f(x+h)$. In any case, we obtain same answer. For example, we have

$$\frac{d^a}{dx^a} x^n = \frac{n!}{\Gamma(n+1-a)} x^{n-a}.$$

The fractional order indefinite integral $\mathbf{I}^a_{[0,x]}$ is defined by the same way, and we can regard it to be $\frac{d^{-a}}{dx^{-a}}$.

Note. Unless considered the variables of f to be positive real numbers, fractional derivation and indefinite integral map single valued functions to many valued functions. Detailed discussion for this point is given in next section.

By using fractional calculus, we define the operators $:\frac{\partial^\infty}{\prod \partial x_n}:$ and $\int_{Q(x)} : d^\infty x :$, $Q(x) = \{\sum t_n e_n | 0 \le t_n \le x_n\}$ by

$$:\frac{\partial^\infty}{\prod \partial x_n}: \; = \; (\prod_{n=1}^{\infty} \frac{1}{\Gamma(1+\mu_n^s)} \frac{\partial^{\mu_n^s}}{\partial x_n^{\mu_n^s}})|_{s=0}, \tag{10}$$

$$\int_{Q(x)} f(x) : d^\infty x : \; = \; ((\prod_{n=1}^{\infty} \Gamma(1+\mu_n^s) \mathbf{I}_{[0,x_n)]}^{\mu_n^s} f(x))|_{s=0}, \tag{11}$$

([7]). Here, $F(s)|_{s=0}$ means analytic continuation of F to $s = 0$.

By definitions, $:\frac{\partial^\infty}{\prod \partial x_n}$ and $\int_{Q(x)} : d^\infty x :$ are inverse operators each other. Since we have

$$\prod \Gamma(1+\mu_n^s) \mathbf{I}_{[0,x_n]}^{\mu_n^s} 1 = \prod x_n^{\mu_n^s},$$

we obtain

$$\int_{Q(x)} 1 : d^\infty x : \; = \; : \prod x_n :, \tag{12}$$

$$:\frac{\partial^\infty}{\prod \partial x_n} :: \prod x_n : \; = \; 1. \tag{13}$$

These formula show fractional calculus fit to the study of regularized product. (12) suggests the regularized *volume form* $: d^\infty x :$ might give some explanation of the formulae such as

$$\int e^{(-2\pi(x,Dx))} \mathcal{D}x = \frac{1}{\sqrt{detD}}.$$

3 Logarithm of derivation

Since $\frac{d^a}{dx^a} \cdot \frac{d^b}{dx^b} = \frac{d^{a+b}}{dx^{a+b}}$, $\{\frac{d^a}{dx^a} | a \ge 0\}$ becomes a semi-group. We denote its generating operator by $\log(\frac{d}{dx})$:

$$\log(\frac{d}{dx}) = \lim_{h \to 0} \frac{1}{h}(\frac{d^h}{dx^h} - \mathrm{I}).$$

We study $\log(\frac{d}{dx})$ by using Borel transformation (cf.[1], [2]). For $f(x) = \sum c_n x^n$, its Borel transformation $\mathcal{B}(f) = \mathcal{B}[f(t)](x)$ is defined by

$$\mathcal{B}(f) = \sum \frac{c_n}{n!} x^n = \frac{1}{2\pi i} \oint \frac{f(t)}{t} e^{\frac{x}{t}} dt.$$

By definition, we have

$$\begin{aligned} \frac{d}{dx}\mathcal{B}(f) &= \mathcal{B}(\frac{f}{t}), && (14)\\ \mathcal{B}(f\cdot g) &= \mathcal{B}(f)\sharp\mathcal{B}(g),\ u\sharp v = \frac{d}{dx}\int_0^x u(x-t)v(t)dt, && (15)\\ \mathcal{B}(f) &= \int_{-\infty}^{\infty} J_0(\sqrt{-4\pi\sqrt{-ixt}})\mathcal{F}(t)dt, && (16)\\ \mathcal{B}^{-1}(f) &= \int_0^{\infty} \mathrm{e}^{-x}f(xt)dx. && (17)\end{aligned}$$

Here $\mathcal{F}$ and $\mathcal{B}^{-1}$ mean the Fourier transformation and inverse of Borel transformation, respectively. By (17), we may define

$$\mathcal{B}(\log t)(x) = \log x + \gamma, \quad \gamma \textit{ is the Euler constant.} \qquad (18)$$

A justification of (18) follows from

$$\mathcal{B}(\log(t+c)) = \log x + \gamma - \mathrm{Ei}(-\frac{-x}{c}), \quad \mathrm{Ei}(-x) = \gamma + \log x + \sum \frac{(-x)^n}{n\cdot n!}.$$

Because $\mathrm{Ei}(-z) = \int_z^\infty \mathrm{e}^{-t}t^{-1}dt$, so we have

$$\lim_{c\to 0}\mathrm{Ei}(-\frac{-x}{c}) = 0, \quad -\frac{\pi}{2} < arg.\frac{x}{c} < \frac{\pi}{2},$$

if $\Re\frac{x}{c} > 0$. Another justification of (18) is the following Lemma ([1]).

Lemma 1. *Let* $u^{\sharp n}$ *be* $\overbrace{u\sharp\cdots\sharp u}^{n}$ *and* $\mathrm{e}^{\sharp u} = \sum\frac{u^{\sharp n}}{n!}$. *Then we have*

$$\mathrm{e}^{\sharp t\log x} = \frac{\mathrm{e}^{-\gamma t}}{\Gamma(1+t)}x^t. \qquad (19)$$

Proof. Since $\log\Gamma(1+t) = -\gamma t + \sum_{n=2}^{\infty}(-1)^m\frac{\zeta(m)}{m}t^m$, the Taylor expansion of $\mathrm{e}^{-\gamma t}/\Gamma(1+t)$ is

$$1 + \sum_{n\geq 2}\{\sum_{s=1}^{n/2}\sum_{2\geq J_1\geq\cdots\geq j_s, j_1+\cdots+j_s=n}(-1)^{n-s}\frac{\zeta(j_1)\cdots\zeta(j_s)}{j_1\cdots j_s}\}t^n.$$

We also note since

$$\begin{aligned} &\textstyle\sum_{\sigma(s)=k}\frac{1}{j_{\sigma(1)}\cdots(j_{sigma(1)}+\cdots+j\sigma(s))}\\ =\ &\frac{1}{j_1\cdots j_{k-1}j_{k+1}\cdots j_s(j_{\sigma(1)}+\cdots+j_{\sigma(s)})}\end{aligned},$$

where $\sigma\in\mathfrak{S}_s$ is a permutation,

$$\sum_{\sigma\in\mathfrak{S}_s} 1j_{\sigma(1)}(j_{\sigma(1)}+j_{\sigma(2)})\cdots(j_{\sigma(1)}+\cdots j_{\sigma(s)}) = \frac{1}{j_1\cdots j_s},$$

is hold if $j_1 + \cdots + j_s = n$.

