SOME INEQUALITIES FOR $k-$RICCI CURVATURE OF CERTAIN SUBMANIFOLDS IN SASAKIAN SPACE FORMS

Dragos Cioroboiu

Abstract

In the present paper, we obtain estimates of the scalar curvature and the k-Ricci curvature respectively, in terms of the squared mean curvature

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1 Preliminaries

A $(2m+1)$-dimensional Riemannian manifold $(\tilde{M}, g)$ is said to be a *Sasakian manifold* if it admits an endomorphism ϕ of its tangent bundle $T\tilde{M}$, a vector field ξ and a 1-form η, satisfying:

$$\begin{cases} \phi^2 = -Id + \eta \otimes \xi, \ \eta(\xi) = 1, \ \phi\xi = 0, \ \eta \circ \phi = 0, \\ g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \ \eta(X) = g(X, \xi), \\ (\tilde{\nabla}_X \phi)Y = -g(X, Y)\xi + \eta(Y)X, \ \tilde{\nabla}_X \xi = \phi X, \end{cases}$$

for any vector fields X, Y on $T\tilde{M}$, where $\tilde{\nabla}$ denotes the Riemannian connection with respect to g.

A plane section π in $T_p\tilde{M}$ is called a ϕ*-section* if it is spanned by X and ϕX, where X is a unit tangent vector orthogonal to ξ. The sectional curvature of a ϕ-section is called a ϕ*-sectional curvature*. A Sasakian manifold with constant ϕ-sectional curvature c is said to be a *Sasakian space form* and is denoted by $\tilde{M}(c)$.

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The curvature tensor of $\tilde{M}(c)$ of a Sasakian space form $\tilde{M}(c)$ is given by [1]

$$\tilde{R}(X,Y)Z = \frac{c+3}{4}\{g(Y,Z)X - g(X,Z)Y\}+ \tag{1.1}$$

$$+\frac{c-1}{4}\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi+$$

$$+g(\phi Y,Z)\phi X - g(\phi X,Z)\phi Y - 2g(\phi X,Y)\phi Z\},$$

for any tangent vector fields X, Y, Z on $\tilde{M}(c)$.

As examples of Sasakian space forms we mention $\mathbb{R}^{2m+1}$ and S^{2m+1}, with standard Sasakian structures (see [1]).

In [8], A. Lotta has introduced the following notion of slant immersion in almost contact metric manifolds.

Definition. We call a differentiable distribution $\mathcal{D}$ on M a *slant distribution* if for each $x \in M$ and each nonzero vector $X \in \mathcal{D}_x$, the angle $\theta_{\mathcal{D}}(X)$ between ϕX and the vector subspace $\mathcal{D}_x$ is constant, which is independent of the choice of $x \in M$ and $X \in \mathcal{D}_x$. In this case, the constant angle $\theta_{\mathcal{D}}$ is called the *slant angle* of the distribution $\mathcal{D}$.

Definition. A submanifold M tangent to ξ is said to be *slant* if for any $x \in M$ and any $X \in T_xM$, linearly independent of ξ, the angle between ϕX and T_xM is a constant $\theta \in [0, \frac{\pi}{2}]$, called the *slant angle* of M in $\tilde{M}$.

Invariant and *anti-invariant immersions* are slant immersions with slant angle $\theta = 0$ and $\theta = \frac{\pi}{2}$, respectively. A slant immersion which is neither invariant nor anti-invariant is called a *proper slant immersion*.

Definition. We say that a submanifold M tangent to ξ is a *bi-slant* submanifold of $\tilde{M}$ if there exist two orthogonal distributions $\mathcal{D}_1$ and $\mathcal{D}_2$ on M such that :

i) TM admits the orthogonal direct decomposition $TM = \mathcal{D}_1 \oplus \mathcal{D}_2 \oplus \{\xi\}$.

ii) For any $i = 1, 2$, $\mathcal{D}_i$ is slant distribution with slant angle θ_i.

Let $2d_1 = \dim \mathcal{D}_1$ and $2d_2 = \dim \mathcal{D}_2$.

Remark If either d_1 or d_2 vanishes, the bi-slant submanifold ia a slant submanifold. Thus, slant submanifolds (and, therefore, invariant and anti-invariant submanifolds) are particular cases of bi-slant submanifolds.

