Weak Linearized Gravitational Models Based on Finslerian $(\alpha, \beta)-$Metrics

Vladimir Balan and Panayiotis C. Stavrinos

Abstract

In §1 we consider a generalized Randers-Kropina type $(\alpha, \beta)-$Finslerian metric F on a space-time manifold M and its associated Berwald-type nonlinear connection N on TM. For TM endowed with an (h, v)-metric structure, we build the canonic N-connection dependent on F only, its d-torsions, curvatures, Ricci tensor and scalars of curvature. In §2, a Finslerian perturbation of $(\alpha, \beta)-$type applied to the weak metric yields a pseudo-Riemann - Finslerian $(h, v)-$metric structure on TM for which we derive the Einstein equations and the corresponding conservation laws. In §3 are determined the equations of the stationary curves and of their deviations, outlining the special cases of $h-$ and $v-$paths.

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sectionPreliminaries

The study of weak gravitational Finslerian and generalized Lagrange models was initiated by P.C.Stavrinos [24] by introducing the concept of gravitational waves in a Finsler space. The study was extended in [5] and [6] to the framework of vector bundles endowed with $(h, v)-$metrics (framework established by R.Miron and M.Anastasiei [17], [19]) and to the one of osculator spaces $Osc^k(M)$ of higher-order geometries ([16], [18]) for the particular case when $k = 1$, leading to the B-FWDM, FWDM and CWDM models for General Relativity ([5], [6]) on the tangent bundle (TM, π, M) of a given space-time M.

In this work we study the geometrical structure of a $(h, v)-$metric produced by a Minkowski metric n_{ij} defined on a real 4-dimensional differentiable manifold M, deformed by means of an $(\alpha, \beta)-$Finslerian metric (denoted further as $(a, b)-$metric)

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of Kropina-Randers type. The metric γ_{ij} of the gravitational field is decomposed into the flat Minkowski metric $n_{ij} = diag(-1,1,1,1)$ and a small perturbation ε_{ij} given by a symmetric tensor field satisfying $|\varepsilon_{ij}(x)| << 1$ ([5], [24], [25]),

$$g = \sum_{i=1}^{4}(\delta_{ij}\varepsilon_i + \varepsilon_{ij}(x))dx^i \otimes dx^j, \tag{1}$$

where we denoted $\varepsilon^i = \varepsilon_j = 1 - 2\delta_1^i,\ i = \overline{1,4}$.

From physical point of view, in the linearized version of a given model of the General Relativity, the symmetric tensor field ε_{ij} produces the weak pseudo-Riemannian gravitational field γ_{ij}. In our case, the considered deformations (the weak perturbation and the Finslerian deformation) emerge from a $(a,b)-$fundamental function defined on the tangent bundle [3] $F(a,b)$, with F $p-$homogeneous in a and b, which we assume of the generalized Randers-Kropina form ($p = 1$),

$$F = \alpha a(x,y) + \beta b(x,y) + \gamma \frac{a^2(x,y)}{b(x,y)}, \tag{2}$$

where (x^i, y^a) are local coordinates in a chart $\tilde{U} \subset TM$, and we denote:

$a(x,y) = \sqrt{a_{ij}(x)y^i y^j}$, with $a_{ij}(x)$ Riemannian metric on M;
$b(x,y) = b_i(x)y^i$, with $b_i(x)$ covector field on M.

Remarks.

1. The first two terms of F determine a Randers-type Finsler metric, while the third is a Kropina one.

2. For $\alpha = \beta = 1$ and $\gamma = 0$, the framework becomes the general Randers case studied by R.Miron in [15];

3. For $\alpha = 1,\ \gamma = 0$, and $\beta = \frac{e}{mc^2}$ (with e - electrical charge, m - the mass and c - the speed of light), we get the Randers-type metric used intensively in physical applications [8].

4. For $\gamma = 0$ and $\alpha = \beta = 1$, we have the case studied in [28].

5. For $\alpha = 1, \beta = \gamma = 0$ and $a_{ij} = n_{ij} + \lambda c_i c_j$, with $\lambda \in \mathbb{R}$ and $c_i(x)$ being a 1-form, we get a special Riemannian subcase of the Beil metric case ([6]).

In the following we shall denote briefly $a(x,y)$ and $b(x,y)$ by a and b, respectively, $\partial_i = \frac{\partial}{\partial x^i}$ and $\dot{\partial}_a = \frac{\partial}{\partial y^a}$; the indices "$,i$" and "$;a$" will represent, respectively, the partial differentiation with respect to x^i and y^a. and the zero subscript will represent the transvection with y. Throughout the paper, the Latin indices $i,j,k,\ldots,a,b,c,\ldots$ run implicitly in the range $\overline{1,4}$, while the Greek ones $\alpha,\beta,\gamma,\ldots$, in the range $\overline{1,8}$.

The indices will be raised in the linearized approach via the flat metric n_{ij}, e.g., $\varepsilon^{rs} = n^{ri}n^{sj}\varepsilon_{ij}$.

We note that the space $F^n = (M, L(x,y))$ is a Finsler space, if the fundamental tensor field $h_{ij} = \frac{1}{2}\frac{\partial^2 F^2(x,y)}{\partial y^i \partial y^j}$ is non-degenerate. Its generic form is [11]

$$h_{ij} = a^{-1}F_a k_{ij} + a^{-2}F_{aa}y_i y_j + a^{-1}F_{ab}y_{\{i}b_{j\}} + F_{bb}b_i b_j, \tag{3}$$

where we denote $F_a = \frac{\partial F}{\partial a}$ and $F_{ab} = \frac{\partial^2 F}{\partial a \partial b}$ the partials of F as function of $a(x,y)$ and $b(x,y)$, and k_{ij} is the angular metric

$$k_{ij} = a_{ij} - a^{-2} y_i y_j. \tag{4}$$

More specific, we have the following

Theorem [3].