To show Lemma, first we note $\int_0^x \log(x-t)(\log t)^{n-1}dt$ is expressed as

$$\log x \int_0^x (\log t)^{n-1}dt - \sum_{m\geq 1} \frac{x^m}{m} \int_0^x t^m (logt)^{n-1}dt.$$

Hence to set $\log x \sharp (\log x)^{n-1} = \sum_{k=0}^n a_{n,k}(\log x)^k$, we have

$$\begin{aligned} a_{n,n} = 1,\ a_{n,n-1} &= 0,\ a_{n,0} = (-1)^{n-1}(n-1)!\zeta(n), \\ a_{n,k} &= \frac{(n-1)!}{k!(n-k-1)!} a_{n-k,0},\ 2 \geq k \geq n-1. \end{aligned}$$

By these formulae, to set $(\log x)^{\sharp n} = \sum_{k=0}^n b_{n,k}(\log x)^k$, we obtain

$$\begin{aligned} b_{n,n} &= 1,\ b_{n,n-1} = 0,\ b_{n,k} = \frac{n!}{k!(n-k)!} b_{n-k,0},\ 2 \leq k \leq n-1, \\ b_{n,0} &= \sum_{s=1}^{[n/2]} \sum_{j_i \geq 2, j_1+\cdots+j_s = n} (-1)^{n-s} \frac{n!\zeta(j_1)\cdots\zeta(j_s)}{j_1(j_1+j_2)\cdots(j_1+\cdots+j_s)},\ n \geq 2. \end{aligned}$$

Since $b_{n,0}/n!$ is the coefficient of t^n of the Taylor expansion of the entire function $\mathrm{e}^{-\gamma t}/\Gamma(1+t)$, $b_{n,0}/n!$ is $o(c^n)$ for any $c > 0$. Hence we get

$$\begin{aligned} \mathrm{e}^{\sharp \log x} &= \sum_{n=0}^{\infty} \frac{t^n}{n!} (\sum_{k=0}^n b_{n,k}(\log x)^k) \\ &= \sum_{k=0}^{\infty} (\sum_{n=k}^{\infty} \frac{k! b_{n,k}}{n!} t^{n-k}) \frac{t^k}{k!} (\log x)^k \\ &= (1 + \sum_{n=2}^{\infty} \frac{b_{n,0}}{n!} t^n)(\sum_{n=0}^{\infty} \frac{t^n}{n!}(\log x)^n) = \frac{\mathrm{e}^{-\gamma t}}{\Gamma(1+t)} x^t. \end{aligned}$$

By (19), assuming a is not a negative integer, we define

$$\mathcal{B}(x^a) = \frac{x^a}{\Gamma(a+1)}. \tag{20}$$

Since $\mathrm{e}^{\sharp a \log x} \sharp \mathrm{e}^{\sharp b \log x} = \mathrm{e}^{\sharp (a+b) \log x}$, we have

$$x^a \sharp x^b = \frac{\Gamma(a+1)\Gamma(b+1)}{\Gamma(a+b+1)} x^{a+b}, \tag{21}$$

provided any of a, b and $a+b$ is not a negative integer. (21) shows $\lim_{a \to -n} \mathcal{B}(x^a) \sharp f = \frac{d^n f}{dx^n}$. Hence by Lemma 1, we obtain

$$\lim_{t \to -n} \mathrm{e}^{\sharp t(\log x + \gamma)} \sharp f(x) = \frac{d^n f(x)}{dx^n}. \tag{22}$$

Definition. We define algebras $\mathfrak{F} = \mathfrak{F}(w)$ and $\mathbb{F} = \mathbb{F}(w)$, $w = \log x$, by

$$\mathfrak{F} = \{\sum_{n=0}^{\infty} c_n w^n \mid |\frac{c_n}{n!}| < C^n, \; for\ some\ C > 0\}, \tag{23}$$

$$\mathbb{F} = \{\sum_{n=0}^{\infty} c_n w^{\sharp n} \mid |\frac{c_n}{n!}| < C^n, \; for\ some\ C > 0\}. \tag{24}$$

$\mathfrak{F}$ is an algebra of functions, while we consider $\mathbb{F}$ to be an operator algebra acting on $\mathfrak{F}$ by the $\sharp$-product. So as an element of $\mathbb{F}$, we have

$$\mathrm{e}^{\sharp -n(w+\gamma)} = \frac{d^n}{dx^n}.$$

Here we consider $f \in \mathfrak{F}$ to be a function of $x = \mathrm{e}^w$.

Definition. Let f be an element of $\mathfrak{F}$. Then we define fractional derivative and logarithmic derivative of f by

$$\frac{d^a f}{dx^a} = \mathrm{e}^{\sharp -a(w+\gamma)} \sharp f, \tag{25}$$

$$\log(\frac{d}{dx}) f = -(w+\gamma) \sharp f. \tag{26}$$

Since $\mathrm{e}^{\sharp f} \sharp \mathrm{e}^{\sharp(-f)} = 1$, the identity map of $\mathfrak{F}$, we have

$$\mathrm{e}^{\sharp a(w+\gamma)} \sharp f = \mathbf{I}^a_{[0,x]} f.$$

Here $\mathbf{I}^a_{[0,x]}$ means the a-th order (fractional) indefinite integral. Similarly, we define logarithm of indefinite integral $\log(\mathbf{I}_{[0,x]})$ by

$$\log(\mathbf{I}_{[0,x]}) f = (w+\gamma) \sharp f. \tag{27}$$

Examples. The logarithmic derivatives of $x^m, m = 0, 1, \ldots$ are given by

$$\log(\frac{d}{dx}) 1 = -(\log x + \gamma), \tag{28}$$

$$\log(\frac{d}{dx}) x^m = -x^m(\log x + (\gamma - (1 + \frac{1}{2} + \cdots + \frac{1}{m}))), \; m \geq 1. \tag{29}$$

Since $w \sharp w = w^2 - \zeta(2)$ and

$$w \sharp w^{n-1} = w^n - \mathrm{P}_{n-1}(w), \quad n \geq 2,$$

$$\mathrm{P}_n(w) = \sum_{k=0}^{n} (-1)^{n-k} \frac{(n+1)!}{k!} \zeta(n+2-k) w^k, \; n \geq 1,$$

we get

$$\log(\frac{d}{dx})1 = -w-\gamma, \quad \log(\frac{d}{dx})w = -w^2-\gamma w+\zeta(2), \tag{30}$$

$$\log(\frac{d}{dx})w^n = -w^{n+1}-\gamma w+\mathrm{P}_{n-1}(w),\ n\geq 2. \tag{31}$$

By these calculations, we have

Lemma 2. *Let* $\mathbf{C}[w]$ *and* $\mathbf{C}[w]_\sharp$ *be polynomial algebras in* $\mathfrak{F}$ *and* $\mathbb{F}$*, respectively. Then we can identify them as vector spaces over* $\mathbf{C}$*, and* $\mathbf{C}[w]_\sharp$ *acts on* $\mathbf{C}[w]$*.*

4 Algebras $\mathfrak{F}$ and $\mathbb{F}$ of several variables.

For the variables $w_1=\log x_1, w_2=\log x_2,\ldots$, we define the algebra $\mathfrak{F}=\mathfrak{F}(w_1,w_2,\ldots)$ by

$$\mathfrak{F} = \{\sum_I c_I x^I \mid \sum_{|I|=m}|c_I|<C^m,\ for\ some\ C>0\}, \tag{32}$$

$$I = (i_1,i_2,\ldots),\ x^I=x_1^{i_1}x_2^{i_2}\cdots,\ |I|=i_1+i_2+\cdots, \tag{33}$$

We say a power series to be a finite exponential type power series if it satisfies the growth condition (32).

