ON ERGODIC PROPERTIES FOR A CLASS OF ALTERNATING NUMBER EXPANSIONS

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Abstract

In the present paper we give some important results arising from the application of ergodic theory to the alternating-Engel expansion. In particular, we investigate the ergodic behaviour of the operator T, which generates the alternating-Engel expansion, relative to the Lebesgue measure ë. Consequently the ergodic theorem may be used in the study of the ergodic properties occurring in the context of alternating number expansions.

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1 Introduction

Let A be any real number. Then according to a general alternating series algorithm introduced by A.and J. Knophmacher (see [3]), analogous to a positive one of Oppenheim (see [4]), A has a unique finite or infinite representation in terms of a general alternating series of rationals as follows.

Theorem 1 *Every real number A has a unique alternating series representation of the form*

$$A = 1 + \frac{1}{a_1} - \frac{1}{a_1 a_2} + \frac{1}{a_1 a_2 a_3} - \ \ldots, \tag{1.1}$$

where

$a_{i+1} \ \geq \ a_i + 1, \, i \geq 1.$

Proof. We deduce this result from the general alternating algorithm:
$a_0 = [A]$ (the integral part of A) and

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$A_1 = A - a_0 = \{A\}$ (the fractional part of A)
$a_n = [1/A_n] \geq 1$, for $A_n > 0$, $n \geq 1$,
where
$A_{n+1} = (1/a_n - A_n)(c_n/b_n)$, for $a_n > 0$,
by a suitable choice of the parameters $b_n = 1$ and $c_n = a_n$, for $n \geq 1$.

Representation (1.1) is called *alternating-Engel expansion* for real numbers, while the positive integers a_1, a_2, ..., a_n are called *digits* of the above-mentioned expansion (see [1]).

2 Auxiliary results

Let $I = [0, 1]$, $X = I\!N^*$, where $I\!N^* = \{1, 2, \ldots\}$ and B_1 the σ-algebra of all Borel subsets of the unit interval I. Then the functions $a_n(.) : I \to X$, for any $n \in I\!N^*$, defined by the general alternating algorithm may be considered as random variables defined almost surely on I with respect to any probability measure on the σ-algebra B_1 (in particular, with respect to the Lebesgue measure λ).

We focus on the alternating-Engel expansion (1.1) for any real number $x \in I$, a unique finite or infinite representation of the form

$$x = \frac{1}{a_1} + \sum_{n\geq 2} (-1)^{n-1} \frac{1}{a_1 a_2 \cdots a_n}, \tag{2.1}$$

which may also be generated by a transformation T as follows.

Let $T : [0, 1] \to [0, 1)$ be the transformation defined by:

$$\begin{aligned} T(0) &= 0, \\ T(x) &= \left(\frac{1}{\left[\frac{1}{x}\right]} - x \right) / \frac{1}{\left[\frac{1}{x}\right]}, \text{ for any x} \neq 0. \end{aligned} \tag{2.2}$$

If $a_1 = a_1(x) = \left[\frac{1}{x}\right]$ and $a_{n+1} = a_{n+1}(x) = \left[\frac{1}{T^n(x)}\right]$, for $T^n(x) \neq 0$, where

$$T^n(x) = \left(\frac{1}{\left[\frac{1}{T^{n-1}(x)}\right]} - T^{n-1}(x) \right) / \frac{1}{\left[\frac{1}{T^{n-1}(x)}\right]}, \quad n \in I\!N^*,$$

we conclude that the transformation T generates the above representation (2.1) for every $x \in (0, 1]$.

Let us now define by

$$B_n = B_n(k_1, k_2, ..., k_n) = \{x \in (0, 1]/a_1(x) = k_1, a_2(x) = k_2, \ldots, a_n(x) = k_n\}, \tag{2.3}$$

for any $k_1, ..., k_n \in I\!N^*$, the set of all $x \in (0, 1]$ which have a unique expansion of the form (2.1) such that the digits $a_1(x), \ldots, a_n(x)$ have the concrete values $k_1, ..., k_n$ respectively. Here $B_0 = [0, 1]$.