a) The fundamental tensor field h_{ij} associated to the fundamental function $F(x,y)$ has the form

$$h_{ij} = \mu a_{ij} + \lambda b_i b_j + \pi b_{\{i} y_{j\}} + \rho y_i y_j, \tag{5}$$

where the coefficients are given by

$$\begin{cases} \mu = (\alpha^2 ab^2 + 2\gamma^2 a^3 + \alpha\beta b^3 + 2\beta\gamma ab^2 + 3\alpha\gamma a^2 b)/(ab^2) \\ \lambda = (\beta^2 b^4 + 3\gamma^2 a^4 + 2\alpha\gamma a^3 b)/(b^4) \\ \pi = (-4\gamma^2 a^3 + \alpha\beta b^3 - 3\alpha\gamma a^2 b)/(ab^3) \\ \rho = (4\gamma^2 a^3 - \alpha\beta b^3 + 3\alpha\gamma a^2 b)/(a^3 b^2), \end{cases} \tag{6}$$

and where we denoted $\tau_{\{ij\}} \equiv \underset{ij}{S} \tau_{ij} = \tau_{ij} + \tau_{ji}$ and $y_i = a_{ij} y^i$.

b) The fundamental tensor $h_{ij}(x,y)$ in (5) is non-degenerate, provided that $\mu \neq 0$ and

$$\begin{cases} \lambda(\mu + \lambda\tilde{b} + 2\pi b) + \pi^2 a^2 \neq 0 \\ \left[\lambda(\mu + \rho a^2) - \pi^2 a^2\right] \left[\lambda(\mu + \lambda\tilde{b} + 2\pi b) + \pi^2 a^2\right] - (\lambda\rho - \pi^2)(\lambda b + \pi a^2)^2 \neq 0, \end{cases}$$

where a^{ij} is the reciprocal tensor field of the Riemannian metric a_{ij}, $\tilde{b} = a_{ij} b^i b^j$ and $b^i = a^{ij} b_j$.

c) Provided that the conditions in b) are fulfilled, the reciprocal tensor $h^{ij}(x,y)$ has the coefficients

$$h^{ij} = \tilde{\mu} a^{ij} + \tilde{\lambda} b^i b^j + \tilde{\pi} b^{\{i} y^{j\}} + \tilde{\rho} y^i y^j, \tag{7}$$

where,

$$\begin{cases} \tilde{\lambda} = (\lambda^2 p + v^2 q), \quad \tilde{\pi} = -(\lambda\pi p + uvq), \quad \tilde{\rho} = -(\pi^2 p + u^2 q), \\ q = w \cdot [w + (\mu a^2 + 2\pi b a^2) + \lambda(b^2 + a^2 \tilde{b})]^{-1}, \\ p = (\mu w)^{-1}, \quad \tilde{l} = \lambda^2 \tilde{b} + 2\pi\lambda b + \pi^2 a^2, \quad \tilde{\mu} = \mu^{-1}, \\ u = (\mu + \lambda\tilde{b} + \pi b)/p, \quad v = (\lambda b + \pi a^2)/q, \quad w = \mu\lambda + \tilde{l}. \end{cases} \tag{8}$$

Remark. In (5), the coefficients α^2 and γ^2 with (6) multiply exactly the Riemannian and the Kropina fundamental tensor fields, respectively; for $\gamma = 0$, h_{ij} becomes the Randers metric tensor field.

Proof. Indeed, the metric (5) admits the convenient form

$$h_{ij} = \mu a_{ij} + \lambda^{-1} l_i l_j + \theta y_i y_j$$

where $l_i = \lambda b_i + \pi y_i$, $\theta = \rho - \frac{\pi^2}{\lambda}$. To our aim, we apply repeatedly the following main result which generalizes the one in [15].

Lemma 1. Let $(a_{ij}) \in GL(n, \mathbb{R})$, $\alpha, \beta \in \mathbb{R}^*$; $v_i \in \mathbb{R}$, $i = \overline{1,n}$, and the matrix $(b_{ij}) \in \mathcal{M}_n(\mathbb{R})$, having as coefficients

$$b_{ij} = \alpha a_{ij} + \beta v_i v_j,$$

such that $\alpha + \beta v^2 \neq 0$, where $v^2 = a^{ij} v_i v_j$. Then we have

a) *det* $(b_{ij}) = \alpha^{n-1}(\alpha + \beta v^2) \cdot det(a_{ij})$;

b) If $(b_{ij}) \in GL(n, \mathbb{R})$, then the coefficients of its inverse (b^{ij}) are

$$b^{ij} = \frac{1}{\alpha}\left(a^{ij} - \beta \frac{v^i v^j}{\alpha + \beta v^2}\right),$$

where $v^i = a^{ij} v_j$ and (a^{ij}) is the inverse matrix of (a_{ij}).

As we have seen, the Finslerian metric h in (5) provides for the case $\alpha = 1, \beta = \gamma = 0$ a *Beil-type perturbation* of n_{ij},

$$\gamma_{ij}(x) = g_{ij}(x) = a_{ij} = n_{ij} + \varepsilon_{ij}, \tag{9}$$

where $\varepsilon_{ij} = \lambda c_i(x) c_j(x)$, with $\lambda \in \mathbb{R}$ and $c_i dx^i \in \Lambda^1(M)$. Then, the canonic *non-linear connection* N on TM provided by $\gamma_{ij}(x)$ is

$$N_i^a(x, y) = \gamma_{j0}^a, \tag{10}$$

where

$$\gamma_{jk}^i = \gamma^{ih}(\gamma_{\{jh,k\}} - \gamma_{jk,h})/2 \tag{11}$$

are the Christoffel symbols of the Riemannian metric $g_{ij}(x)$; this produces on $\mathcal{X}(\tilde{U})$ the *local adapted basis*

$$\{\delta_i = \partial_i - N_i^b \partial_b,\ \dot{\partial}_a\}_{i,a=\overline{1,4}} \equiv \{\delta_\beta\}_{\beta=\overline{1,8}}, \tag{12}$$

as well as *the dual local basis*

$$\{d^i = dx^i,\ \delta^a = \delta y^a = dy^a + N_j^a dx^j\}_{i,a=\overline{1,4}} \equiv \{\delta^\beta\}_{\beta=\overline{1,8}}. \tag{13}$$