To define $\mathbb{F}=\mathbb{F}(w_1,w_2,\ldots)$, first we need to define $w_n\sharp w_m, n\neq m$, and so on.

Definition. Let $w_n\cdot w_m$ be the ordinary product. Then we define $w_n\sharp w_m$, for $n\neq m$, by

$$w_n\sharp w_m = w_n\cdot w_m+\frac{m-n}{|m-n|}\frac{\pi i}{2}. \tag{34}$$

We define $\mathbb{F}=\mathbb{F}(w_1,w_2,\ldots)$ to be the algebra of finite exponential type power serieses of $w_1,w_2,\ldots$ by the $\sharp$-product.

Note. Later, we mainly interest to the case $x_1,x_2,\ldots$ are the variables of H. In this case,

$$\omega(s)=\sum\mu_n^s(w_n+\gamma),$$

belongs to both of $\mathfrak{F}$ and $\mathbb{F}$ if (and only if) $\Re s>d$.

To define the action of $\mathbb{F}$ to $\mathfrak{F}$, we need to prepare some notations. Let $S=\{p_1,\ldots,p_n\}$ be a set of natural numbers (may not be distinct each other). $\complement T$ be the complement of $T=\{p_{j_1},\ldots,p_{j_k}\}\subseteq S$ in S. For $p_l\in T$, we correspond a natural number $\natural_S(p_l)=\natural(p_l)$ as follows;

$$\natural(p_l) = 1, \quad if\ p_l=minT,$$

$$\natural(p_l) = 2, \quad if\ p_l=min\complement\{p_j;\natural(p_j)=1\}\cap T,$$

and so on. The sign $\mathrm{sgn}T = \mathrm{sgn}\{p_{j_1}, \ldots, p_{j_k}\}$ of T is defined by using the signature of permutation, as follows;

$$\mathrm{sgn}\{p_{j_1}, \ldots, p_{j_k}\} = \mathrm{sgn}\begin{pmatrix} 1, & \ldots, & k \\ \natural(p_{j_1}), & \ldots, & \natural(p_{j_k}) \end{pmatrix}. \tag{35}$$

For example, we have

$$\mathrm{sgn}\{n, m\} = \frac{m-n}{|m-n|}, \quad n \neq m, \qquad \mathrm{sgn}\{n, n\} = 0.$$

Let $\mathbf{C}[w_1, w_2, \ldots] \subset \mathfrak{F}$ be the algebra of polynomials of $w_1, w_2, \ldots$. Then we can regard $w_n \sharp w_m \in \mathbf{C}[w_1, w_2, \ldots]$. So we assume $w_{p_1}\sharp \cdots \sharp w_{p_k} \in \mathbf{C}[w_1, w_2, \ldots]$, $k < n-1$. Let $w_n \sharp' w_m$ be $w_n \sharp w_m$ if $n \neq m$, and $w_n \cdot w_n = w_n^2$ if $n = m$. Then we define $w_{p_1}\sharp' \cdots \sharp' w_{p_n}$ by

$$w_{p_1} \cdots w_{p_n} + \sum_{n-2m \geq 0} (\frac{\pi i}{2})^m \mathrm{sgn}\complement\{p_{j_1}, \ldots, p_{j_{n-2m}}\} w_{p_{j_1}} \cdots w_{j_{n-2m}}. \tag{36}$$

For example, if $j < k < l$, we have

$$w_j \sharp w_k \sharp w_l = w_j \cdot w_k \cdot w_l + \frac{\pi i}{2}(w_j + w_k + w_l).$$

Assuming $j < k < l < m$, $w_j \sharp w_k \sharp w_l \sharp w_m$ is given by

$$w_j w_k w_l w_m + \frac{\pi i}{2}(w_j w_k + w_j w_l + w_j w_m + w_k w_l + w_k w_m + w_l w_m) - \frac{\pi^2}{4}.$$

By this definition, we can regard $\mathbb{F}$ to be an operator algebra acting on $\mathfrak{F}$ by the $\sharp$-product. Similar to (25) and (26), we define fractional partial derivation $\frac{\partial^a}{\partial x_n^a}$ and logarithm of partial derivation $\log(\frac{\partial}{\partial x_n})$ by

$$\frac{\partial^a}{\partial x_n^a} f = \mathrm{e}^{\sharp - a(w_n+\gamma)} \sharp f, \tag{37}$$

$$\log(\frac{\partial}{\partial x_n}) f = -(w_n + \gamma)\sharp f. \tag{38}$$

The following Lemma also follows from this definition.

Lemma 3. *Let* $\mathbf{C}[w_1, w_2, \ldots]$ *and* $\mathbf{C}[w_1, w_2, \ldots]_\sharp$ *be polynomial algebras in* $\mathfrak{F}$ *and* $\mathbb{F}$*, respectively. Then we can identify them as vector spaces.*

We denote the subspaces of $\mathfrak{F}$ and $\mathbb{F}$ consisted by polynomials of order at most n and homogeneous polynomials of order n by $\mathfrak{F}^n$, $\mathbb{F}^n$, $\mathfrak{F}_n$ and $\mathbb{F}_n$, respectively. As vector spaces, we have

$$\mathfrak{F}^n \cong \mathbb{F}^n, \quad \mathfrak{F}_n \cong \mathbb{F}_n. \tag{39}$$

$\mathfrak{F}$ is commutative, but $\mathbb{F}$ is a non commutative algebra. By the commutator $[f,g]=[f,g]_\sharp=f\sharp g-g\sharp g$, $\mathbb{F}^1$, $\mathbb{F}^2$ and $\mathbb{F}_2$ become Lie algebras. As Lie algebras, $\mathbb{F}^1$ is an ideal of $\mathbb{F}^2$ and we have

$$\mathbb{F}^2=\mathbb{F}^1\oplus\mathbb{F}_2. \tag{40}$$

Since we have

$$[w_n,w_m]=\mathrm{sgn}\{n,m\}\pi i, \tag{41}$$

we get by Campbell-Haussdorff formula (or by direct calculation)

$$\mathrm{e}^{\sharp a(w_n+\gamma)}\sharp\mathrm{e}^{\sharp b(w_m+\gamma)}=\mathrm{e}^{\sharp(a(w_n+\gamma)+b(w_m+\gamma)+\mathrm{sgn}\{n,m\}ab(\frac{\pi i}{2})}.$$

Hence we obtain

Proposition 1. (i). *If $n\neq m$, then*

$$\mathrm{e}^{\sharp a(w_n+\gamma)}\sharp\mathrm{e}^{\sharp b(w_m+\gamma)}=(-1)^{ab}\mathrm{e}^{\sharp b(w_m+\gamma)}\sharp\mathrm{e}^{\sharp a(w_n+\gamma)}, \tag{42}$$

where $-1=\mathrm{e}^{\pi i}$ if $m>n$, and $-1=\mathrm{e}^{-\pi i}$ if $m<n$.

