NUMERICAL SOLUTION OF THE FLOW OF A VISCOUS CONDUCTING FLUID PRODUCED BY AN OSCILLATING PLANE WALL SUBJECTED TO A TRANSVERSE MAGNETIC FIELD

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Abstract

A numerical analysis is carried out in this paper in order to study the flow of a laminar incompressible viscous electrically conducting fluid, caused by the oscillation of an inflexible plane wall subjected to a uniform magnetic field acting vertically to the flow direction. The motion of the flow is unsteady and unidirectional. First, we non-dimensionalize the basic equation, governing the motion of such a flow. Then this equation is solved numerically by a finite difference method with two sets of non-dimensional boundary and initial conditions. The developments of the non-dimensional velocity field in time for different values of a magnetic parameter are presented graphically. Finally, the results are discussed.

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Key words: Viscous conducting fluid, unsteady and undirectional flow, transverse magnetic field, non-dimensionalization, finite difference method, Crank-Nicolson scheme.

1 Introduction

The unsteady flow of a viscous fluid, which sets up from rest when a plane wall oscillates with a prescribed velocity, was first studied by Stokes according to Bansal [1] and is termed as Stokes's second problem (Schlichting [2]). In the literature either it is called Stoke's first problem (Zeng and Weinbaum [3]) or simply Rayleigh's problem (Bansal [1]).

Kerezek and Davis [4] performed the linear stability analysis of the Stoke's layers on the oscillating surface.

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The transient solution for the flow due to the oscillating plate has been given by Panton [5]. He has assumed that for large times steady-state flow is set-up with the same frequency as the velocity of the plane boundary. In order to obtain the starting solution, since the problem is linear, a transient solution must be added to the steady-state solution. Panton [5] has presented a closed-form solution to the transient component of Stoke's problem using the steady state component as the initial profile, however, he has not given the expression for the starting velocity field and the expression of the transient solution for the cosine oscillation of the plane boundary. Recently, Erdogan [6] obtained an analytical solution describing the flow at small and large times after the start of the boundary by the Laplace transform method. He has considered the flow of a viscous fluid produced by a plane boundary moving in its own plane with a sinusoidal variation of velocity. He obtained the steady solution and also the transient solution by subtracting the steady state solution from the starting solution.

The study on the flow of a viscous fluid over an oscillating plate is not only of fundamental theoretical interest but also of practical importance as it occurs in many applied problems. It arises in acoustic streaming around an oscillating body. Another example is an unsteady boundary layer with fluctuation in the free-stream velocity [7].

In this paper, we study numerically the flow of an incompressible viscous fluid, caused by the oscillation of a rigid plane wall. Considering the flow over a plane solid wall y=0 to be unsteady and unidirectional the equation governing the motion can be written as

$$\frac{\partial u}{\partial t} = \frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2}, \tag{1}$$

where $u(y,t)$ is the velocity component in the x-direction, ρ is the density of the fluid and p is the pressure.

We can set the pressure gradient $\frac{\partial p}{\partial x} = 0$ as the flow is the only kept in motion by the oscillation of the wall. Accepting this, equation (1) reduces to

$$\frac{\partial u}{\partial t} = \nu\frac{\partial^2 u}{\partial y^2}. \tag{2}$$

The solution of (2) is given by (White [8]) and has the form

$$u\left(y,t\right) = f\left(y\right)\exp\left(i\omega t\right), \tag{3}$$

where ω is the frequency of the oscillatory motion, $i = \sqrt{-1}$ and $f\left(y\right)$ has both real and imaginary parts. So long as the calculations involve only linear operations on the velocity u, we may proceed as if u were complex, taking the real part of the final result. The complex form is used because u is periodic in t. The no-slip condition, in complex notation is

$$u\left(0,t\right) = U_0 e^{i\omega t}, \tag{4}$$

where U_0 is the maximum velocity of the wall.

As the fluid in the far field is at rest we take $u\,(\infty, t) = 0$. From (3) it follows that $f\,(0) = U_0$.

The purpose of this paper is to investigate the effects of a transverse magnetic field on the incompressible electrically conducting viscous fluid flow over a plane solid wall oscillating in its own plane. This paper has four paragraphs. Each of them is analyzed as follows:

The first paragraph is the introduction.

The mathematical formulation of the problem is given in the second paragraph.

The numerical calculations have been performed in the third paragraph by a finite difference method, using the well-known Crank-Nicholson Implicit scheme.

The last paragraph contains the results and discussions about the solutions and some graphic representations of these solutions are also given.

