ENERGY-DEPENDENT MINKOWSKI METRIC IN SPACE-TIME

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Abstract

The geometrical properties of the space-time endowed with a metric depending on the energy E of the considered process are obtained using the generalized Lagrange space methods. The tensor of the electromagnetic field is introduced in order to include both electromagnetic and gravitational fields in the same model. An example of energy-dependent metric is given and a comparison with other results is presented.

AMS Subject Classification: 53B40, 53C60
Key words: generalized Lagrange space, N-connection, energy, (deformed) Minkowski metric.

1 Introduction

In a recent paper [1], Cardone and Mignani starting from the Minkowski metric

$$\begin{gathered} ds^2 = c^2dt^2 - \left\{ \left(dx^1\right)^2 + \left(dx^2\right)^2 + \left(dx^3\right)^2 \right\} = \eta_{ij}dx^i dx^j; \\ i,j = 0,1,2,3; \qquad x^0 = ct, \end{gathered} \tag{1.1}$$

have considered a "deformed" Minkowski metric of the form:

$$\begin{aligned} d\check{s}^2 = \tilde{b}_0^2(E)c^2dt^2 - \left\{ b_1^2(E)\left(dx^1\right)^2 + b_2^2(E)\left(dx^2\right)^2 + b_3^2(E)\left(dx^3\right)^2 \right\} = \\ = \check{\eta}_{ij}dx^i dx^j. \end{aligned} \tag{1.2}$$

Here, E is the energy of the considered process. The energy E is viewed as a "phenomenological variable", i.e. E is the energy measured by detectors via the electromagnetic interaction in the Minkowski space M, and $b_i(E) > 0$ $(i = 0,1,2,3)$.

Recently, many authors have considered deformed Lorentz metrics in space-time M, $ds^2 = a_{ij}dx^i dx^j$, with the coefficients a_{ij} depending on the coordinates $\left(x^i\right)$,

Editor Gr.Tsagas *Proceedings of The Conference of Applied Differential Geometry - General Relativity and The Workshop on Global Analysis, Differential Geometry and Lie Algebras, 2001,* 109-123

the energy E, the velocity and of other characteristics of the studied process. Such metrics are, evidently, not Riemannian ones, but of Finsler type or, more generally, they are generalized Lagrange metrics [2]. The generalized Lagrange metrics have been introduced by R. Miron [2], and applied by G.S. Asanov, S. Ikeda, R. Tavakol, J. Roxburgh, R. Miron, T. Kawaguchi, V. Balan, P. Stavrinos, G. Zet and many others to the Relativistic Optics and also to the General Relativity [3], [7], [10 − 19].

In this paper we study the geometrical properties of the space-time M endowed with the metric (1.2) depending of the energy E of the considered process. We adopt the definition:

$$E = \eta_{ij} \frac{dx^i}{dt} \frac{dx^j}{dt}. \tag{1.3}$$

In addition, we introduce the tensor $\check{F}^i{}_j = g^{ik} F_{kj}$ of the electromagnetic field in order to include in the same model both the gravitational and electromagnetic properties based on the deformed metric (1.2).

2 Generalized Lagrange Space GL^n

If M is a differentiable manifold and (TM, π, M) is the tangent bundle, then the coordinates of $x \in M$ will be denoted by (x^i), $i, j = 1, 2, \ldots, n = \dim(M)$, and those of $u \in TM$, $\pi(u) = x$, by (x^i, y^i). In the differentiable manifold TM we have the following local transformations of coordinates:

$$\begin{cases} \tilde{x}^i = \tilde{x}^i(x), \quad \det\left(\dfrac{\partial \tilde{x}^i}{\partial x^j}\right) \neq 0, \\ \\ \tilde{y}^i = \dfrac{\partial \tilde{x}^i}{\partial x^j}\, y^j. \end{cases} \tag{2.1}$$

The natural basis of the tangent space $T_u(TM)$ at a point $u = (x, y) \in TM$ is $\left\{\dfrac{\partial}{\partial x^i}, \dfrac{\partial}{\partial y^j}\right\}$, $i, j = 1, 2, \ldots, n$. Taking into account (2.1), we have:

$$\begin{cases} \dfrac{\partial}{\partial x^i} = \dfrac{\partial \tilde{x}^s}{\partial x^i} \dfrac{\partial}{\partial \tilde{x}^s} + \dfrac{\partial \tilde{y}^s}{\partial x^i} \dfrac{\partial}{\partial \tilde{y}^s}, \\ \\ \dfrac{\partial}{\partial y^i} = \dfrac{\partial \tilde{y}^s}{\partial y^i} \dfrac{\partial}{\partial \tilde{y}^s}. \end{cases} \tag{2.2}$$

Therefore, the vector fields $\left\{\dfrac{\partial}{\partial y^i}\right\}$, $i = 1, 2, \ldots, n$, generate locally a distribution V. As it results from (2.2), the distribution V is defined everywhere on the tangent manifold TM and is integrable, too. V is named the *vertical distribution* on TM.