(ii). *We have*

$$\mathrm{e}^{\sharp -a(w_n+\gamma)}\sharp\mathrm{e}^{\sharp a(w_m+\gamma)}\sharp 1 = (-1)^{-a^2}\frac{\sin(\pi a)}{\pi a}(\frac{x_m}{x_n})^a, \tag{43}$$

$$\mathrm{e}^{\sharp -(w_n+\gamma)}\sharp\mathrm{e}^{\sharp (w_m+\gamma)}\sharp 1 = \mathrm{e}^{\sharp (w_m+\gamma)}\sharp\mathrm{e}^{\sharp -(w_n+\gamma)}\sharp 1=\delta_{n.m}. \tag{44}$$

Here, -1 is same as above, and $\delta_{n.m}$ is Kronecker's δ.

Note. $\mathfrak{F}$ is the algebra of functions on the universal covering space of $(\mathbf{C}\setminus\{0\})\times(\mathbf{C}\setminus\{0\})\times\cdots$. So the operaotrs such as $\log(\frac{\partial}{\partial x_1}+\frac{\partial}{\partial x_2})$ do not belong to $\mathbb{F}$.

5 Fractional degree differential forms

In this section, we assume the variables $x_1=\mathrm{e}^{w_1},x_2=\mathrm{e}^{w_2}\ldots$ are the coordinate functions of H.

Let $\sum c_nw_n$ be an element of $\mathbb{F}$, then we have $\sum|c_n|<\infty$. Since

$$\sum_{n=1}^{\infty}\sum_{m=n+1}^{\infty}c_nc_m=\frac{1}{2}((\sum c_n)^2-\sum c_n).$$

Because to set $\sum c_n=C$, we can rewrite $\sum_{n=1}^{m}=C-\sum_{n=m+1}^{\infty}$. Hence we obtain

$$\mathrm{e}^{\sharp c_1w_1}\sharp\mathrm{e}^{\sharp c_2w_2}\sharp\cdots=\mathrm{e}^{\sharp(\sum c_ne_n+\frac{C(C-1)}{2}\pi i)}\mathrm{e}^{\sharp\sum c_nw_n}. \tag{45}$$

By (45), we obtain

$$\mathrm{e}^{\sharp - \omega(s)} = \mathrm{e}^{\frac{\zeta(G,s)(\zeta(G,s)-1)}{2}\pi i}\mathrm{e}^{\sharp - \mu_1^s(w_1+\gamma)}\sharp \mathrm{e}^{\sharp - \mu_2^s(w_2+\gamma)}\sharp \cdots . \tag{46}$$

Definition. We define regularized infinite product : $\prod_{n=1}^{\infty,\rightarrow} \sharp \mathrm{e}^{\sharp-(w_n+\gamma)}$: thought to be the regularization of $\mathrm{e}^{\sharp-(w_1+\gamma)}\sharp \mathrm{e}^{\sharp-(w_2+\gamma)}\sharp \cdots$, by

$$:\prod_{n=1}^{\infty,\rightarrow} \sharp \mathrm{e}^{\sharp-(w_n+\gamma)} : \sharp f = \mathrm{e}^{-\frac{\nu(\nu-1)}{2}\pi i}\mathrm{e}^{\sharp-\omega(s)}\sharp f|_{s=0}. \tag{47}$$

: $\prod_{n=1}^{\infty,\rightarrow} \sharp \mathrm{e}^{\sharp-(w_n+\gamma)}$: may not acts on $\mathfrak{F}$. So we can not consider it to be an element of $\mathbb{F}$. But if $f \in \mathbf{C}[w_1, w_2, \ldots]$, then

$$:\prod_{n=1}^{\infty,\rightarrow} \sharp \mathrm{e}^{\sharp-(w_n+\gamma)} : \sharp f = \lim_{N\to\infty} \frac{\partial^N}{\partial x_1 \cdots \partial x_N} f.$$

So : $\prod_{n=1}^{\infty,\rightarrow} \sharp \mathrm{e}^{\sharp-(w_n+\gamma)}$: is densely defined in $\mathfrak{F}$.

Since $\log \Gamma(1+\mu_n^s) = -\gamma \mu_n^s + O(\mu_n^2 s)$, we have

$$-\mu_n^s(w_n - \gamma) - \log\Gamma(1+\mu_n^s) = -\mu_n^s w_n + O(\mu_n^2 s).$$

So we have

$$\Gamma(1+\mu_n^s)^{-1}\mathrm{e}^{\sharp-\mu_n^s(w_n+\gamma)} = \mathrm{e}^{\sharp-\mu_n^s w_n} + O(\mu_n^{2s}).$$

Therefore, we may consider

$$:\frac{\partial^\infty}{\prod \partial x_n} : f = \mathrm{e}^{\frac{\nu(\nu-1)}{2}\pi i}\mathrm{e}^{\sharp - \sum \mu_n^s w_n}\sharp f|_{s=0}. \tag{48}$$

We set

$$\mathbb{F}_{1,-} = \{\sum c_n w_n \in \mathbb{F}_1 | \Re c_n \leq 0\}, \tag{49}$$

$$\mathbb{F}_{1,+} = \{\sum c_n w_n \in \mathbb{F}_1 | \Re c_n \geq 0\}, \tag{50}$$

and denote $\mathrm{Exp}(\mathbb{F}_{1,-})$ and $\mathrm{Exp}(\mathbb{F}_{1,+})$, the algebras generated by $\{\mathrm{e}^{\sharp u} | u \in \mathbb{F}_{1,\pm}\}$ and 1, respectively. The sub vector spaces of $\mathrm{Exp}(\mathbb{F}_{1,\pm})$ generated by

$$\mathrm{e}^{\sharp u},\ u = \sum_{j=1}^k c_{n_j} w_{n_j}, \quad c_{n_1} \neq 0, \ldots, c_{n_k} \neq 0,$$

are denoted by $\mathrm{Exp}(\mathbb{F}_{1,\pm})_k$. Then $\mathrm{Exp}(\mathbb{F})_{1,\pm} = \cup_{k=0}^\infty \mathrm{Exp}(\mathbb{F}_{1,\pm})$ are the algebras generated by $\mathrm{Exp}(\mathbb{F}_{1,\pm})$ and 1.