The Finslerian deformation of the weak metric γ_{ij} considered in the next section will provide a certain $(h, v)-$metric on TM. Generally, a $(h, v)-$metric on the tangent bundle (TM, π, M)

$$G = g_{ij}(x, y) dx^i \otimes dx^j + h_{ab}(x, y) \delta y^a \otimes \delta y^b, \tag{14}$$

defines a *canonical* $N-$*connection* $\mathbf{D}$, dependent only on G and N, having in the adapted basis (12) the coefficients $\{\Gamma^{\alpha}_{\beta\gamma}\} \equiv \{L^i_{jk}, \tilde L^a_{bk}, \tilde C^i_{ja}, C^a_{bc}\}$ given by ([17])

$$\begin{array}{ll} L^i_{jk} = \frac{1}{2}g^{is}(\delta_{\{j}g_{sk\}} - \delta_s g_{jk}), & \tilde L^a_{bk} = \dot\partial_b N^a_k + \frac{1}{2}h^{ac}(\delta_k h_{bc} - h_{c\{d}\dot\partial_{b\}}N^d_k) \\ \tilde C^i_{ja} = \frac{1}{2}g^{ih}\dot\partial_a g_{jh}, & C^a_{bc} = \frac{1}{2}h^{ad}(\dot\partial_{\{b}h_{dc\}} - \dot\partial_d h_{bc}), \end{array} \tag{15}$$

which preserves the $h-v$ splitting produced by N, is metrical and $h-$ and $v-$ symmetrical.

Its *torsion tensor field* $\mathcal{T} \in \mathcal{T}^1_2(TM)$ has the coefficients

$$\mathcal{T}(\delta_\alpha, \delta_\beta) = \mathcal{T}_{\beta\ \alpha}^{\ \kappa}\delta_\kappa, \qquad \mathcal{T}_{\beta\ \alpha}^{\ \kappa} = \Gamma^{\alpha}_{[\beta\kappa]} + B^{\alpha}_{[\beta\kappa]}, \tag{16}$$

where we denoted $\tau_{[\alpha\beta]} = \tau_{\alpha\beta} - \tau_{\beta\alpha}$ and where *the non-holonomy coefficients* $B_{\alpha\ \beta}^{\ \gamma}$ are uniquely defined by the relations $[\delta_\alpha, \delta_\beta] = B_{\alpha\ \beta}^{\ \gamma}\delta_\gamma$. The $h, v-$splitting of $\mathcal{T}$ provides the *torsion* $N-$*tensor fields* [17]

$$\left\{ \begin{array}{ll} T^i_{jk} = d^i\mathcal{T}(\delta_k, \delta_j) = L^i_{[jk]}, & R^a_{kl} = \delta^a\mathcal{T}(\delta_l, \delta_k) = \delta_{[l}N^a_{k]}, \\ P^i_{ja} = d^i\mathcal{T}(\dot\partial_a, \delta_j) = \tilde C^i_{ja}, & P^a_{bk} = \delta^a\mathcal{T}(\delta_k, \dot\partial_b) = \dot\partial_b N^a_k - \tilde L^a_{bk}, \\ S^a_{bc} = \delta^a\mathcal{T}(\dot\partial_c, \dot\partial_b) = C^a_{[bc]}, & \end{array} \right. \tag{17}$$

Similarly, the *curvature tensor field* $\mathcal{R} \in \mathcal{T}^1_3(TM)$ of the $N-$connection $\mathbf{D}$ has the coefficients given by

$$\mathcal{R}(\delta_\alpha, \delta_\beta)\delta_\gamma = \mathcal{R}_{\gamma\ \beta\alpha}^{\ \lambda}\delta_\lambda, \qquad \mathcal{R}_{\beta\ \gamma\theta}^{\ \alpha} = \delta_{[\theta}-^{\alpha}_{\beta\gamma]} + -^{\phi}_{\beta[\gamma}-^{\alpha}_{\phi\theta]} + -^{\alpha}_{\beta\phi}\mathcal{B}^{\phi}_{\gamma\theta}, \tag{18}$$

and its $h, v-$splitting of $\mathcal{R}$ provides the *curvature* $N-$*tensor fields*

