STRONG INEQUALITIES FOR PLANE-NORMAL CURVATURE

Wladimir-Georges Boskoff and Constantin Costara

Abstract

The aim of this paper is to prove that if in the interior of $S^n \subset R^{n+1}$ there is a $n-$dimensional compact Riemannian manifold M with convex interior, and in each point the $\pi-$normal curvatures $\lambda^{\pi,M}$ are strictly positive and $\lambda^{\pi,M} \leq \lambda^{\pi,S^n} = 1$ for all 2- normal planes π, then $M = S^n$.

AMS Subject Classification: 53C20.
Key words: convex set, curve, curvature.

The main result of this note is the coincidence theorem, which can be found below as Theorem 2. For the proof of this theorem we need first of all a planar lemma, which contains the core of the proof. We understand by "convex" the convexity in the set theory sense. We'll say a manifold has a convex interior if its interior is a convex set. Let M be a $n-$dimensional manifold isometrically immersed into the $(n+1)-$euclidean space, and let us denote by g the metric induced by the canonical metric. Let p be a point which belongs to S^n. Consider both the normal line in p to S^n and a 2-plane which contains the previous normal line. Denote by C_1 and by C_2 the curves $\pi \cap S^n$ and $\pi \cap M$ respectively. We call $\pi-$normal curvature in a point q_i belonging to a curve C_i the curvature of the curve at the point in the sense of the theory of curves. Let us start by the following lemma:

Lemma 1. *Let us consider a simple closed curve* $c : \mathbb{R} \to \mathbb{R}^2$ *with convex interior. Suppose this curve is enclosed in the closure of the unit disk, and that in any point of* c *the curvature is strictly positive and less than or equal to one. Then the image of the curve* c *is the unit circle.*

Proof. Consider $c : \mathbb{R} \to \overline{int\ S^1}$ given by $c(t) = (x(t), y(t))$, where x and y are $\mathcal{C}^2(\mathbb{R}, \mathbb{R})$ periodic functions, having the main period T strictly positive. We may suppose that the point $(x(t), y(t))$ is moving in the counterclockwise sense. (1) By the four vertex theorem, there exists two consecutive values, say $0 < t_1 < t_2 \leq T$ such that $\dot{x}(t_1) = \dot{x}(t_2) = 0$. Without losing generality, we may suppose that $y(t_1) \geq y(t_2)$, and in this case we shall consider the line d parallel to the $x-$axis,

---

Editor Gr.Tsagas *Proceedings of The Conference of Applied Differential Geometry - General Relativity and The Workshop on Global Analysis, Differential Geometry and Lie Algebras, 2001,* 53-58

passing through $T_2\left(x\left(t_2\right), y\left(t_2\right)\right)$. Otherwise, the same considerations can be made for the other point $T_1\left(x\left(t_1\right), y\left(t_1\right)\right)$.

If $y\left(t_2\right) \leq 0$ we can work in the inferior half plane. Contrary, the curve $\bar{c}\ (t) = \left(x\left(t\right), -y\left(t\right)\right)$ which is the symmetric of c with respect to $x-$axis has all the properties of c (convex interior, is contained in $\overline{int\ S^1}$ and its curvature is strictly positive and less than or equal to one at any point). Furthermore, it has the second component negative in t_2, therefore we may suppose $y\left(t_2\right) \leq 0$. (2)

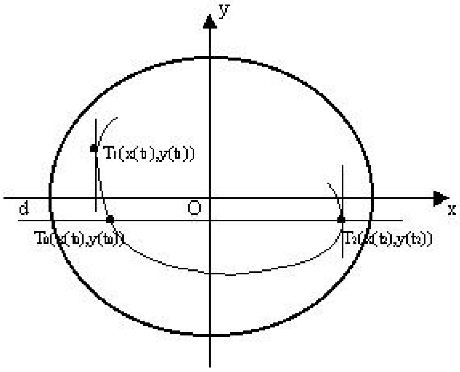

Figure 1.