Let $\phi \in \mathrm{Exp}(\mathbb{F}_{1,-})$ and $\psi \in \mathrm{Exp}(\mathbb{F}_{1,+})$. Then $\phi\sharp\psi\sharp 1$ is a function of $x_1, x_2, \ldots.\ x_n$ is expressed as $r_n \mathrm{e}^{i\theta_n}$ by polar coordinate. Then the constant part $c_0((\upsilon)$ of $\upsilon = \phi\sharp\psi\sharp 1$ is given by

$$\lim_{N\to\infty}(\lim_{n_1\to\infty}\frac{1}{2n_1\pi}\int_0^{2n_1\pi} d\theta_1 \ldots \lim_{n_N\to\infty}\frac{1}{2n_N\pi}\int_0^{2n_N\pi} d\theta_N \upsilon(\theta_1,\ldots)).$$

Definition. We define the pairing (ϕ, ψ) of $\phi \in \mathrm{Exp}(\mathbb{F}_{1,-})_k$ and $\psi \in \mathrm{Exp}(\mathbb{F}_{1,+})_k$ by

$$(\phi, \psi) = c_0(\psi\sharp\phi\sharp 1). \tag{51}$$

By this definition and Proposition 1, we have

$$(\mathrm{e}^{\sharp - a(w_n+\gamma)}, \mathrm{e}^{\sharp b(w_m+\gamma)}) = 0, \quad a \neq b, \ \textit{or}\ n \neq m, \tag{52}$$

$$(\mathrm{e}^{\sharp - a(w_n+\gamma)}, \mathrm{e}^{\sharp a(w_n+\gamma)}) = -(-1)^{-(a-1)^2}\frac{\sin(\pi a)}{\pi a}, \quad a \notin \mathbf{Z}. \tag{53}$$

Hence in $\mathrm{Exp}(\mathbb{F}_{1,\pm})_1$, $\mathrm{e}^{\sharp - a(w_n+\gamma)}$ and $\mathrm{e}^{\sharp a(w_n+\gamma)}$ are dual basis each other.

In $\mathrm{Exp}(\mathbb{F}_{1,+})$, the ideal generated by $\mathrm{e}^{\sharp 2(w_n+\gamma)}, n = 1, 2, \ldots$ is denoted by $\mathbb{I}_+ = \mathbb{I}$. For simple, $\mathrm{Exp}(\mathbb{F})_{1,+} \cap \mathbb{I}$ is also denoted by $\mathbb{I}$. Similar ideal in $\mathrm{Exp}(\mathbb{F}_{1,-})$ is also denoted by $\mathbb{I}_- = \mathbb{I}$.

Definition. We set

$$\mathrm{Fr}\Lambda H = \mathrm{Exp}(\mathbb{F})_{1,+}/\mathbb{I}, \tag{54}$$

and say the algebra of fractional differential forms on H. The element of $\mathrm{Fr}\Lambda H$ *is said to be a fractional order differential form.*

We also use notations

$$\mathrm{Fr}\Lambda^p H = \mathrm{Exp}(\mathbb{F}_{1,+})_p/(\mathbb{I} \cap \mathrm{Exp}(\mathbb{F}_{1,+})_p), \tag{55}$$

$$\mathrm{Fr}\widetilde{\Lambda H} = \mathrm{Exp}(\mathbb{F}_{1,+})/\mathbb{I}. \tag{56}$$

Let $u^\flat$ be the class of $u \in \mathrm{Exp}(\mathbb{F}_{1,+})$ in $\mathrm{Fr}\widetilde{\Lambda H}$. Then we denote

$$u^\flat \wedge v^\flat = (u\sharp v)^\flat. \tag{57}$$

Definition. We denote the class of $\mathrm{e}^{\sharp a(w_n+\gamma)}$ in $\mathrm{Fr}\Lambda H$ by $d^a x_n$, and say the a-th order differential form.

By the definition of $\mathbb{I}$, we may assume $0 \leq \Re a < 2$ in the expression of $d^a x_n$. If $a = 1$, we denote dx_n instead of $d^1 x_n$. By (42), we have

$$d^a x_n \wedge d^b x_m = (-1)^{ab} d^b x_m \wedge d^a x_n, \quad n \neq m. \tag{58}$$

Here $-1 = e^{\pi i}$ if $n < m$ and $-1 = e^{-\pi i}$ if $n > m$. Since $dx_n \wedge dx_n$ is the class of $e^{\sharp 2(w_n+\gamma)} \in \mathbb{I}$, we have $dx_n \wedge dx_n = 0$. So together with (58), we recover usual commutation law of differential forms.

Note. Similarly, we can define fractional vector fields. Then by (51), we obtain the pairing of fractional differential forms and vector fields.

$\omega(s)$ does not belong to $\mathrm{Exp}(\mathbb{F})_{1,+}$, but belongs to $\mathrm{Exp}(\mathbb{F}_{1,+})$. So $\omega(s)^\flat$ does not belong to $\mathrm{Fr}\Lambda H$ but belongs to $\mathrm{Fr}\widetilde{\Lambda H}$. Similar to $\prod_{n=1}^{\infty,\to} \sharp e^{\sharp-\mu_n^s(w_n+\gamma)}$, we define infinite wedge product $\prod_{n=1}^{\infty,\to} \wedge d^{\mu_n^s} x_n$, thought to be $d^{\mu_1^s} \wedge d^{\mu_2^s} x_2 \wedge \dots$ by

$$\prod_{n=1}^{\infty,\to} \wedge d^{\mu_n^s} x_n = e^{\frac{\zeta(G,s)(\zeta(G,s)-1)}{2}\pi i} (e^{\sharp\omega(s)})^\flat. \tag{59}$$

Let $\psi \in \mathrm{Exp}(\mathbb{F})_{1,+}$ be $e^{\sharp u}$, $u = \sum c_n w_n$. Then as the representative of $\psi^\flat$, we take $e^{\sharp u^\natural}$, where

$$u^\natural = \sum \widetilde{c_n} w_n, \quad \tilde{c} \cong c, \ mod.2, \ 0 \le \Re\tilde{c} < 2,$$

and define the action $\psi^\flat * f$ of $\psi^\flat$ to $f \in \mathfrak{F}$ by

$$\psi^\flat * f = e^{\sharp u^\natural} \sharp f. \tag{60}$$

Similar to (47), we define regularized infinite wedge product $: \Lambda_{n=1}^{\infty,\to} dx_n :$ by

$$: \Lambda_{n=1}^{\infty,\to} dx_n : * f = e^{\frac{\zeta(G,s)(\zeta(G,s)-1)}{2}\pi i} (e^{\sharp\omega(s)})^\flat * f|_{s=0}. \tag{61}$$

$: \Lambda_{n=1}^{\infty,\to} dx_n :$ can not act on $\mathfrak{F}$, but acts on $\mathbf{C}[w_1, w_2, \dots]$. So it is densely defined in $\mathfrak{F}$.