Let N be a distribution on TM supplementary to V, i.e.

$$T_u(TM) = N_u \oplus V_u, \quad \forall u \in TM. \tag{2.3}$$

Then, N is named a *horizontal distribution*, or a *nonlinear connection* on TM. An addapted basis to the distributions N and V (or addapted to the direct sum decomposition (2.3)) is of the form $\left\{\dfrac{\delta}{\delta x^i}\right\}$ in N and $\left\{\dfrac{\partial}{\partial y^i}\right\}$ in V, where:

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N_i^j(x,y)\,\frac{\partial}{\partial y^j}. \tag{2.4}$$

Here, $N_i^j(x,y)$ are the *coefficients* of the nonlinear connection N.

Because the vector fields $\dfrac{\delta}{\delta x^i}$ are contained in the distribution N, it follows then that N is an integrable distribution if and only if the brackets $\left[\dfrac{\delta}{\delta x^i},\dfrac{\delta}{\delta x^j}\right]$, $i,j = 1,2,\ldots,n$, determine vector fields included in N. But, we can write [3]:

$$\left[\frac{\delta}{\delta x^i},\frac{\delta}{\delta xj}\right] = R^s{}_{ij}\,\frac{\partial}{\partial y^s}, \tag{2.5}$$

where

$$R^i{}_{jk} = \frac{\delta N^i{}_j}{\delta x^k} - \frac{\delta N^i{}_k}{\delta x^j} \tag{2.6}$$

is a field of *d-tensors* (or *distinguished tensors*). This means that the components $R^i{}_{jk}$ transform like the components of a tensor of $(1,2)$-type on the base space M under the coordinates transformation (2.1) on TM:

$$\tilde{R}^i{}_{jk} = \frac{\partial \tilde{x}^i}{\partial x^s}\,\frac{\partial x^r}{\partial \tilde{x}^j}\,\frac{\partial x^p}{\partial \tilde{x}^k}\,R^s{}_{rp}.$$

The tensor of torsion of the nonlinear connection is:

$$t^i_{jk} = \frac{\partial N^i{}_j}{\partial y^k} - \frac{\partial N^i{}_k}{\partial y^j}.$$

Remark. This will be considered the general rule of definition for distinguished tensors (d-tensors) on the manifold TM.

From (2.5) it follows:

Proposition 2.1. *The horizontal distribution N (the nonlinear connection) is integrable if and only if the d-tensor $R^i{}_{jk}$ on TM is vanishing.*

In that follows, we will consider the dual basis $\{dx^i, \delta y^j\}$ of the addapted basis $\left\{\dfrac{\delta}{\delta x^i},\dfrac{\partial}{\partial y^j}\right\}$. We obtain:

$$\delta y^i = dy^i + N^i{}_j dx^j. \tag{2.7}$$

It is important to remark that $\left\{\dfrac{\delta}{\delta x^i}\right\}$ is a d-field of covariant vectors and $\left\{\dfrac{\partial}{\partial y^i}\right\}$ has the same property. Analogous, $\{dx^i\}$ is a d-field of contravariant vectors and $\{\delta y^i\}$ has the same property.

A generalized Lagrange space is a pair $GL^n = (M, g_{ij}(x,y))$, where $g_{ij}(x,y)$ is a d-field of tensors on the manifold TM (or, sometimes, on the manifold $\widetilde{TM} = TM\backslash\{0\}$), covariant, symmetric, non-degenerate and of constant signature. The d-field $g_{ij}(x,y)$ is non-degenerate if

$$\operatorname{rank} \|g_{ij}(x,y)\| = n.$$

In the next section we will show that the space endowed with the metric $d\check{s}^2$ in (1.2) is a generalized Lagrange space.