Denote by $T_0\left(x\left(t_0\right), y\left(t_0\right)\right)$ the second point of intersection between d and c (i.e. $y\left(t_0\right) = y\left(t_2\right)$). Consider D the closure of the convex domain bounded by the line d and the curve c, the points belonging to D having the second component negative (see *Figure no. 1*). The point $M\left(x\left(t\right), y\left(t\right)\right)$ belonging to $c \cap D$ has the property $y\left(t\right) \leq y\left(t_2\right)$. We will reparamatrize the previous described piece of curve. Taking into account that $\dot{x}\ (t) \neq 0$ for all $t \in \left(t_0, t_2\right)$, the coordinate $x : \left[t_0, t_2\right] \rightarrow [u, v] \subset [-1, 1]$ is a strictly increasing function, where $x\left(t_0\right) =: u$, $x\left(t_2\right) =: v$. (3) Therefore it exists $x^{-1} : [u, v] \rightarrow \left[t_0, t_2\right]$ its inverse and we may consider $c\left(s\right) := \left(s, y\left(x^{-1}\left(s\right)\right)\right)$, $s \in [u, v]$. Let us observe that the images of the points $c\left(u\right)$ and $c\left(v\right)$ belong to the straight-line d. Considering $f\left(s\right) := y\left(x^{-1}\left(s\right)\right)$, $f : [u, v] \rightarrow \mathbb{R}$ and taking into account that D is a convex set we have $f''\ (s) > 0$ for all $s \in [u, v]$. So f' is a strictly increasing function. If we suppose by contrary that f' has constant sign on $[u, v]$, it results that f is a strictly monotonic function, in contradiction with $f\left(u\right) = f\left(v\right)$. Consequently there is $c_1 \in (u, v)$ such that $f'\ (c_1) = 0$. Therefore $f'\ (s) < 0$ for $s \in \left(u, c_1\right)$ and $f'\ (s) > 0$ for $s \in \left(c_1, v\right)$.

Let us consider the line interval $\left[c_1, v\right]$: the curvature of c is given by the formula

$$K\left(s\right) = \frac{\left|f''\left(s\right)\right|}{\left(1+\left(f'\left(s\right)\right)^2\right)^{\frac{3}{2}}}$$

that is

$$K\left(s\right) = \frac{f''\left(s\right)}{\left(1+\left(f'\left(s\right)\right)^2\right)^{\frac{3}{2}}} .$$

Since $K\left(s\right)\leq 1$ and $f^{'}\left(s\right)>0$ on $\left(c_1,v\right)$, it results

$$\frac{f^{"}(s)f^{'}(s)}{\left(1+\left(f'(s)\right)^2\right)^{\frac{3}{2}}}\leq f^{'}\left(s\right)\ ,\ \left(\forall\right)\ s\in\left(c_1,v\right)\ .$$

We obtain

$$\left(f\left(s\right)+\frac{1}{\sqrt{1+\left(f'(s)\right)^2}}\right)^{'}\geq 0\ ,$$

that is

$$h\left(s\right):=f\left(s\right)+\frac{1}{\sqrt{1+\left(f'(s)\right)^2}}$$

is an increasing function on (c_1,v). (4)
From (3) and (4) it results

$$\left\{\begin{array}{c} h\left(c_1\right)\leq\lim\limits_{s\to v}h\left(s\right)=f\left(v\right)+\lim\limits_{s\to v}\frac{1}{\sqrt{1+\left(f'(s)\right)^2}}=f\left(v\right)\\ h\left(c_1\right)=f\left(c_1\right)+\frac{1}{\sqrt{1+\left(f'(c_1)\right)^2}}=f\left(c_1\right)+1\end{array}\right. \qquad (5)$$

so

$$f\left(c_1\right)+1\leq f\left(v\right)\leq 0. \qquad (6)$$

Since $\left(c_1,f\left(c_1\right)\right)\in\overline{intS^1}$ we have

$$f\left(c_1\right)\geq -1. \quad (7)$$

Using (6) and (7) we obtain consequently $f\left(v\right)=0$, $f\left(c_1\right)=1$ and $c_1=0$. According to (4) the function h is increasing, and using (6) we obtain $h\left(c_1\right)=0$ and $\lim\limits_{s\to v}h\left(s\right)=0$. This yields $h\left(s\right)=0$ for $s\in[0,v)$.
Next we will show that the equation

$$f\left(s\right)+\frac{1}{\sqrt{1+\left(f'(s)\right)^2}}=0\ ,\ s\in\left(0,v\right) \qquad (8)$$

admits a solution f such that $f^2\left(s\right)+s^2=1$, $s\in\left(0,v\right)$, therefore the image of the curve c is contained in the unit circle S^1 for all $s\in[0,v]=[0,1]$. (9)
We remark that (8) may be written in the form

$$\frac{f(s)f^{'}(s)}{\sqrt{1-f^2(s)}}=\pm 1\ ,\ s\in\left(0,v\right)\ . \qquad (10)$$

By integrating, we get

$$\sqrt{1-f^2\left(s\right)}=\pm s+k\ ,\ s\in\left(0,v\right) \qquad (11)\,,$$

and the condition $f(0) = -1$ leads to $k = 0$. Squaring the equality (11) we obtain the assertion (9).