6 Coordinate transformation of $\mathbb{F}$

Let T be a linear operator acting on $\mathfrak{F}$. We assume T can be written as $T = e^S$, for some linear operator S acting on $\mathfrak{F}$. If $u^{\sharp u}$, $u \in \mathbb{F}$ belongs to $\mathbb{F}$, then we can consider $T \cdot e^{\sharp u}$, or $e^{\sharp u} \cdot T$. Let $t \in \mathbf{C}$ be a parameter. Then by Campbell-Haussdorff formula, we can set

$$e^{tS} \cdot e^{\sharp tu} = e^{t(S+\sharp u)+t^2 CH(S,\sharp u;t)}, \tag{62}$$

where $CH(S, \sharp u; t)$ is a Taylor series of t with the convergence radius $r > 0$. r may be smaller than 1. But since $t \cdot e^{\sharp u}$ exists, there is a path $\beta = \beta(s)$ in $\mathbf{C}$ such that $\beta(0) =$, $\beta(1) = 1$ and $CH(S, \sharp u; t)$ allows analytic continuation along β. We denote $CH(S, \sharp u; \beta; 1)$ the value of the analytic continuation of along β of $CH(S, \sharp u; t)$ at 1 (cf. [3]). Then we have

$$T \cdot e^{\sharp u} = e^{\sharp u + S + CH(S,\sharp u;\beta;1)}. \tag{63}$$

Definition. We define the action of S to $u \in \mathbb{F}$ with respect to β by

$$S +_\beta u = u + S + CH(S, \sharp u; \beta; 1). \tag{64}$$

Note. The action of S to u does not determined absolutely. It depends on the choice of β. Later, we need to get logarithm of the action of T to $e^{\sharp u}$. In this case, the action also depends on the choice of S, the logarithm of T.

Definition. We denote $S+_\beta$ the action of a linear operator S of $\mathfrak{F}$ and a path β of $\mathbf{C}$ joining 0 and 1 defined by $S +_\beta u, u \in \mathbb{F}$.

Note. $S+_\beta$ may not defined on $\mathbb{F}$. But densely defined in $\mathbb{F}$.

If $T_1 = e^{S_1},\ T_2 = E^{S_2}$, then we can set

$$e^{tS_1} \cdot e^{tS_2} = e^{t(S_1+S_2)+t^2 CH(S_1,S_2;t)}.$$

Let $CH(S_1, S_2; \beta; 1)$ be same as above. Then we may assume β satisfies

$$T_1 \cdot T_2 = e^{S_1+S_2+CH(S_1,S_2;\beta;1)}. \tag{65}$$

The notation $S_1 +_\beta S_2$ is also used in this case. Precisely, we use the following notations;

$$\begin{aligned} S_1 +_\beta S_2 &= S_1 + S_2 + c_\beta(S_1, S_2), & (66)\\ c_\beta(S_1, S_2) &= CH(S_1, S_2; \beta; 1). & (67) \end{aligned}$$

By definition, if $S_1 S_2 = S_2 S_1$, then $c_\beta(S_1, S_2) = 0$ for any β. Especially,we have

$$c_\beta(S, -S) = 0, \tag{68}$$

for any S and β. Similarly, we obtain

$$(e^{S_1})^{c_\beta(S_1+_\alpha S_2, -S_2)} = e^{S_1},$$

for any $S_1,\ S_2$ and $\alpha,\ \beta$. Since $e^{S_1+_\alpha S_2} = e^{S_1+_\beta S_2} = T_1T_2$ holds and $S_1 +_\alpha S_2 = (S_1 +_\beta S_2) + c_\alpha(S_1, S_2) - c_\beta(S_1, S_2)$, we get

$$(T_1T_2)^{c_\alpha(S_1,S_2)-c_\beta(S_1,S_2)} = T_1T_2. \tag{69}$$

We also have

$$\begin{aligned} S_1 +_\alpha (S_2 +_\beta S_3) &= S_1 + S_2 + S_3 + c_\beta(S_2, S_3) + c_\alpha(S_1, S_2 +_\beta S_3),\\ (S_1 +_\alpha S_2) +_\beta S_3 &= S_1 + S_2 + S_3 + c_\beta(S_1 +_\alpha S_2, S_3) + c_\alpha(S_1, S_2). \end{aligned}$$

We set $e^{S_i} = T_i,\ i = 1, 2, 3$. Then by definition, we have

$$e^{S_1+_\alpha(S_2+_\beta S_3)} = T_1(T_2T_3), \quad e^{(S_1+_\alpha S_2)+_\beta S_3} = (T_1T_2)T_3.$$

We define $\delta_{\alpha,\beta}c(S_1, S_2, S_3)$ by

$$c_\beta(S_2, S_3) - c_\beta(S_1 +_\alpha S_2, S_3) + c_\alpha(S_1, S_2 +_\beta S_3) - c_\alpha(S_1, S_2). \tag{70}$$

Then, since $T_1(T_2T_3) = (T_1T_2)T_3$, we have

$$\mathrm{e}^{S+\delta_{\alpha,\beta}c(S_1,S_2,S_3)} = \mathrm{e}^S, \quad \mathrm{e}^S = \mathrm{e}^{S_1}\mathrm{e}^{S_2}\mathrm{e}^{S_3}. \tag{71}$$

Since Taylor expansion of $[f, w_n]_\sharp$ in w_n begins at least order 2-term. So the ideal $\mathbb{I}$ is invariant by the action $f +_\beta$. Hence we can transfer the action $f + \beta$ of $\mathbb{F}$ to the action of $\mathrm{Fr}\Lambda H$ or $\mathrm{Fr}\widetilde{\Lambda H}$.

Definition. We define the action $T \circ_{S,\beta} d^a x_n = \mathrm{e}^S \circ_\beta d^a x_n$ by

$$T \circ_{S,\beta} d^a x_n = \mathrm{e}^{S +_\beta \sharp a w_n}. \tag{72}$$

The action $T \circ_{S,\beta} \psi$ of $T = \mathrm{e}^S$ and β for an element ψ of $\mathrm{Fr}\Lambda H$ is similarly defined.

Note. We also use the notation $\mathrm{e}^S \circ_\beta$, which means $T \circ_{S,\beta}$, when $T = \mathrm{e}^S$.

Since we have

$$\begin{aligned} \mathrm{e}^{S_1} \circ_\alpha (\mathrm{e}^{S_2} \circ_\beta d^a x_n) &= \mathrm{e}^{S_1+S_2+\sharp a w_n + c_\beta(S_2,\sharp a w_n) + c_\alpha(S_1, S_2 +_\beta \sharp a w_n)}, \\ (\mathrm{e}^{S_1}\mathrm{e}^{S_2}) \circ_\beta d^a x_n &= \mathrm{e}^{S_1+S_2+\sharp a w_n + c_\alpha(S_1,S_2) + c_\beta(S_1 +_\alpha S_2, \sharp a w_n)}, \end{aligned}$$

the obstruction to the associativity of the action of linear operators $T_i = \mathrm{e}^{S_i},\ i = 1, 2$ on $\mathfrak{F}$ to $d^a x_n$ is given by $\delta_{\alpha,\beta}c(S_1, S_2, \sharp a w_n)$.