Clearly, if the d-tensor $g_{ij}(x,y)$ do not depend on the variables y^i, then $GL^n = (M, g_{ij}(x,y))$ is a pseudo-Riemannian space (or a Riemannian one). When $g_{ij}(x,y)$ depends only of the variables y^i (in prefered charts), this space is a generalized Lagrange space, locally Minkowski.

A function

$$L : (x,y) \in TM \longrightarrow L(x,y) \in \mathbb{R}, \tag{2.8}$$

differentiable on $\widetilde{TM}$ and continuous on the null section of π is named a *regular Lagrangian* if the Hessian of L with respect to the variables y^i is non-singular.

A generalized Lagrange space $GL^n = (M, g_{ij}(x,y))$ is named *reducible to a Lagrange space* if there is a regular Lagrangian L such that:

$$g_{ij} = \frac{1}{2}\frac{\partial^2 L}{\partial y^i \partial y^j}. \tag{2.9}$$

on $\widetilde{TM}$. A necessary condition that GL^n be reducible to a Lagrange space is that the d-tensor $\dfrac{\partial g_{ij}}{\partial y^k}$ is totally symmetric. If the previous condition is satisfied and g_{ij} are 0-homogeneous in the variables y^i, then the function $L = g_{ij}(x,y)y^i y^j$ is a solution of the system of equations (2.9) (with partial derivatives). In this case the pair (M, L) is a Finsler space (M, F), with $F^2 = L$. We say that GL^n is reducible to a Finsler space.

A linear N-connection on TM (or on $\widetilde{TM}$) is defined by a pair of geometrical objects $C\Gamma(N) = \left(L^i{}_{jk}, C^i{}_{jk}\right)$ on TM with the property that $L^i{}_{jk}(x,y)$ transform with respect to (2.1) like the coefficients of a linear connection on the base manifold M, and $C^i{}_{jk}(x,y)$ transform under (2.1) like a d-tensor of $(1,2)$-type.

We can define then two types of covariant derivatives: *a covariant h-derivative* denoted by "$|$" and **a** *covariant v-derivative* denoted by "$\big|$". For example, in the case of the field of d-tensors $g_{ij}(x,y)$, we have respectively:

$$\begin{cases} g_{ij|k} = \dfrac{\delta g_{ij}}{\delta x^k} - g_{sj}L^s{}_{ik} - g_{is}L^s{}_{jk}, \\[2ex] g_{ij}\big|_k = \dfrac{\partial g_{ij}}{\partial y^k} - g_{sj}C^s{}_{ik} - g_{is}C^s{}_{jk}. \end{cases} \tag{2.10}$$

We remark that $g_{ij|k}$ and $g_{ij}\big|_k$ are d-tensors of type $(0,3)$.

Proposition 2.2. *The following Ricci identities hold:*

$$\begin{cases} g_{ij|k|h} - g_{ij|h|k} & = -g_{sj} R_i{}^s{}_{kh} - g_{is} R_j{}^s{}_{kh} - T^s{}_{kh} g_{ij|s} - R^s{}_{kh} g_{ij}\big|_s, \\ g_{ij|k}\big|_h - g_{ij}\big|_{h|k} & = -g_{sj} P_i{}^s{}_{kh} - g_{is} P_j{}^s{}_{kh} - C_k{}^s{}_h g_{ij|s} - P^s{}_{kh} g_{ij}\big|_s, \\ g_{ij}\big|_k\big|_h - g_{ij}\big|_h\big|_k & = -g_{sj} S_i{}^s{}_{kh} - g_{is} S_j{}^s{}_{kh} - S^s{}_{kh} g_{ij}\big|_s. \end{cases} \tag{2.11}$$

Here, $R_j{}^i{}_{kh}, S_j{}^i{}_{kh}$ and $P_j{}^i{}_{kh}$ are the d-tensors of curvature:

$$\begin{aligned} R_j{}^i{}_{kh} &= \frac{\delta L^i{}_{jk}}{\delta x^h} - \frac{\delta L^i{}_{jh}}{\delta x^k} + L^r{}_{jk}\, L^i{}_{rh} - L^r{}_{jh}\, L^i{}_{rk} + C^i_{jr} R^r{}_{kh}, \\ P_j{}^i{}_{kh} &= \frac{\partial L^i{}_{jk}}{\partial y^h} - C^i{}_{jh|k} + C^i_{jr} P^r{}_{kh}, \\ S_j{}^i{}_{kh} &= \frac{\partial C^i{}_{jk}}{\partial y^h} - \frac{\partial C^i{}_{jh}}{\partial y^k} + C^r{}_{jk} C^i_{rh} - C^r{}_{jh} C^i_{rk}, \end{aligned} \tag{2.12}$$