Now we are under the condition of *Figure no. 2* :

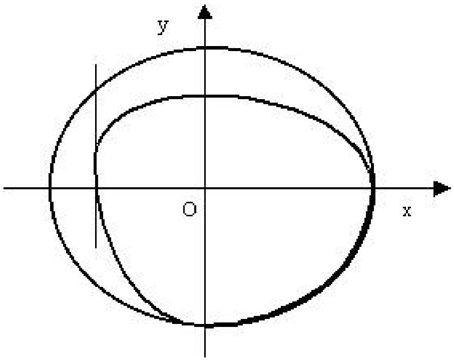

Figure 2.

Using a symmetry with respect to the origin of coordinates we obtain *Figure no. 3,*

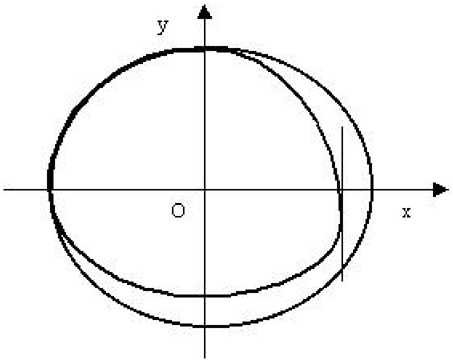

Figure 3.

and we can repeat the considerations made for *Figure no. 1*. We obtain that c has two arc components included in S^1 i.e. the situation is described by *Figure no. 4*.

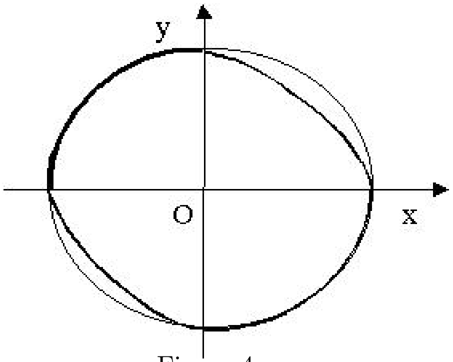

Figure 4.

A new symmetry with respect to Ox and same considerations leads to *Figure no. 5.*

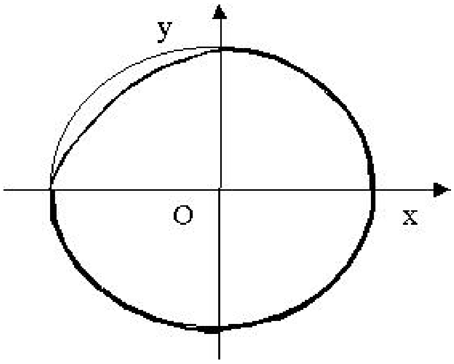

Figure 5.

A last symmetry with respect to the origin and the same considerations end the proof. $\square$

Denote by S^2 the unit sphere included into the three dimensional Euclidean space.

Theorem 1. *Let M be a two-dimensional compact manifold with convex interior, $M \subseteq \overline{intS^2}$. Suppose that the $\pi-$normal curvatures $\lambda^{\pi,M}$ are strictly positive and $\lambda^{\pi,M} \leq \lambda^{\pi,S^n} = 1$ for all 2- normal planes π. Then $M = S^2$.*

Proof. Consider $p \in S^2$ and let T_pS^2 be the tangent space to S^2 at p. Let N be the normal direction to T_pS^2 and denote by C_1 and by C_2 the curves $\pi \cap S^2$ and $\pi \cap M$ respectively, where π is an arbitrary 2$-$plane which includes N. Let $\lambda^{\pi,M}$ be the curvature of the curve C_2 at the point $q \in C_2$. Therefore, for any π and any point $q \in C_2 := \pi \cap M$ we have $\lambda^{\pi,M} \leq 1$. Taking into account that the interior of M is a convex set it results that $int C_2$ is a convex set included into the interior of a circle of radius one. We are in the conditions of Lemma 1. It follows that $C_2 = \pi \cap S^2$, that is $M = S^2$.$\square$

Previous considerations are available and may be extended to the $n-$dimensional case, so the following result holds:

Theorem 2. (The Coincidence Theorem)

Let M be a $n-$dimensional compact Riemannian manifold, $n \geq 1$, with convex interior and $M \subseteq \overline{intS^n}$, where S^n is the $n-$dimensional unit sphere included into the $(n+1)-$dimensional Euclidean space. Suppose that the $\pi-$normal curvatures $\lambda^{\pi,M}$ are strictly positive and $\lambda^{\pi,M} \leq \lambda^{\pi,S^n} = 1$ for all 2- normal planes π. Then $M = S^n$.

References

[1] M. Berger, B. Gostiaux, *Differential Geometry: Manifolds, Curves, and Surfaces,* Springer-Verlag, New York, 1988.

Authors' address:

Wladimir-Georges Boskoff and Constantin Costara
Faculty of Mathematics
"Ovidius" University Constantza
124 Bd. Mamaia, 8700 Constantza, ROMANIA
E-mail: math.ovidius@impromex.ro