2 Mathematical formulation

Consider the unsteady flow of an electrically conducting incompressible viscous fluid (with electrical conductivity σ) past a rigid plane wall coinciding with the plane y=0. The fluid over the plane wall is initially at rest and it sets in motion due to the oscillation of the wall subjected to a uniform transverse magnetic field . The fluid is taken occupy the upper half-plane ($y \succeq 0$) with the wall on the x-axis.

The governing equation of the motion is

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2} - \frac{\sigma B_0}{\rho} u, \tag{5}$$

where ν is the kinematic viscosity, σ is the electrical conductivity, B_0 is the applied magnetic field perpendicular to the flow direction, y is the co-ordinate and t is the time.

The first set of boundary and initial conditions for the cosine oscillations is

$$\begin{cases} u = U_0 \cos(\omega t) & at \;\; y = 0, \;\; for \;\; t > 0, \\ u = 0 & at \;\; t = 0, \;\; for \;\; y > 0, \\ u \to 0 & as \;\; y \to \infty, \end{cases} , \tag{6}$$

where ω is the frequency of the velocity of the wall.

The other set of boundary and initial conditions for the sinusoidal oscillations is

$$\begin{cases} u = U_0 \sin(\omega t) & at \;\; y = 0, \;\; for \;\; all \;\; t, \\ u = 0 & at \;\; t = 0, \;\; for \;\; y > 0, \\ u \to 0 & as \;\; y \to \infty \end{cases} . \tag{7}$$

The equation (5) is transformed, in terms of non-dimensional quantities as

$$\frac{\partial \overline{u}}{\partial \tau} = \frac{\partial^2 \overline{u}}{\partial \eta^2} - M\overline{u}, \tag{8}$$

where $\overline{u}=\frac{u}{U_0}$, $\tau=\omega t$, $\eta=y\left(\frac{\omega}{\nu}\right)^{\frac{1}{2}}$ are the non-dimensional quantities of velocity u, time t and co-ordinate y and $M=\frac{\sigma B_0^2}{\rho\omega}$ is the non-dimensional magnetic parameter.

Correspondingly, the two sets of boundary and initial conditions (6) and (7) are transformed, to

$$\begin{cases} \overline{u}=\cos\tau & at\ \ \eta=0,\ \ for\ \ \tau>0, \\ \overline{u}=0 & at\ \ \tau=0,\ \ for\ \ \eta>0, \\ \overline{u}\to 0 & as\ \ \eta\to\infty \end{cases} \tag{9}$$

and

$$\begin{cases} \overline{u}=\sin\tau & at\ \ \eta=0,\ \ for\ \ all\ \ \tau, \\ \overline{u}=0 & at\ \ \tau=0,\ \ for\ \ \eta>0, \\ \overline{u}\to 0 & as\ \ \eta\to\infty \end{cases} . \tag{10}$$

3 Numerical Computations

The equation (8) with boundary and initial conditions, given by (9) and (10) are solved separately by a finite difference technique.

We use Crank - Nicholson implicit scheme as appropriate for the parabolic type of equation. In this scheme, the time derivative term $\frac{\partial\overline{u}}{\partial\tau}$ is represented by forward - differences as given by

$$\frac{\partial\overline{u}}{\partial\tau}=\left.\overline{u}_\tau\right|_{i,j}=\frac{\overline{u}_{i,j+1}-\overline{u}_{i,j}}{k}+0\left(R\right), \tag{11}$$

while the space derivative $\frac{\partial^2\overline{u}}{\partial\eta^2}$ is represented by the average central differences at the present and new time step (Niyogi [9]) is given by:

$$\frac{\partial^2\overline{u}}{\partial\eta^2}=\left.\overline{u}_{\eta\eta}\right|_{i,j}=\frac{\overline{u}_{i+1,j}-2\overline{u}_{i,j}+\overline{u}_{i-1,j}}{h^2}+0^2\left(R\right). \tag{12}$$

Accordingly, we write (8) in the form:

$$\begin{aligned}\frac{1}{4\tau}\left(\overline{u}_j^{n+1}-\overline{u}_j^n\right) \quad = \quad & \frac{1}{2}\left[\frac{1}{4\eta^2}\left(\overline{u}_{j+1}^{n+1}-2\overline{u}_j^{n+1}+\overline{u}_{j-1}^{n+1}\right)+\right. \\ & \left.\frac{1}{4\eta^2}\left(\overline{u}_{j+1}^{n}-2\overline{u}_j^{n}+\overline{u}_{j-1}^{n}\right)\right]-\frac{M}{2}\left(\overline{u}_j^{n+1}-\overline{u}_j^n\right),\end{aligned} \tag{13}$$

or,

$$-r\,\overline{u}_{j-1}^{n+1}+\left(1+2r+MrP\right)\overline{u}_j^{n+1}-r\,\overline{u}_{j+1}^{n+1}=r\left(\overline{u}_{j-1}^{n}-\overline{u}_{j+1}^{n}\right)+\left(1-MrP-2r\right)\overline{u}_j^{n}, \tag{14}$$

where

$$r = \frac{4\tau}{24\eta^2}, \quad P = 4\eta^2. \tag{15}$$

The system of algebraic equations in tridiagonal form that follows from (14) is solved by Thomas algorithm for each time level. In this problem 25x25 grid-points have been considered for numerical computation. $\overline{u}$ is obtained at each grid-points at each time level.