$$\begin{array}{l} R_{j\ kl}^{\ i} = d^i\mathcal{R}(\delta_{\updownarrow}, \delta_{\|})\delta_{|} = \delta_{[\updownarrow}\mathcal{L}^{\rangle}_{|\|]} + \mathcal{L}^{\langle}_{|[\|}\mathcal{L}^{\rangle}_{\langle\updownarrow]} + \heartsuit\!\mathcal{C}^{\rangle}_{|\dashv}\mathcal{R}^{\dashv}_{\|\updownarrow} \\ \tilde R_{b\ kl}^{\ a} = \delta^a\mathcal{R}(\delta_{\updownarrow}, \delta_{\|})\eth_{\lfloor} = \delta_{[\updownarrow}\heartsuit\!\mathcal{L}^{\dashv}_{\lfloor\|]} + \heartsuit\!\mathcal{L}^{\rfloor}_{\lfloor[\|}\heartsuit\!\mathcal{L}^{\dashv}_{\rfloor\updownarrow]} + \mathcal{C}^{\dashv}_{\lfloor\rfloor}\mathcal{R}^{\rfloor}_{\|\updownarrow} \\ P_{j\ kc}^{\ i} = d^i\mathcal{R}(\eth_{\rfloor}, \delta_{\|})\delta_{|} = \eth_{\rfloor}\mathcal{L}^{\rangle}_{|\|} - (\delta_{\|}\heartsuit\!\mathcal{C}^{\rangle}_{|\rfloor} + \mathcal{L}^{\rangle}_{\langle\|}\heartsuit\!\mathcal{C}^{\langle}_{|\rfloor} - \mathcal{L}^{\langle}_{|\|}\heartsuit\!\mathcal{C}^{\rangle}_{\langle\rfloor} - \heartsuit\!\mathcal{L}^{\lfloor}_{\rfloor\|}\heartsuit\!\mathcal{C}^{\rangle}_{|\lfloor}) + \heartsuit\!\mathcal{C}^{\rangle}_{|\lfloor}\mathcal{P}^{\lfloor}_{\|\rfloor} \\ \tilde P_{b\ kc}^{\ a} = \delta^a\mathcal{R}(\eth_{\rfloor}, \delta_{\|})\eth_{\lfloor} = \eth_{\rfloor}\heartsuit\!\mathcal{L}^{\dashv}_{\lfloor\|} - (\delta_{\|}\mathcal{C}^{\dashv}_{\lfloor\rfloor} + \heartsuit\!\mathcal{L}^{\dashv}_{\lceil\|}\mathcal{C}^{\lceil}_{\lfloor\rfloor} - \heartsuit\!\mathcal{L}^{\lceil}_{\lfloor\|}\mathcal{C}^{\dashv}_{\lceil\rfloor} - \heartsuit\!\mathcal{L}^{\lceil}_{\rfloor\|}\mathcal{C}^{\dashv}_{\lceil\lfloor}) + \mathcal{C}^{\dashv}_{\lfloor\lceil}\mathcal{P}^{\lceil}_{\|\rfloor} \\ \tilde S_{j\ bc}^{\ i} = d^i\mathcal{R}(\eth_{\rfloor}, \eth_{\lfloor})\delta_{|} = \eth_{[\rfloor}\heartsuit\!\mathcal{C}^{\rangle}_{|\lfloor]} + \heartsuit\!\mathcal{C}^{\langle}_{|[\lfloor}\heartsuit\!\mathcal{C}^{\rangle}_{\langle\rfloor]} \\ S_{b\ cd}^{\ a} = \delta^a\mathcal{R}(\eth_{\lceil}, \eth_{\rfloor})\eth_{\lfloor} = \eth_{[\lceil}\mathcal{C}^{\dashv}_{\lfloor\rfloor]} + \mathcal{C}^{\rceil}_{\lfloor[\rfloor}\mathcal{C}^{\dashv}_{\rceil\lceil]}. \end{array} \tag{19}$$

Based on these $N-$tensor fields, we shall derive the Einstein equations of the linearized deformed model defined in the following section.

1 The (a, b) - Finslerian deformed weak model

The $(a, b)-$type deformation of the weak metric γ_{ij} is produced by a Finslerian perturbation $\tilde\varepsilon_{ij}(x, y) = h_{ij}(x, y)$ as in (5) of the pseudo-Riemannian gravitational field γ_{ij}, which leads to the generalized Finslerian metric [24]

$$f_{ij}(x, y) = \gamma_{ij}(x) + \tilde\varepsilon_{ij}(x, y), \tag{20}$$

with γ_{ij} given in (9). Here the $(a,b)-$type Finslerian perturbation $\tilde{\varepsilon}_{ij}(x,y)$ is postulated to satisfy the condition $|\tilde{\varepsilon}_{ij}(x,y)| \ll 1$ in order that f_{ij} be non-degenerate. Moreover, the tensor

$$\varepsilon^*{}_{ij}(x,y) = \varepsilon_{ij}(x) + \tilde{\varepsilon}_{ij}(x,y) = \lambda c_i(x)c_j(x) + h_{ij}(x,y) \tag{21}$$

provides a weak Finslerian perturbation of the Minkowski metric n_{ij}, and vanishes iff γ_{ij} is flat. This point of view permits us to consider $(h,v)-$metric $v-$Finslerian approaches.

Physically significant perturbations $\tilde{\varepsilon}_{ij}$ considered in (20) may belong to the geometrical framework developed by R.G.Beil [10], to the Kaluza-Klein ansatz or the one of the Randers-type Yang-Mills theory [8], [9]; in all cases, the Finslerian perturbation of the pseudo-Riemannian metric is given by the electromagnetic field, or by a gauge or spinor extension of the pseudo-Riemannian gravitational field. In each of these models, the original pseudo-Riemannian model appears as a limiting case.

We note that the deformed metric $f_{ij}(x,y)$ is of Finsler type itself, providing on TM a particular case of a generalized Lagrange structure $GL^n = (M, f_{ij})$ in the sense of R.Miron [17]. For N being the Cartan non-linear connection attached to F, this canonically gives the *almost Hermitian model* on TM, given by the $N-$lift of f_{ij} to TM and by the canonic adapted complex structure $J \in End(\mathcal{X}(TM))$ of associated matrix in local frames (12) $[J] = \begin{pmatrix} 0 & -I_4 \\ I_4 & 0 \end{pmatrix}$, which is in fact a *Kahler structure.*

As an alternative approach which we shall follow hereafter, we build on TM the $(h,v)-$metric provided by the two adjusted components of the Finslerian metric f_{ij} in (9), $g = n + \varepsilon^{(1)}$ and $\tilde{\varepsilon} = h_{ij} \cdot F^{-2}$ of the weak Finslerian metric, given by

$$G = \underbrace{(n_{ij} + \lambda c_i(x)c_j(x))}_{g_{ij}(x)} dx^i \otimes dx^j + \tilde{\varepsilon}_{ab}(x,y)\delta y^a \otimes \delta y^b, \tag{22}$$

with $\tilde{\varepsilon}_{ij}(x,y) = h_{ij}(x,y)F^{-2}(x,y)$, where in view of preserving the 0-homogeneity of G in y, the metric $h_{ij}(x,y)$ is scaled by the conformal factor $F^{-2}(x,y)$. We call the metric structure (TM, G), *the (a, b)− deformed weak model* (AB-DWM).