Then we obtain the following

Theorem 2 *The set* $B_n = B_n(k_1, k_2, ..., k_n)$ *is bounded and its corresponding bounds are given by the relations*

$$\begin{aligned} M_n &\equiv \sup B_n(k_1,k_2,...,k_n) = \frac{1}{k_1} - \frac{1}{k_1k_2} + \cdots + (-1)^{n-1}\frac{1}{k_1k_2\cdots k_n} + \frac{1}{k_1k_2\cdots k_n}, \\ m_n &\equiv \inf B_n(k_1,k_2,...,k_n) = \frac{1}{k_1} - \frac{1}{k_1k_2} + \cdots + (-1)^{n-1}\frac{1}{k_1k_2\cdots k_n} + \frac{1}{k_1k_2\cdots k_n}, \end{aligned}$$

where n is even. If n is an odd number, then the above relations are inverted. Moreover its Lebesgue measure is given by

$$\lambda(B_n) = \prod_{j=1}^{n} \frac{1}{k_j},$$

for any $n \in I\!N^*$.

Proof. For the proof we refer the reader to Ganatsiou (Prop. 2, [2], p.36-37).□

In the next we shall give some useful notions and concepts from the ergodic theory.

Definition 3 Let $(\Omega_1, \mathcal{F}_1, P_1)$ and $(\Omega_2, \mathcal{F}_2, P_2)$ be two probability spaces.

(i) A map $T : \Omega_1 \to \Omega_2$ is called a transformation.

(ii))A transformation $T : \Omega_1 \to \Omega_2$ is said to be measurable if $(\forall)$ $E \in \mathcal{F}_2$, then $T^{-1}E \in \mathcal{F}_1$.

(iii) A measurable transformation $T : \Omega_1 \to \Omega_2$ is said to be non-singular if $(\forall)$ $E \in \mathcal{F}_2$ with $P_2(E) = 0$, then $P_1(T^{-1}E) = 0$.

We take now the two above - mentioned probability spaces be identical.

Definition 4 A measurable non-singular transformation T is called *ergodic* if the relation $T^{-1}E = E$, for $E \in \mathcal{F}$, implies $P(E) = 0$ or $P(E) = 1$. If the weaker assumption $T^{-1}E \subset E$, for $E \in \mathcal{F}$, already implies $P(E) = 0$ or 1, then T is called *strongly ergodic.*

Under the above definitions we take the following useful classical criterion for ergodicity of Knopp.

Theorem 5 *Let E be a Lebesgue - measurable subset of* $(0,1]$ *with* $P(E) > 0$. *Assume that there is a collection J of subintervals of* $(0,1)$ *with the following properties:*

(i) every open subinterval of $(0,1)$ *is at most a denumerable union of disjoint elements of J (P a. s.) and*

(ii) for every $B \in J$, $P(EB) \geq cP(B)$ *with a constant* $c > 0$.

Then $P(E) = 1$.

3 Ergodic properties of the basic operator

Using the above - mentioned auxiliary results we are able to prove the following statement.

Theorem 6 *The operator* $T : [0,1] \to [0,1)$ *is ergodic relative to the Lebesgue measure* λ.

Proof. At first we define a linear function $h_n = h_n(k_1, \ldots, k_n) : (0,1] \to B_n$ by the relation

$$h_n(v) = \sum_{j=1}^{n} \frac{(-1)^{j-1} \cdot \lambda(B_{j-1})}{k_j} + (-1)^n \cdot v \ \cdot \lambda(B_n) =$$

$$= \sum_{j=1}^{n} \frac{(-1)^{j-1}}{k_1 k_2 \cdots k_{j-1}} \cdot \frac{1}{k_j} \left(1 + \tfrac{(-1)^{j-2}}{k_j}\right) + (-1)^n \cdot v \cdot \prod_{j=1}^{n} \frac{1}{k_j}.$$

If $x \in B_n$ then

$$x = \sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{a_1 a_2 \cdots a_{j-1}} \frac{1}{a_j}$$

$$= \sum_{j=1}^{n} \frac{(-1)^{j-1} \cdot \lambda\left(B_{j-1}\right)}{k_j} + \lambda(B_n) \cdot \sum_{j=n+1}^{\infty} \frac{(-1)^{j-1}}{a_{n+1} \cdots a_{j-1}} \cdot \frac{1}{a_j} = h_n\left(T^n(x)\right).$$

So $h_n^{-1} = T^n : B_n \to I$.