4 Results and discussions

Solutions for the velocity $\overline{u}(\eta, \tau)$ as obtained by solving the finite-difference equation (14) with the specified sets of initial and boundary conditions (9) and (10) are represented by graphs over a cycle, respectively in figures 1-2 and 3-4.

Case i) $\overline{u} = \cos\tau$ at $\eta = 0$

In fig.1 velocity profiles for times $\tau = 0.1$ and $\tau = 2.5$ are shown. For both the cases the values of the magnetic parameter being taken M=0 and M=5 separately. In the right half-cycle, where is suppression of velocity due to the influence of the magnetic field (M=5). The difference between the velocities with M=5 and M=0 disappears when they vanish at a height $\eta = 1.7$ above the wall. In the left half-cycle, it is seen that for time $\tau = 2.5$ the velocity is enhanced for the case with M=5. The two velocity curves, for M=5 and M=0 cross each other at $\eta = 1.2$. The velocities corresponding to M=5 and M=0 vanish, respectively at $\eta = 2.4$ and $\eta = 5$ above the wall.

In fig.2, velocity profiles for times $\tau = 0.1$ and $\tau = 2.5$ and with M=0 and M=10 are shown. In this case the velocity profiles in both right and left half-cycles for M=10 are qualitatively similar to that of the case M=5. Suppression of velocity in the right half cycle with M=10 is little more than that of the same with the case M=5. Enhancement of velocity is also little more with M=10 than that with the case M=5.

Case ii) $\overline{u} = \sin\tau$ at $\eta = 0$

In fig.3, velocity profiles for times $\tau = 2.5$ and $\tau = 5$ are shown. For both cases the values of magnetic parameter has being taken M=0 and M=5 separately. It may be noticed that in the right half-cycle ($\tau = 2.5$) large amount of suppression of the velocity occurs with M=5 and M=10. Such suppression of velocity is continued up to $\eta = 3$ where velocity in the case M=5 vanishes. Velocity with the case M=0 is found to vary at $\eta = 5$. In the left half-cycle, there is large enhanced of velocity at $\tau = 5$ and M=5 up to $\eta = 2.4$ above the wall. The velocity for $\tau = 5$ and M=5 vanishes at $\eta = 3$. Whereas, the velosity for $\tau = 5$ and M=0 vanishes at $\eta = 5$.

In fig.4, velocity profiles for times $\tau = 2.5$ and $\tau = 5$ with M=0 and M=10 are shown. In this case the suppression of velocity in the right half-cycle is larger than that of the previous case namely, with $\tau = 2.5$ and M=5. The velocity for $\tau = 2.5$ with M=10 vanishes at a lower high $\eta = 2.2$ at the wall. The velocity for $\tau = 2.5$ and M=0 vanishes at $\eta = 0$ as before.

In the left half-cycle the enhancement in the velocity for $\tau = 5$ and M=10 is larger than that of the case with $\tau = 5$ and M=5. The velocity for $\tau = 5$ and M=10 is found to vanish at $\eta = 2.1$. The velocity for $\tau = 5$ and M=0 vanishes at $\eta = 5$ above the wall as before.

Thus in the right half-cycle as we go on increasing the value of M the velocity is suppressed more and more. Also the velocity is enhanced more and more with the inverse of M in the left half-cycle.

It is observed that the unsteady behavior of the velocity field of the sine oscillations differs much from the same with the cosine oscillations.

The reason may be attributed to the fact that in the case of the sine oscillations the fluid is at rest at time $\tau = 0$.