We note that though the corresponding associated deformation of type

$$\tilde{f}_{ij}(x,y) = \underbrace{n_{ij} + \varepsilon_{ij}(x)}_{g_{ij}(x,y)} + \underbrace{h_{ij}(x,y)F^{-2}(x,y)}_{\tilde{\varepsilon}_{ij}(x,y)}$$

is nolonger proper Finslerian (due to the lack of 0-homogeneity in the last term coefficients, which are -2 homogeneous), the metric $G \in \mathcal{T}^0_2(TM)$ in (22) is 0-homogeneous, hence dependent on direction, and living on the projectivized space PTM.

In particular, if $\tilde{\varepsilon}$ depends on y only, then G is a *pseudo-Riemann - locally Minkowski* $(h,v)-$metric, and the gravitational field of this space is called *weak Riemannian-locally Minkowski gravitational field.* In the *linear approach*, the Christoffel symbols γ^i_{jk} in (11) are approximated by the linearized Christoffel symbols $\bar{\gamma}^i_{jk}$ of the weak metric γ_{ij} ([24], [5])

$$\bar{\gamma}^i_{jk} = \frac{1}{2} n^{is}(\partial_{\{j}\varepsilon_{sk\}} - \partial_s \varepsilon_{jk}) = \frac{\lambda}{2} n^{is}(c_{\{j,k\}}c_s + c_{[s,k]}c_j + c_{[s,j]}c_k) \approx \gamma^i_{jk}. \tag{23}$$

The nonlinear connection is also approximated by the *weak nonlinear connection*

$$\bar{N}^a_i = \varepsilon^a_{ib} y^b = \frac{\lambda}{2} n^{as}(c_{\{i,0\}}c_s + c_{[s,0]}c_i + c_{[s,i]}c_0) \approx N^a_i. \tag{24}$$

In particular, if exists $c \in \mathcal{F}(M)$ such that $c_i = c_{,i}$, i.e., if $c_i(x)dx^i = dc$ is a *potential* 1-form, then

$$\bar{\gamma}^i_{jk} = \varepsilon^i c_{j,k} \qquad \bar{N}^a_i = \varepsilon^a c_{j,0}. \tag{25}$$

For obtaining the Einstein equations of the linearized deformed model, we determine first the canonic linear connection, provided generally by

Lemma 2. *a) The coefficients of the Berwald canonic linear $N-$connection* $\mathbf{D}$ *of the linearized AB-DWM are*

$$L^i_{jk} = \tilde{L}^i_{jk} = \bar{\gamma}^i_{jk} \approx \gamma^i_{jk}, \qquad \tilde{C}^i_{ja} = 0;\ C^a_{bc} = \frac{1}{2}\tilde{\varepsilon}^{ad} C_{dbc}, \tag{26}$$

where $C_{abc} = \dot{\partial}_a \tilde{\varepsilon}_{bc}/2$ is the Cartan tensor field associated to $\tilde{\varepsilon}_{ij}$ ([11], [6]); this is given by

$$C_{ijk} = [\frac{F_{ab}}{a}(k_{\{ij}p_{k\}} + k_{ki}p_j) + F_{bbb}p_i p_j p_k]/2 - \theta_{ijk} + \kappa'_{ijk}, \tag{27}$$

where

$$\begin{aligned} \theta_{ijk} &= (h_{i\{k}F_{;j\}} - h_{jk}F_{;i})/F \\ \kappa'_{ijk} &= -2F^{-3}(h_{j\{k}F_{;i\}} - h_{ik}F_{;j}), \end{aligned}$$

$p_i = b_i - by_i a^{-2}$ and k_{ij} is the angular metric in (4). Explicitly, we have

$$\begin{aligned} C^i_{jk} &= \tilde{\mu}(\alpha_0 \delta^i_{\{j}b_{k\}} + \alpha_1 \delta^i_{\{j}y_{k\}}) + \\ &+ \lambda_0 b^i a_{jk} + \lambda_1 y^i a_{jk} + \mu_0 b^i b_j b_k + \mu_1 b^i b_{\{j}y_{k\}} + \\ &+ \mu_2 b^i y_j y_k + \nu_1 y^i b_j b_k + \nu_2 y^i b_{\{j}y_{k\}} + \nu_3 y^i y_j y_k - h^{is}\theta_{sjk} + \kappa''^i_{jk}, \end{aligned} \tag{28}$$

where $\kappa''^i_{jk} = -F^{-1}(\delta^i_{\{j}F_{;k\}} - h_{jk}h^{is}F_{;s})$ and

$$\begin{cases} \lambda_0 = \tilde{\mu}\alpha_0 + \tilde{\lambda}u_0 + \tilde{\pi}u_1, & \lambda_1 = \tilde{\mu}\alpha_1 + \tilde{\pi}u_0 + \tilde{\rho}u_1 \\ \mu_0 = \tilde{\mu}\beta_0 + \tilde{\lambda}v_0 + \tilde{\pi}w_0, & \mu_1 = \tilde{\mu}\beta_1 + \tilde{\lambda}v_1 + \tilde{\pi}w_1,\ \mu_2 = \tilde{\mu}\beta_2 + \tilde{\lambda}v_2 + \tilde{\pi}w_2, \\ \nu_1 = \tilde{\mu}\beta_0 + \tilde{\pi}v_0 + \tilde{\rho}w_0, & \nu_2 = \tilde{\mu}\beta_1 + \tilde{\pi}v_1 + \tilde{\rho}w_1,\ \nu_3 = \tilde{\mu}\beta_2 + \tilde{\pi}v_2 + \tilde{\rho}w_2, \\ u_0 = \alpha_1 b + \alpha_0 \tilde{b}, & u_1 = \alpha_1 a^2 + \alpha_0 b, \\ v_0 = 2\alpha_0 + \beta_0 \tilde{b} + \beta_1 b, & w_0 = \beta_0 b + \beta_1 a^2 \\ v_1 = \alpha_1 + \beta_1 \tilde{b} + \beta_2 b, & w_1 = \alpha_0 + \beta_1 b + \beta_2 a^2 \\ v_2 = \beta_2 \tilde{b} + \beta_3 b, & w_2 = \beta_2 b + \beta_3 a^2. \end{cases}$$