and $T^i{}_{jk}, S^i{}_{jk}, R^i{}_{jk}, C^i{}_{jk}, P^i{}_{jk}$ are the d-tensors of torsion of the connection $C\Gamma(N)$:

$$T^i{}_{jk} = L^i{}_{jk} - L^i{}_{kj}, \quad S^i{}_{jk} = C^i{}_{jk} - C^i{}_{kj}, \quad P^i{}_{jk} = \frac{\partial N^i_j}{\partial y^k} - L^i{}_{kj}, \tag{2.13}$$

and $R^i{}_{jk}$ is given in (2.6). The quantities $C^i{}_{jk}$ are the v-coefficients of the connection $C\Gamma(N)$.

Theorem 2.1. 1^0. *For every generalized Lagrange space*

$$GL^n = (M, g_{ij}(x,y))$$

there is only one linear N*-connection* $C\Gamma(N) = \left(L^i_{jk}, C^i_{jk}\right)$ *which satisfies the following axioms*

1^0. N *is a nonlinear connection a priori given;*
2^0. $g_{ij|k} = 0$, *(*$C\Gamma(N)$ *is a h-metric);*
3^0. $g_{ij}\big|_k = 0$, *(*$C\Gamma(N)$ *is a v-metric);*
4^0. $T^h{}_{ij} = 0$, *(*$C\Gamma(N)$ *is h-symmetric);*
5^0. $S^h{}_{ij} = 0$, *(*$C\Gamma(N)$ *is v-symmetric).*

2^0. *The coefficients of connection* $C\Gamma(N)$ *in* 1^0 *are given by the generalized Christoffel symbols:*

$$\begin{aligned} L^i{}_{jk} &= \frac{1}{2} g^{is} \left(\frac{\delta g_{sj}}{\delta x^k} + \frac{\delta g_{ks}}{\delta x^j} - \frac{\delta g_{jk}}{\delta x^s} \right), \\ C^i{}_{jk} &= \frac{1}{2} g^{is} \left(\frac{\partial g_{sj}}{\partial y^k} + \frac{\partial g_{ks}}{\partial y^j} - \frac{\partial g_{jk}}{\partial y^s} \right). \end{aligned} \tag{2.14}$$

The tensors of deflection associated to the connection $C\Gamma(N)$ *are defined by:*

$$\begin{cases} D^i{}_j = y^i{}_{|j} = -N^i_j + y^s L^i{}_{sj}, \\ d^i{}_j = y^i\big|_j = \delta^i{}_j + y^s C^i{}_{sj}, \end{cases} \tag{2.15}$$

where y^i *is the Liouville vector field on* $\widetilde{TM}$*, i.e.* $y^i \dfrac{\partial}{\partial y^i}$.

We will use the connection $C\Gamma(N)$, which is named the metrical canonical N-connection of the generalized Lagrange space GL^n, to study the deformed Minkowski metric (1.2).

3 Energy-Dependent Metric of Minkowski Space

Le us consider the deformed Minkowski metric (1.2)–(1.3):

$$d\check{s}^2 = \check{\eta}_{ij}(E)\, dx^i dx^j, \qquad (i, j, \ldots = 0, 1, 2, 3;\ x^0 = t), \tag{3.1}$$

where the d-tensor $\check{\eta}_{ij}(E)$ is given by the matrix:

$$\check{\eta}_{ij}(E) = \begin{pmatrix} b_0^2(E)\, c^2 & 0 & 0 & 0 \\ 0 & -b_1^2(E) & 0 & 0 \\ 0 & 0 & -b_2^2(E) & 0 \\ 0 & 0 & 0 & -b_3^2(E) \end{pmatrix}. \tag{3.2}$$

The functions $b_i(E)$ are positive and the variable E is the energy of the considered process:

$$\begin{aligned} E &= \eta_{ij} \frac{dx^i}{dt} \frac{dx^j}{dt} = \\ &= c^2 \left(\frac{dx^0}{dt}\right)^2 - \left\{ \left(\frac{dx^1}{dt}\right)^2 + \left(\frac{dx^2}{dt}\right)^2 + \left(\frac{dx^3}{dt}\right)^2 \right\}. \end{aligned} \tag{3.3}$$

It is supposed that the metric (3.1) is locally defined in the region of the space-time where the process occurs.