Definition. We say that M tangent to ξ is a *semi-slant* submanifold of $\tilde{M}$ if there exist two orthogonal distributions $\mathcal{D}_1$ and $\mathcal{D}_2$ on M such that :

i) TM admits the orthogonal direct decomposition $TM = \mathcal{D}_1 \oplus \mathcal{D}_2 \oplus \{\xi\}$.

ii) The distribution $\mathcal{D}_1$ is an invariant distribution, i.e., $\phi(\mathcal{D}_1) = \mathcal{D}_1$.

ii) The distribution $\mathcal{D}_2$ is slant with angle $\theta \neq 0$.

Let $2d_1 = \dim \mathcal{D}_1$ and $2d_2 = \dim \mathcal{D}_2$.

In [2], the invariant distribution of a semi-slant submanifold is a slant distribution with zero angle. Thus, it is obvious that, in fact, semi-slant submanifolds are particular cases of bi-slant submanifolds. Moreover, it is clear that, if $\theta = \frac{\pi}{2}$, then the semi-slant submanifold is a semi-invariant submanifold.

(a) If $d_2 = 0$, then M is an invariant submanifold.

(b) If $d_1 = 0$ and $\theta = \frac{\pi}{2}$, then M is an anti-invariant submanifold.

(c) If $d_1 = 0$ and $\theta \neq \frac{\pi}{2}$, then M is a proper slant submanifold, with slant angle θ.

We say that a semi-slant submanifold is *proper* if $d_1 d_2 \neq 0$ and $\theta \neq \frac{\pi}{2}$.

Examples (see [2]).

1. For any constant k,

$$x(u, v, t) = 2(u, k\cos v, v, k\sin v, t)$$

defines a slant submanifold M with slant angle $\theta = \arccos\frac{1}{\sqrt{1+k^2}}$, scalar curvature $\tau = \frac{-1}{3(1+k^2)}$, constant mean curvature given by $\|H\| = \frac{|k|}{3(1+k^2)}$. Moreover, the following statements are equivalent:

(a) $k = 0$;
(b) M is invariant;
(c) M is minimal;
(d) M has parallel mean curvature vector.

2. For any $\theta_1 \in [0, \frac{\pi}{2}]$, we chose $\theta_2 \in (0, \frac{\pi}{2}]$, such that $\cos\theta_2 = \frac{\cos\theta_1}{\sqrt{2}}$. Then

$$x(u, v, w, s, t) = 2(u, 0, w, 0, v\cos\theta_1, v\sin\theta_1, s\cos\theta_2, s\sin\theta_2, t)$$

defines a five-dimensional bi-slant submanifold M in $(\mathbf{R}^9, \phi_0, \xi, \eta, g)$, with both slant angles equal to θ_2, but it is not slant submanifold. In fact we can chose a local orthonormal frame $\{e_1, ..., e_5\}$ of TM such that

$$e_1 = \frac{1}{\sqrt{2}}\{2(\frac{\partial}{\partial x^1} + y^1\frac{\partial}{\partial z}) + 2(\frac{\partial}{\partial x^4} + y^4\frac{\partial}{\partial z}), \qquad e_2 = \cos\theta_1(2\frac{\partial}{\partial y^1}) + \sin\theta_1(2\frac{\partial}{\partial y^2}),$$

$$e_3 = 2(\frac{\partial}{\partial x^3} + y^3\frac{\partial}{\partial z}), \qquad e_4 = \cos\theta_2(2\frac{\partial}{\partial y^3}) + \sin\theta_2(2\frac{\partial}{\partial y^4}),$$

$$e_5 = 2\frac{\partial}{\partial z} = \xi.$$

Now we define the distributions $\mathcal{D}_1 = \langle e_1, e_2\rangle$ and $\mathcal{D}_2 = \langle e_3, e_4\rangle$. It is easy to see that both $\mathcal{D}_1$ and $\mathcal{D}_2$ are slant distribution with the same slant angle θ_2. Nevertheless, we can obtain that M is not slant since $\theta_2 \neq 0$.

3. Let $\mathbf{R}^6$ be the Euclidian space of dimension 6, with the standard metric and the almost complex structure given by $J(\frac{\partial}{\partial x^i}) = \frac{\partial}{\partial x^i}$, for any $i = 1, 2, 3$, where (x^i, y^i) denote the cartesian coordinates.