Definition. We denote $T \circ_{S,\beta}$ or $\mathrm{e}^S \circ_\beta$ the action of T and a path β to $\mathrm{Fr}\Lambda H$ defined by $T \circ_{S,\beta} \psi,\ \psi \in \mathrm{Fr}\Lambda H$.

By this notation, we can define $\delta_{\alpha,\beta}(S_1, S_2, \psi)$, which can be regarded as a function of ψ. Regarding $\delta_{\alpha,\beta}(S_1, S_2, \psi)$ to be a function, we denote $\delta_{\alpha,\beta}(S_1, S_2)$ instead of $\delta_{\alpha,\beta}(S_1, S_2, \psi)$.

7 Fractional differential forms on mapping spaces

Let $Map(X, M)$ be the space of maps from a compact Riemannian manifold X to a smooth almost complex manifold M of complex dimension d. We fix a non-degenerate selfadjoint elliptic (pseudo) differential operator D on X (acting on scalar fields). The operator $D \otimes I$, I the identity matrix in $GL(d, \mathbf{C})$, is also denoted by D. By assumption, D determines the Sobolev metric in $L^2(X) \otimes \mathbf{C}^d$. Denoting $W^k(X)$ the Sobolev k-space constructed by D and $L^2(X) \otimes \mathbf{C}^d$, $Map(X, M)$ becomes a Sobolev manifold modeled by $W^k(X)$ and we may consider $W^k(X)$ is equipped with D. $L^2(X)\otimes \mathbf{C}^d$ is H and the Green operator of D is G in the formalism of section 2.

Let $\tau = \tau(Map(X,M))$ and $\tau* = \tau*(Map(X,M))$ be the tangent bundle and cotangent bundle of $Map(X,M)$, respectively. Then the fibre of τ is $W^k(X)\otimes \mathbf{C}^d$ and the fibre of $\tau*$ is $W^{-k}(X)\otimes \mathbf{C}^d$. We may assume the structure groups of τ and $\tau*$ are the loop group $\Omega U(N)$. In general, we can not equip D to the fibres of τ and $\tau*$. But to add connection term A_U to D, we have

$$g_{UV}(D+A_V) = (D+A_U)g_{UV},$$

where $\{g_{UV}\}$ is the transition function of τ. Since we can take A_U to be an Hermitian operator, $\{A_U\}$ also becomes a connection of D with respect to $\tau*$. Since $g_{UV}(x)$, $x\in U\cap V$ is a unitary operator, spectres of $D+A_U(x)$ does not depend on U. Moreover, if D is positive, we can take A_U such that $D+A_U(x)$ is positive for any U and $x\in U$.

The complete ortho-normal basis of $L^2(X)\otimes \mathbf{C}^d$ thought to be the (co)tangent space of $Map(X,M)$ should be the proper functions $\{e_{U,n}\}; e_{U,n} = e_{U,x,n}$ of $D+A_U(x)$, we need to take $w_{U,n} = \log x_{U,n}$, $(x_{U,1}, x_{U.2},\ldots)$ is the coordinate of $L\ 2(X)\otimes \mathbf{C}^d$ determined by $\{e_{U,n}\}$, as the basis of $\mathbb{F}$ instead of $\{e_n\}$. We note since $g_{UV}(D+A_V) = (D+A_U)g_{UV}$, we have

$$e_{U,n} = g_{UV}e_{V,n}.$$

To develop global analysis on $Map(X,M)$, we need to assume integrity of the regularized dimension $\nu(x) = \zeta(D+A_U(x),0)$. Let k be the order of D and $l=[j/k]$, where $[\ ]$ is the Gauss notation and $j = dimX$, then take the mass term $m = m(x)$ to be a solution of

$$\begin{aligned}\tfrac{Res_{s=l}\zeta(D+A_U(x),s)}{l}m^l - \tfrac{Res_{s=l-1}\zeta(D+A_U(x),s)}{l-1}m^{l-1} + \cdots\\ \cdots + (-1)^{l-1}Res_{s=1}\zeta(D+A_U(x),s)m + (-1)^l\nu(x) \qquad &= n,\end{aligned}$$

we have $\zeta(D+A_U(x)+m,0) = n$([5], cf. [8]). Since $\zeta(D+A_U(x),s)$ does not depend on U, the residue $r(x,k)$ of $\zeta(D+A_U(x),s)$ at $s=k$ is a smooth function of x. Hence the equation

$$\frac{r(x,l)}{l}m^l - \frac{r(x,l-1)}{l-1}m^{l-1} + \cdots + \nu(x) = n,$$

is globally defined on $Map(X,M)$. If the discriminant of this equation does not vanish on $Map(X,M)$ *e.g.* $l=1$, we can choose $m=m(x)$ to be a globally defined function on $Map(X,M)$. So we can choose n as the virtual dimension of $Map(X,M)$. Otherwise, we can not choose n uniformly on $Map(X,M)$.

Since τ is a loopgroup bundle, its transition function g_{UV} can be set $e^{h_{UV}}$ if $\oint tr(g_{UV}^{-1}dg_{UV}) = 0$. In general, the 1-dimensional cohomology class $s^1(\tau)$ in $H^1(Map(X,M),\mathbf{Z})$ represented by $\{\frac{1}{2\pi i}\oint tr(g_{UV}^{-1}dg_{UV})\}$ gives the complete obstruction to the expression $\{g_{UV}\} = \{e^{h_{UV}}\}$. In the rest, *we assume* $s^1(\tau)=0$. We note in this case, we have

$$h_{UV} +_\beta h_{VW} = h_{UW},$$

for any β. So we denote $h_{UV} +_b h_{VW}$ instead of $h_{UV} +_\beta h_{VW}$.The operation $+_b$ may not be commutative, but associative.

Note. If $X = S^3$, assuming $s^2(\tau) = 0$, we can set $g_{UV} = e^{h_{UV}}$ directly. So we proceed same discussion below, without reducing the structure group of τ to the loopgroup $\Omega U(N)$.

The associate $\mathbb{F}$-bundle of τ can be constructed if and only if

$$h_{UV} +_\alpha (h_{VW} +_\beta u) = (h_{UV} +_b h_{VW}) +_\beta u \ (= h_{UW} +_\beta u), \quad u \in \mathbb{F}. \tag{73}$$