b) The $N-$fields of torsion of the linearized AB-DWM are

$$R^a{}_{jk} = r_c{}^a{}_{jk} y^c = \lambda n^{is} \left[(c_s c_{[j})_{,ok]} + (c_0 c_{[k})_{,sj]} \right] /2, \qquad (29)$$
$$T^i{}_{jk} = 0, \quad \tilde{C}^i{}_{ja} = 0, \quad P^a{}_{kb} = 0, \quad S^a_{bc} = 0.$$

c) The curvature of the linearized AB-DWM has the components

$$\begin{array}{rcl} R_j{}^i{}_{kl} &=& r_j{}^i{}_{kl}, \quad \tilde{R}_b{}^a{}_{kl} = r_b{}^a{}_{kl}, \quad P_j{}^i{}_{kc} = 0, \\ \tilde{P}_b{}^a{}_{kc} &=& -(\delta_k C^a_{bc} + \bar{\gamma}^a_{dk} C^d_{bc} - \bar{\gamma}^d_{\{bk} C^a_{dc\}}) \\ S_j{}^i{}_{bc} &=& 0, \quad \tilde{S}_b{}^a{}_{cd} = C^s_{b[d} C^a_{c]s}, \end{array}$$

where $r_j{}^i{}_{kl}$ is the linearized weak curvature,

$$\begin{aligned} r_j{}^i{}_{kl} &= \partial_{[l} \bar{\gamma}^i_{jk]} = n^{is} (\partial^2_{[lj} \varepsilon_{sk]} + \partial^2_{[ks} \varepsilon_{jl]})/2 = \\ &= \lambda n^{is} \left[c_s c_{[k,jl]} + c_{s,[l} c_{k],j} + c_{s,j} c_{[k,l]} + c_{[k} c_{s,jl]} + (c_j c_{[l})_{,sk]} \right] /2, \end{aligned} \qquad (30)$$

which in the potential case becomes $r_j{}^i{}_{kl} = \lambda n^{is} (c_{s,[l} c_{k,j]})/2$.

Generally, the $hh-$Ricci $N-$tensor field and the $h-$scalar of curvature are [25], [5]

$$\begin{array}{l} R_{ij} \equiv R_i{}^k{}_{jk} = r_i{}^k{}_{jk} = r_{jk} = \frac{\lambda}{2} (\Box \varepsilon_{ij} + \partial^2_{ij} \varepsilon - \partial^2_{\{js} \varepsilon^s_{i\}}), \\ R = r = \Box \varepsilon - \partial^2_{ij} \varepsilon^{ij}, \end{array} \qquad (31)$$

where $\varepsilon = n^{ij} \varepsilon_{ij}$, and "$\Box$" denotes the d'Alambertian

$$\Box = -\partial^2_{00} + \partial^2_{11} + \partial^2_{22} + \partial^2_{33} \equiv -\partial^2_{tt} + \partial^2_{xx} + \partial^2_{yy} + \partial^2_{zz}.$$

In the considered model (21), we have

Lemma 3. *a) The Ricci $N-$tensor fields of the linearized AB-DWM are*

$$\begin{array}{l} R_{ij} = \frac{\lambda}{2} \varepsilon^s \{ c_s c_{[j,is]} + c_{s,[s} c_{j],i} + c_{s,i} c_{[j,s]} + c_{[j} c_{s,is]} + (c_i c_{[s})_{,sj]} \}, \\ P_{jb} \equiv P_j{}^k{}_{kb} = 0, \qquad \tilde{P}_{bk} \equiv \tilde{P}_b{}^d{}_{kd} = -(\delta_k C^a_{ba} - \varepsilon^d_{bk} C^a_{da}), \\ S_{ab} \equiv S_a{}^d{}_{bd} = C^e_{a[d} C^d_{b]e}, \end{array} \qquad (32)$$

In the linearized gravitational potential case we have $r_{jk} = \lambda \varepsilon^i (c_{i,i} c_{k,j} - c_{i,k} c_{i,j})/2$.

b) The Ricci scalars of curvature of the linearized AB-DWM are

$$\begin{array}{l} R = \frac{\lambda}{2} \varepsilon^i \varepsilon^j (c_i c_{[j,ji]} + c_{i,[i} c_{j,j]} + c_{i,j} c_{[j,i]} + c_{[j} c_{i,ji]} + (c_j c_{[i})_{,ij]}), \\ S = C^e_{b[d} C^d_{c]e} \tilde{\varepsilon}^{bc}. \end{array} \qquad (33)$$

S plays the role of parameter of anisotropy of the weak gravitational field. In the linearized potential case we have $r = \varepsilon^i \varepsilon^j (c_{i,j[j}c_{j]} + c_{[i,j}c_{i,j]})/2$.

Theorem 1. *The Einstein equations of the linearized AB-DWM are given by, in virtue of (32), (33):*

$$\begin{aligned}
R_{ij} - \tfrac{1}{2}(R+S)n_{ij} &\equiv \varepsilon^s\{c_s c_{[j,is]} + c_{s,[s}c_{j],i} + c_{s,i}c_{[j,s]} + c_{[j}c_{s,is]} \\
&+ (c_i c_{[s]})_{,sj]}\} - n_{ij}(R+S) = \kappa T_{ij} \\
S_{ab} - \tfrac{1}{2}(R+S)\tilde{\varepsilon}_{ab} &\equiv C^e_{a[d}C^d_{b]e} - \tfrac{1}{2}\tilde{\varepsilon}_{ab}(R+S) = \kappa T_{ab} \\
\tilde{P}_{jb} &\equiv 0 = -\kappa T_{jb}, \\
P_{bk} &\equiv -(\delta_k C^a_{ba} - \varepsilon^d_{bk}C^a_{da}) = \kappa T_{bk},
\end{aligned} \tag{34}$$

where $\kappa \in \mathbb{R}$ and $T_{ij}, T_{ab}, T_{jb}, T_{bk}$ are the energy-momentum $N-$tensor fields.