Let $\mathbf{R}^5 \hookrightarrow \mathbf{R}^6$ be the usual immersion. Then, $C = \frac{\partial}{\partial y^3}$ is the unit normal to $\mathbf{R}^5$ and so, $\xi = -JC = \frac{\partial}{\partial x^3}$.

Now, for any $\theta \neq 0$, we can consider the immersions:
$\varphi_1 : \mathbf{R}^4 \longrightarrow \mathbf{R}^6 : (u, v, t, s) \longmapsto (u\cos\theta, u\sin\theta, t, v, 0, s)$,
$\varphi_2 : \mathbf{R}^3 \longrightarrow \mathbf{R}^5 : (u, v, t) \longmapsto (u\cos\theta, u\sin\theta, t, v, 0)$.
We can directly prove that φ_1 is a semi-slant immersion, with complex distribution $\mathcal{D}_1 = \left\langle \frac{\partial}{\partial x^3}, \frac{\partial}{\partial y^3} \right\rangle$ and slant distribution, with angle θ, $\mathcal{D}_2 = \left\langle \cos\theta \frac{\partial}{\partial x^1} + \sin\theta \frac{\partial}{\partial x^2}, \frac{\partial}{\partial y^1} \right\rangle$. On the other hand, φ_2 is a $\theta-$slant immersion, where $\mathbf{R}^5$ has the almost contact metric structure induced by the described almost Hermitian structure on $\mathbf{R}^6$.

For the other properties and examples of slant, bi-slant and semi-slant submanifolds in Sasakian manifolds, we refer to [2].

Let M be an n-dimensional submanifold of a Riemannian manifold $\tilde{M}$. We denote by $K(\pi)$ the sectional curvature of M associated with a plane section $\pi \subset T_pM, p \in M$, and ∇ the Riemannian connection of M. Also, let h be the second fundamental form and R the Riemann curvature tensor of M.

Then the equation of Gauss is given by

$$\tilde{R}(X, Y, Z, W) = R(X, Y, Z, W) + \\ + g(h(X, W), h(Y, Z)) - g(h(X, Z), h(Y, W)), \tag{1.2}$$

for any vectors X, Y, Z, W tangent to M.

Let $p \in M$ and $\{e_1, ..., e_n\}$ an orthonormal basis of the tangent space T_pM. We denote by H the mean curvature vector, that is

$$H(p) = \frac{1}{n}\sum_{i=1}^{n} h(e_i, e_i). \tag{1.3}$$

Also, we set

$$h_{ij}^r = g(h(e_i, e_j), e_r) \tag{1.4}$$

and

$$\|h\|^2 = \sum_{i,j=1}^{n} g(h(e_i, e_j), h(e_i, e_j)). \tag{1.5}$$

For any tangent vector field X to M, we put $\phi X = PX + FX$, where PX and FX are the tangential and normal components of ϕX, respectively. We denote by

$$\|P\|^2 = \sum_{i,j=1}^{n} g^2(Pe_i, e_j). \tag{1.6}$$

Suppose L is a k-plane section of T_pM and X a unit vector in L. We choose an orthonormal basis $\{e_1, ..., e_k\}$ of L such that $e_1 = X$.

Define the *Ricci curvature* Ric_L of L at X by

$$Ric_L(X) = K_{12} + K_{13} + ... + K_{1k}, \tag{1.7}$$

where K_{ij} denotes the *sectional curvature of the* 2*-plane section spanned by* e_i, e_j. We simply called such a curvature a k-Ricci curvature.

The *scalar curvature* τ *of the* k*-plane section* L is given by

$$\tau(L) = \sum_{1\leq i<j\leq k} K_{ij}. \tag{1.8}$$

For each integer k, $2 \leq k \leq n$, the Riemannian invariant Θ_k on an n-dimensional Riemannian manifold M is defined by

$$\Theta_k(p) = \frac{1}{k-1} \inf_{L,X} Ric_L(X), \quad p \in M, \tag{1.9}$$

where L runs over all k-plane sections in T_pM and X runs over all unit vectors in L.

Recall that for a submanifold M in a Riemannian manifold, the *relative null space of* M *at a point* $p \in M$ is defined by

$$\mathcal{N}_p = \{X \in T_pM |\, h(X,Y) = 0, \text{ for all } Y \in T_pM\}. \tag{1.10}$$

2 k-Ricci curvature

In this section, we show a relationship between the k-Ricci curvature and the squared mean curvature for slant, bi-slant and semi-slant submanifolds in a Sasakian space form.