Moreover

$$M_n = h_n(1), m_n = h_n(0), \text{ for any } n = 2, 4, ...,$$

with

$$\begin{aligned} h_n(0) &= \frac{1}{k_1} - \frac{1}{k_1 k_2} + \frac{1}{k_1 k_2 k_3} - \cdots + (-1)^{n-2} \frac{1}{k_1 k_2 \cdots k_{n-1}} + (-1)^{n-1} \frac{1}{k_1 k_2 \cdots k_n} \\ &= \sum_{j=1}^{n} \frac{(-1)^{j-1} \lambda\left(B_{j-1}\right)}{k_j}, \\ h_n(1) &= \frac{1}{k_1} - \frac{1}{k_1 k_2} + \frac{1}{k_1 k_2 k_3} - \cdots + (-1)^{n-1} \frac{1}{k_1 k_2 \cdots k_n} + (-1)^{n} \frac{1}{k_1 k_2 \cdots k_n} \\ &= \sum_{j=1}^{n} \frac{(-1)^{j-1} \lambda\left(B_{j-1}\right)}{k_j} + (-1)^n \lambda\left(B_n\right). \end{aligned}$$

(If n is odd then the above relations are inverted).

So for any interval $(a,b] \subseteq I$ we have:

$$\begin{aligned} \lambda\left(T^n\left(a,b\right] \cap B_n\right) &= \lambda\left(h_n\left(a,b\right] \cap B_n\right) = \left|h_n(b) \ - \ h_n(0)\right| = \\ &= (b-a)\,\lambda\left(B_n\right) = \lambda(a,b] \cdot \lambda\left(B_n\right). \end{aligned}$$

Therefore

$$\lambda\left(T^n E \cap B_n\right) = \lambda(E) \lambda\left(B_n\right) \tag{3.1}$$

for any set E in the Boolean ring R of all finite disjoint unions of intervals $(a, b] \subseteq I$. It can be easily seen by using standard measure theory that the same equation holds for any Borel set E in I. Let now E be a Borel set in I such that $T^{-1}E = E$. Then we obtain that $T^{-n}E = E$, for any $n \geq 1$.

Relation (3.1) gives that

$$\lambda(E \cap B_n) = \lambda(E) \cdot \lambda(B_n),$$

or

$$\lambda(E \cap B_n) = c \cdot \lambda(B_n), \text{ with } c = \lambda(E) > 0.$$

If J is the collection of all cylinders B_n, $n \geq 1$ and, for any $j \geq 1$, then any open subinterval of $(0, 1)$ is an at most denumerable (or countable) disjoint union of elements of J (λ a.s). Therefore

$$\lambda(E \cap B) = c \cdot \lambda(B), \text{ with } c = \lambda(E) > 0, \tag{3.2}$$

for any set B which is a countable disjoint union of the fundamental intervals B_n. Thus by using Theorem 5, Definition 4 and the formula (3.2) the proof is complete. □

Since T is an ergodic but not a λ- measure-preserving transformation on the probability space $([0, 1], B_{[0,1]}, \lambda)$ a version of the Birkhoff-Riesz theorem (due to Dunford and Miller) now implies that for any integrable function g on I the a.s. limit

$$\lim_{n\to\infty} \frac{1}{n} \cdot \sum_{k=0}^{n-1} g\left(T^k(x)\right)$$

exists.

As a consequence, by applying the above Theorem in the study of the metric properties of the alternating-Engel expansion, we are able to obtain important results given in the following result.

Theorem 7

(i) For all $x \in I$ outside a set of Lebesgue measure 0, the limit $\lim\limits_{n\to\infty} (a_1 a_2 \cdots a_n \cdots)^{\frac{1}{n}}$ exists.

(ii) The sequence $(T^n(x))_n$ is uniformly distributed in I, and has asymptotic (infinite) versions of arithmetic and geometric means given by the following limits respectively:

$$\lim_{n\to\infty} \frac{1}{n} \cdot \sum_{k=0}^{n-1} T^k(x) \quad , \qquad \lim_{n\to\infty} \left(\prod_{k=0}^{n} T^k(x)\right)^{1/n} .$$

Proof.

(i) If $g(x) = \log a_1(x)$, then almost surely the limit

$$\lim_{n\to\infty} \frac{1}{n} \cdot \sum_{k=0}^{n-1} \log a_k(x)$$

or the limit

$$\lim_{n\to\infty} \left(\prod_{k=0}^{n} a_k(x) \right)^{1/n}$$

exists.

(ii) By choosing g to be the identity function $g(x) = x$ or the function $g(x) = \log x$ respectively, the above form of the Ergodic Theorem implies, for almost all x, the above required result.

References

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