Theorem 2. *The conservation laws for the Einstein equations of the linearized AB-DWM are*

$$\begin{aligned}
E^i_{j|i} &\equiv \left(R^i_j - \tfrac{1}{2}(R+S)\varepsilon^i_j\right)_{|i} = \kappa\varepsilon^i T^i_{j|i}, \\
E^a_b|_a &\equiv \left(S^a_b - \tfrac{1}{2}S\varepsilon^a_b\right)|_a = 0 \\
P^a_k|_a &= \kappa T^a_k|_a,
\end{aligned} \tag{35}$$

where R^i_j, S^a_b and P^a_k are given by (32), and $|_i$, $\Big|_a$ are respectively the $h-$ and the $v-$covariant derivations induced by the $N-$connection ∇ ([17]).

2 The stationary curves of the linearized AB-DWM

Assume that $c : I = [a,b] \subset \mathbb{R} \to TM$ is a smooth curve, such that its image lies in a chart $\tilde{U} \subset TM$,

$$c(t) = (x^i(t), y^a(t)) \equiv (y^\alpha(t)), \forall t \in I,$$

and let $\mathbf{D}$ be a linear $N-$connection on TM given in the adapted basis (12) by $\mathbf{D}_{\delta_\beta}\delta_\gamma = \Gamma^\alpha_{\beta\gamma}\delta_\alpha$.

Definitions. a) The *covariant velocity* field $\mathcal{V}$ and the *covariant force* $\mathcal{F}$ on the curve c are the fields defined on c by

$$\begin{aligned}
&\mathcal{V} = \mathcal{V}^\alpha\delta_\alpha, && \mathcal{V}^\alpha = \frac{\delta y^\alpha}{dt} \\
&\mathcal{F} = \frac{\mathbf{D}\mathcal{V}}{dt} = \mathcal{F}^\alpha\delta_\alpha, && \mathcal{F}^\alpha = \frac{\delta\mathcal{V}^\alpha}{dt} + \Gamma^\alpha_{\beta\kappa}\mathcal{V}^\beta\mathcal{V}^\kappa,\ \alpha = \overline{1,8},
\end{aligned} \tag{36}$$

b) The curve c is said to be *stationary* with respect to $\mathbf{D}$ iff $\mathcal{F}=0$ along the curve.
c) The curve c is called $h-$*curve*, if

$$\pi_v(\mathcal{V})=0 \quad\Leftrightarrow\quad \frac{\delta y^a}{dt}=0 \quad\Leftrightarrow\quad \dot{y}^a+N^a_i\dot{x}^i=0,\ a=\overline{1,4},$$

where we denote $\dot{y}^\alpha=\frac{dy^\alpha}{dt}$, $\ddot{y}^\alpha=\frac{d^2y^\alpha}{dt^2}$, $\alpha=\overline{1,8}$, and is called $v-$*curve*, if

$$\pi_h(\mathcal{V})=0 \quad\Leftrightarrow\quad \dot{x}^i=0 \quad\Leftrightarrow\quad x^i=x^i_0,\ i=\overline{1,4},$$

where by π_h and π_v we denote respectively the $h-$ and $v-$projectors of the canonic splitting induced by N. If a h-/v-curve satisfies also the extra condition $\mathcal{F}=0$, then it is called *h-/v-path*, respectively.

In physical applications, the covariant force determines the non-linear connection.

I. In the linearized approach, the $h-$paths project onto geodesics of M; they are solutions of the Volterra-Hamilton-type second-order differential system

$$\begin{cases} \dot{y}^a+N^a_j(x(t),y(t))\dot{x}^j=0 \\ \ddot{x}^i+L^i_{jk}(x(t),y(t))\dot{x}^j\dot{x}^k=0, \quad a,i=\overline{1,4}, \end{cases} \tag{37}$$

which in the linearized AB-DWM rewrites as the first-order differential system

$$\begin{cases} \dot{y}^i=-\bar{\gamma}^i_{jk}(x(t))\, y^j(t)z^k(t), \quad \dot{x}^i=z^i(t) \\ \dot{z}^i=-\bar{\gamma}^i_{jk}(x(t))\, z^j(t)z^k(t), \quad i=\overline{1,4}, \end{cases} \tag{38}$$

which describes via (23) the gravitational interaction caused by electromagnetic fields provided by the potentials $c_i(x)$. The Cauchy problem associated to (37) is numerically solvable. In the particular case $\dot{x}^i=y^i$, i.e., when the fibre variable describes the velocity of the $h-$path, then these become the geodesics of the Riemannian deformed $h-$metric, satisfying

$$\ddot{x}+\bar{\gamma}^i_{00}=0, \quad (y=\dot{x}).$$

II. The $v-$*paths* of the linearized AB-DWMcoincide with the $v-$paths of the Finsler space $(M,F(x,y))$, with $F^2=\varepsilon^{(2)}{}_{ab}(x,y)y^ay^b$. A $v-$path $c:I\subset\mathbb{R}\to\tilde{U}\subset TM$, $c(t)=(x^i_0,y^a(t))$ is a solution of the second-order differential system

$$\ddot{y}^a+C^a_{bc}(x_0,y(t))\dot{y}^b\dot{y}^c=0. \tag{39}$$

III. Let be a family of stationary curves $c:I_1\times I_2\subset\mathbb{R}^2\to\tilde{U}\subset TM$, having the arc-length parameter t, and the deviation parameter u ([22], [12]),

$$c(t,u)=(x^i(t,u),y^a(t,u))=(y^\alpha(t,u))\in\tilde{U}, \quad \forall(t,u)\in I_1\times I_2.$$