We state an inequality between the scalar curvature and the squared mean curvature for submanifolds tangent to ξ.#

Theorem 2.1. *Let* M *be an* $(n = 2k+1)-$*dimensional* $\theta-$*slant submanifold in a* $(2m+1)$*-dimensional Sasakian space form* $\widetilde{M}(c)$ *tangent to* ξ. *Then we*

$$\|H\|^2 \geq \frac{2\tau}{n(n-1)} - \frac{c+3}{4} - \frac{[3(n-1)\cos^2\theta - 2n + 2](c-1)}{4n(n-1)}. \tag{2.1}$$

Theorem 2.2. *Let* M *be an* $(n = 2d_1 + 2d_2 + 1)-$*dimensional semi-slant submanifold in a* $(2m+1)$*-dimensional Sasakian space form* $\tilde{M}(c)$ *tangent to* ξ. *Then we have*

$$\|H\|^2 \geq \frac{2\tau}{n(n-1)} - \frac{c+3}{4} - \frac{[3(d_1 + d_2\cos^2\theta) - n + 1](c-1)}{2n(n-1)}. \tag{2.2}$$

Theorem 2.3. *Let* M *be an* $(n = 2k+1)-$*dimensional* $\theta-$*slant submanifold in a* $(2m+1)$*-dimensional Sasakian space form* $\widetilde{M}(c)$ *tangent to* ξ. *Then, for any integer* $k, 2 \leq k \leq n$, *and any point* $p \in M$, *we have*

$$\|H\|^2(p) \geq \Theta_k(p) - \frac{c+3}{4} - \frac{[3(n-1)\cos^2\theta - 2n + 2](c-1)}{4n(n-1)}. \tag{2.3}$$

Theorem 2.4. *Let M be an $(n = 2d_1 + 2d_2 + 1)-$dimensional bi-slant submanifold in a $(2m+1)$-dimensional Sasakian space form $\tilde{M}(c)$ tangent to ξ. Then, for any integer $k, 2 \leq k \leq n$, and any point $p \in M$, we have*

$$\|H\|^2(p) \geq \Theta_k(p) - \frac{c+3}{4} - \frac{[3(d_1 \cos^2\theta_1 + d_2 \cos^2\theta_2) - n + 1](c-1)}{2n(n-1)}. \tag{2.4}$$

Theorem 2.5. *Let M be an $(n = 2d_1 + 2d_2 + 1)-$dimensional semi-slant submanifold in a $(2m+1)$-dimensional Sasakian space form $\tilde{M}(c)$ tangent to ξ. Then, for any integer $k, 2 \leq k \leq n$, and any point $p \in M$, we have*

$$\|H\|^2(p) \geq \Theta_k(p) - \frac{c+3}{4} - \frac{[3(d_1 + d_2 \cos^2\theta) - n + 1](c-1)}{2n(n-1)}. \tag{2.6}$$

Corollary 2.6. *Let M be an n-dimensional invariant submanifold of a Sasakian space form $\tilde{M}(c)$. Then, for any integer $k, 2 \leq k \leq n$, and any point $p \in M$, we have*

$$\Theta_k(p) \leq \frac{c+3}{4} + \frac{c-1}{4n}. \tag{2.7}$$

Corollary 2.7. *Let M be an n-dimensional anti-invariant submanifold of a Sasakian space form $\tilde{M}(c)$. Then, for any integer $k, 2 \leq k \leq n$, and any point $p \in M$, we have*

$$\|H\|^2(p) \geq \Theta_k(p) - \frac{c+3}{4} + \frac{c-1}{2n}. \tag{2.8}$$

Corollary 2.8. *Let M be an n-dimensional contact CR submanifold ($\theta_1 = 0$, $\theta_2 = \frac{\pi}{2}$) of a Sasakian space form $\tilde{M}(c)$. Then, for any integer $k, 2 \leq k \leq n$, and any point $p \in M$, we have*

$$\|H\|^2(p) \geq \Theta_k(p) - \frac{c+3}{4} - \frac{(3d_1 - n + 1)(c-1)}{2n(n-1)}. \tag{2.9}$$

where $2d_1 = \dim \mathcal{D}_1$.

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Author's address:

Dragos Cioroboiu
Univ. Politehnica of Bucharest
Splaiul Independenţei 313
77206 Bucharest, ROMANIA
E-mail: tiabaprov@pcnet.ro