Since β-dependence of $+_\beta$ comes from the many valuedness of u, the difference

$$\delta h_{UVW}(\beta)u = (-h_{UW} +_\beta (h_{UV} +_\beta (h_{VW} +_\beta u))).$$

takes the value in $\pi_1(\{\sum x_n e_n \in H^-(finite)|x_n \neq 0\}) = \mathbf{Z}^\infty$. Let β_n be the class of $\{z_n e_n \in H^-(finite)||z_n| = 1\}$. We define $n(\tau)_{UVW;\beta_n} \in \mathbf{Z}$ and the map $n(\tau)_{UVW} : \mathbf{Z}^\infty \to \mathbf{Z}$ by

$$n(\tau)_{UVW;\beta_n} = \frac{1}{2\pi i}\delta h_{UVW}(\beta_n)w_{w,n}. \tag{74}$$

$$n(\tau)_{UVW}(\beta_n) = n(\tau)_{UVW;\beta_n}. \tag{75}$$

By definition, $\{n(\tau)_{UVW}\}$ is a 2-cocycle taking the values in $(\mathbf{Z}^\infty)^*$, the dual of $\mathbf{Z}^\infty$. Since $\mathbf{Z}^\infty$ is the fundamental group $\pi_1(U_x^*)$ of U_x^*,where U_x^* means $\{\sum c_n e_{U,x,n}|c_n \neq 0\}$,

$$(\mathbb{Z}^\infty)^* = \cup_{x \in Map(X,M)}\pi_1(U_x^*,$$

becomes a local system *if the spectres of* $D + A_U(x)$ *has no crossing.* Here we regard $\{e_{U,x,n}\}$ to be the local coordinate of $Map(X,M)$ at x. Under this assumption, we obtain

Proposition 2. (i). *The cohomology class* $o(\tau;\mathbb{F})$ *of* $\{n(\tau)_{UVW}\}$ *in* $H^2(Map(X,M),(\mathbb{Z}^\infty)^*)$ *is determined by* τ.
(ii). *We can construct associate* $\mathbb{F}$*-bundle of* τ *if and only if* $o(\tau;\mathbb{F}) = 0$.

Note 1. If ξ is a loopgroup bundle over $Map(X,M)$ such that $s^1(\xi) = 0$. Then we can define $o(\xi;\mathbb{F})$ by the same way and the associate $\mathbb{F}$-bundle of ξ exists if and only if $o(\xi;\mathbb{F}) = 0$.

Especially, since τ and $\tau*$ have same transition function, $o(\tau;\mathbb{F}) = o(\tau*;\mathbb{F})$. Hence associate $\mathbb{F}$-bundle of τ exists if and only if associate $\mathbb{F}$-bundle of $\tau*$ exists.

Note 2. If $X = S^1$, taking $D = \frac{1}{i}\frac{d}{dt} + c$, t is the loopvariable, the spectres of $D + A_U(x)$ are $\{2n\pi + c(x)|n \in \mathbf{Z}\}$. Since $c(x)$ is a smooth function on ΩX, spectres of $D + A_U(x)$ do not cross. Hence we can use Proposition 2 in this case.

Definition. We denote the analytic continuation of $e^{tf} \cdot e^{tg}$ along β to $t = s,\ s < 1$, by $e^{f +_{\beta,s} g}$.

Symbolically, we may consider $e^{f+_{\beta,s}g} = e^{sf} \circ_{s\beta} e^{sg}$. Here $s\beta$ means the path defined by $s\beta(t) = \beta(st)$. So, if $e^f = \phi$ and $e^g = \psi$, we may consider

$$e^{f+_{\beta,s}g} = \phi^s \circ_{s\beta} \psi^s.$$

Definition. We define the action $g_{UV;\beta}d^a x_n$ of g_{UV} to $d^a x_n$ with respect to β by

$$g_{UV;\beta}d^a x_n = e^{h_{UV}+_{\beta,a}aw_n}. \tag{76}$$

Note. This definition of the action of g_{UV} to $d^a x_n$ needs to assume $a \leq 1$. When $1 < a < 2$, we define the action of g_{UV} to $d^a x_n$ by using the relation $d^a x_n = dx_n \wedge d^{a-1}x_n$.

By using the action $g_{UV;\beta}$, we can construct associate $\mathrm{Fr}\Lambda H$-bundle of $\tau*$, if $o(\tau;\mathbb{F}) = 0$. So we can consider fractional differential forms on $Map(X,M)$ if $s^1(\tau) = o(\tau,\mathbb{F}) = 0$. In this case, associate $\mathrm{Fr}\widetilde{\Lambda H}$-bundle of $\tau*$ is also defined.

If the associate $\mathbb{F}$-bundle of $\tau*$ exists,

$$\omega_U(s) = \sum \mu_n^s(w_{U,n} + \gamma), \quad (D + A_U)e_{U,n} = \mu_n^{-1}e_{U,n},$$

is a cross-section of this bundle. $e^{\sharp\omega_U(s)}$ induces a cross-section of the associate $\mathrm{Fr}\widetilde{\Lambda H}$-bundle of $\tau*$. But it is not a cross-section of the associate $\mathrm{Fr}\Lambda H$-bundle of $\tau*$. Hence it defines a line bundle over the associate $\mathrm{Fr}\Lambda H$-bundle of $\tau*$.

If each $D + A_U$ is taken to be positive, and if we can choose the virtual dimension $\nu(x) = \zeta(D + A_U(x), 0)$ to be an integer n, uniformly,then this bundle is trivial. Therefore to define $\Lambda^{\infty,\to}dx_{U,n}$ similar to $\Lambda^{\infty,\to}dx_n$, we get the regularized volume form of $Map(X,M)$ as a cross-section of the trivial line bundle over the associate $\mathrm{Fr}\Lambda H$-bundle of $\tau*$.

If we can not chose $\nu(x)$ to be a constant integer, the sign $(-1)^{\nu(\nu-1)/2}$ in the definition of $:\Lambda^{\infty,\to}dx_{U,n}:$ causes many valuedness of $:\Lambda^{\infty,\to}dx_{U,n}:$. Hence the line bundle determined by $:\Lambda^{\infty,\to}dx_{U,n}:$ may be non-trivial if $\pi_1(Map(X.M))$ is non-trivial.

If D is not positive *e,g.*, D is the Dirac operator, then the situation is more complicated. But as for a loopspace ΩM, we have the following Theorem (cf. [11]).

Theorem 2. *Regularized volume form exists on ΩM if the first and second string classes of ΩM vanishes.*

Proof. Let $g : \Omega M \to U(N)$ be the characteristic map of τ, and

$$S(g) = \{x \in \Omega M | det(g(x) - I) = 0\}, \tag{77}$$

$$S^r(g) = \{x \in \Omega M | rank(g(x) - I) = N - r\} \tag{78}$$

Then $\tau|(\Omega M \setminus S(g))$ is trivial and the dual classes

$$\langle S^{2r-1}(g)\rangle \in H^{2r-1}(\Omega M, \mathbf{Z}),$$

of $S^{2r-1}(g)$ represent string classes $s^r(\tau)$ of τ ([4]). Hence, if $s^1(\tau) = s^2(\tau) = 0$, we may assume $codim(S(g)) > 3$, and there exists a neighborhood $U(x)$ of $x \in S(g)$ such that $H^2(U(x) \setminus S(g), \mathbf{Z}) = 0$. Since $\tau|(\Omega M \setminus S(g))$ is trivial, regularized volume form exists on $\Omega M \setminus S(g)$ and it defines a line bundle E. Since $E|p^{-1}(U(x) \setminus S(g))$, p the projection of τ. is trivial, we can extend E to $p^{-1}(U(x) \setminus S(g))$. This shows Theorem.

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