Consider the *deviation vector field*

$$\mathcal{Z} = \mathcal{Z}^\alpha \delta_\alpha, \qquad \mathcal{Z}^i = \partial_u x^i,\ \mathcal{Z}^a = \partial_u y^a + N^a_i \partial_u x^i,$$

and *velocity vector field*

$$\mathcal{V} = \mathcal{V}^\alpha \delta_\alpha, \qquad \mathcal{V}^i = \partial_t x^i,\ \mathcal{V}^a = \partial_t y^a + N^a_i \partial_t x^i.$$

For any vector field $\mathcal{W} = \mathcal{W}^\alpha \delta_\alpha$, defined on the family of curves having the same image-patch $Im(c)$, we consider the partial covariant derivatives along the patch

$$\delta_t \mathcal{W}^\alpha = \partial_t \mathcal{W}^\alpha + \Gamma^\alpha_{\beta\gamma} \mathcal{W}^\beta \mathcal{V}^\gamma, \quad \delta_u \mathcal{W}^\alpha = \partial_u \mathcal{W}^\alpha + \Gamma^\alpha_{\beta\gamma} \mathcal{W}^\beta \mathcal{Z}^\gamma. \tag{40}$$

The equations of deviations of the family with respect to the connection $\mathbf{D}$ characterize the tidal force $\mathcal{Z}$, and have the form ([1], [5], [7])

$$\delta_t^2 \mathcal{Z}^\alpha + \delta_t \mathcal{T}^\alpha = \rho^\alpha + \delta_u \mathcal{F}^\alpha, \quad \alpha = \overline{1,8}, \tag{41}$$

for $\mathcal{T}^\alpha = \mathcal{T}_{\beta\ \gamma}^{\ \alpha} \mathcal{V}^\beta \mathcal{Z}^\gamma$ and $\rho^{\,\alpha} = \mathcal{R}_{\beta\ \gamma\lambda}^{\ \alpha} \mathcal{V}^\beta \mathcal{Z}^\gamma \mathcal{V}^\lambda$. These equations rewrite

$$\partial_t^2 \mathcal{Z}^\alpha + X^\alpha_\gamma \partial_t \mathcal{Z}^\gamma + Y^\alpha_\gamma \mathcal{Z}^\gamma + L^\alpha = 0, \quad \alpha = \overline{1,8}, \tag{42}$$

where we denoted

$$\begin{cases} X^\alpha_\gamma = (\mathcal{T}^\alpha_{\beta\gamma} + 2\Gamma^\alpha_{\gamma\beta})\mathcal{V}^\beta, \ \ L^\alpha = -\delta_u \mathcal{F}^\alpha, \quad \alpha, \gamma = \overline{1,8}, \\ Y^\alpha_\gamma = \delta_t[(\mathcal{T}^\alpha_{\beta\gamma} + \Gamma^\alpha_{\gamma\beta})\mathcal{V}^\beta] + (\mathcal{T}^\alpha_{\sigma\beta} + \Gamma^\alpha_{\beta\sigma})\Gamma^\beta_{\gamma\mu} \mathcal{V}^\sigma \mathcal{V}^\mu - -\mathcal{R}_{\beta\ \gamma\lambda}^{\ \alpha} \mathcal{V}^\beta \mathcal{V}^\lambda. \end{cases}$$

Then, denoting $\{\xi^A\}_{A=\overline{1,16}} = \{\mathcal{Z}^\alpha, \partial_t \mathcal{Z}^\beta\}$ and $\{L^A\}_{A=\overline{1,16}} = \{0, \partial_u \mathcal{F}^\alpha\}$ as $16-$column vectors, and considering the $8 \times 8-$matrices $X = (X^\alpha_\gamma)$, $Y = (Y^\alpha_\gamma)$ and $P = \begin{pmatrix} 0 & -I_{2n} \\ Y & X \end{pmatrix}$, the system (42) rewrites in matrix form

$$\partial_t \xi + P\xi + L = 0 \qquad \Leftrightarrow \qquad \partial_t \xi^A + P^A_B \xi^B + L^A = 0, \quad A = \overline{1,16}. \tag{43}$$

The equations of deviations of paths presented above are particular cases of the ones in ([4], [1], [5], [7]), of the extended Finslerian case developed in [2]. Alternatively, the study of deviation of geodesics for the Finslerian case $n + \varepsilon^{(1)} + \varepsilon^{(2)}$ was performed in [24], [26].

Conclusions. We study an extension of the weak pseudo-Riemannian gravitational model by imposing an $(a, b)-$type deformation of the weak pseudo-Riemannian metric γ_{ij} of the 4-dimensional space M, which provides an $(h, v)-$ metric on TM. The obtained Finslerian-type model fits in the general theory of $(h, v)-$metric structures on vector bundles developed in [17], [19], [20], [5], [4]. In the linearized framework, the explicit Einstein equations, the associated conservation laws and the equations of stationary curves and of their deviations are determined for the canonic linear $N-$connection, where N is the Berwald-type nonlinear connection considered in linearized approach. The study of the modelled weak gravitational field provides further information on the gravitational waves in these spaces, the wave vectors of the weak field theory being intrinsically incorporated in such spaces as the considered ones, in which the elements depend on both position and direction.

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Authors' addresses:

V. Balan
Department Mathematics I
University Politehnica of Bucharest
Splaiul Independenţei 313,
RO-060042 Bucharest, Romania
e-mail: vbalan@mathem.pub.ro

P.C.Stavrinos
Department of Mathematics,
University of Athens,Panepistemiopolis,
Athens 15784, Greece
e-mail: pstavrin@atlas.uoa.gr