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Figure Captions

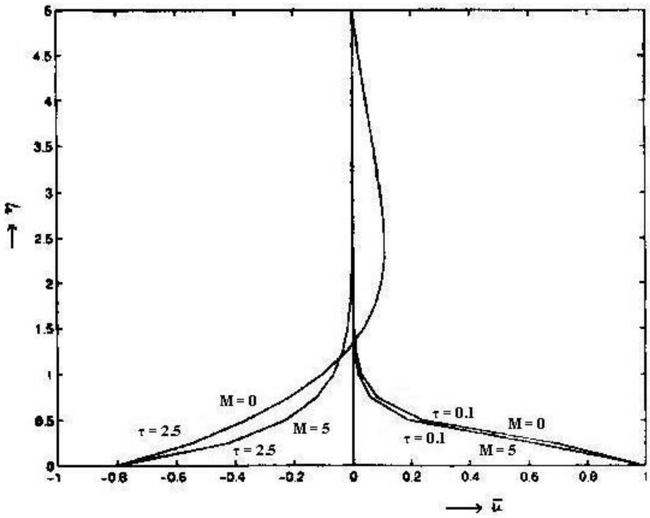

Fig.1: Diagram of velocity profiles for times $\tau = 0.1$ and $\tau = 2.5$ with $M = 0$ and $M = 5$

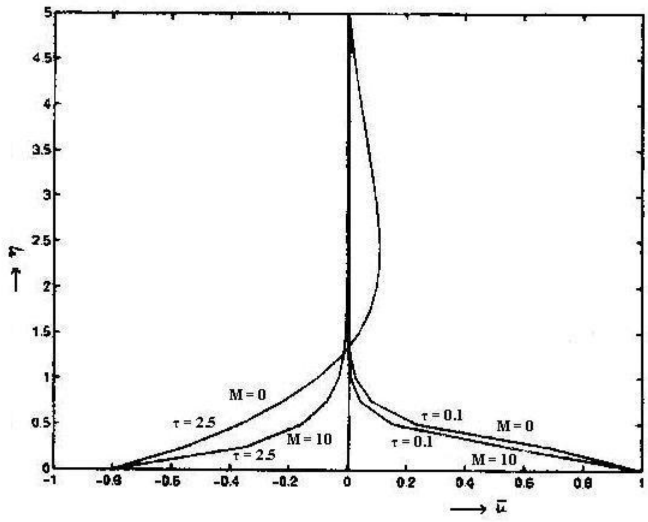

Fig.2: Diagram of velocity profiles for times $\tau = 0.1$ and $\tau = 2.5$ with $M = 0$ and $M = 10$

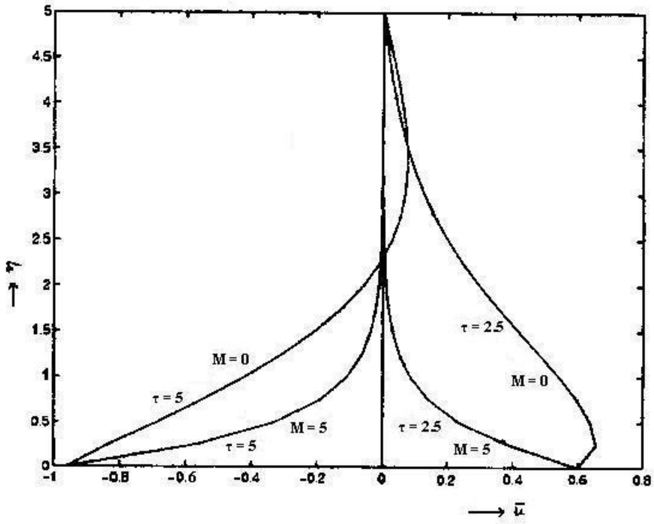

Fig.3: Diagram of velocity profiles for times $\tau = 2.5$ and $\tau = 5$ with $M = 0$ and $M = 5$

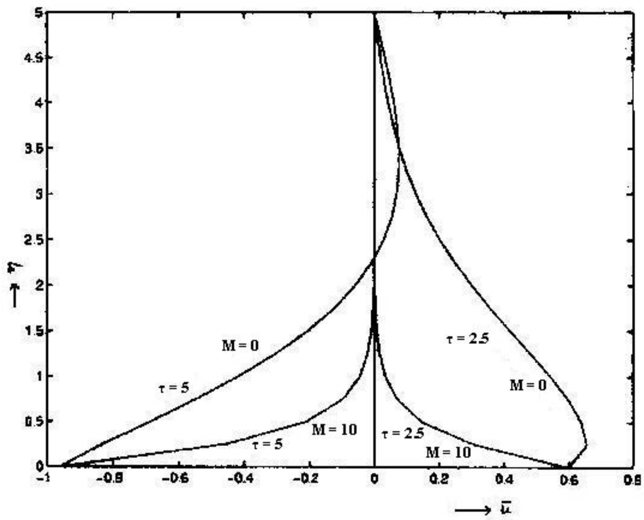

Fig.4: Diagram of velocity profiles for times $\tau = 2.5$ and $\tau = 5$ with $M = 0$ and $M = 10$

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