The derivative $\left(\dfrac{dx^i}{dt}\right)$ defines a contravariant vector tangent to the manifold M in the point x. We will denote this vector by $y = \left(y^i\right)$. Then, (x, y) is a point of the tangent bundle TM. Now, let us consider the generalized Lagrange space $GL^n = (M, g_{ij}(x, y))$ where

$$\begin{cases} g_{ij}(x, y) = \check{\eta}_{ij}\big(E(x, y)\big), \\ E(x, y) = \eta_{ij}\, y^i y^j. \end{cases} \tag{3.4}$$

Then, we can prove the following theorem:

Theorem 3.1. *The pair* $GL^n = (M, g_{ij}(x, y))$*, with* g_{ij} *given by (3.4) is a generalized Lagrange space which is not reducibe to a Riemann space, or to a Finsler space, or to a Lagrange space.*

To prove the theorem let us observe that g_{ij} in (3.4) depend essentially of y^i and therefore GL^n is not a Riemann space. In order to show that GL^n is not reducible to a Lagrange space, we must prove that the d-tensor field $\dfrac{\partial g_{ij}}{\partial y^k}$ is not totally symmetric, i.e. the equation

$$\frac{\partial g_{ij}}{\partial y^k} = \frac{\partial g_{ik}}{\partial y^i}, \tag{3.5}$$

does not holds. We assume, by absurdum, that the equality (3.5) is verified. Then, for $i = j \neq k$ we have

$$\frac{\partial g_{ij}}{\partial y^k} = 0, \qquad i \neq k. \tag{3.6}$$

Now, using (3.4), the equation (3.6) becomes:

$$\frac{\partial \check{\eta}_{ii}}{\partial E}\,\frac{\partial E}{\partial y^k} = 0, \qquad \forall i, k;\ i \neq k. \tag{3.7}$$

This result implies the property $\dfrac{\partial E}{\partial y^k} = 0$ which is impossible. Therefore, the theorem is proved.

The following two particular cases are interesting:
1^0. The space GL^n has spatial symmetry:

$$b_1(E) = b_2(E) = b_3(E).$$

2^0. The space is local conform with Minkowski spaces:

$$b_0(E) = b_1(E) = b_2(E) = b_3(E).$$

This is a particular case of the Miron-Tavakol metric [4] which satisfy the Ehlers-Pirani-Schild axiomatics of Special Relativity.

Another important result is:

Theorem 3.2. *The generalized Lagrange space GL^n endowed with the fundamental metric tensor (3.4) is a local Minkowski space.*

Indeed [5], the fundamental tensor $g_{ij}(x, y)$ do not depend on the variables x^i (in the existent chart) because $\dfrac{\partial E}{\partial x^i} = 0$, q.e.d.

Therefore, we have

$$\frac{\partial g_{ij}}{\partial x^k} = 0, \qquad \frac{\partial E}{\partial x^k} = 0. \tag{3.8}$$

Let us consider then the Minkowski space M endowed with the fundamental tensor η_{ij} and denote by $F_{ij}(x)$ its electromagnetic tensor. We will put

$$\check{F}^i{}_j(x, y) = \check{\eta}^{is}\, F_{sj}(x), \qquad \forall x \in M, \tag{3.9}$$

where

$$F_{ij}(x) = \frac{\partial A_j}{\partial x^i} - \frac{\partial A_i}{\partial x^j}, \tag{3.10}$$

with $A_i(x)$–the potentials of the electromagnetic field. Therefore $\check{F}^i{}_j(x,y)$ from (3.9) is a d-tensor of type $(1,1)$, called the electromagnetic tensor of the generalized Lagrange space GL^n.

Consequently, in GL^n, we can a priori give the nonlinear connection N with local coefficients:

$$N^i{}_j = \left\{ {i \atop jk} \right\} y^k - \check{F}^i{}_j(x,y), \tag{3.11}$$

where $\left\{ {i \atop jk} \right\}$ are the Christoffel symbols of the Riemannian metric tensor η_{ij}. That means $\left\{ {i \atop jk} \right\} = 0$.

Therefore, we have to consider the non-linear connection N in the generalized Lagrange space $GL^n = (M, \check{\eta}_{ij})$ having the local coefficients:

$$N^i{}_j(x,y) = -\check{F}^i{}_j(x,y). \tag{3.12}$$

The addapted basis of the distribution N, $\left\{ \frac{\delta}{\delta x^i} \right\}$, $i = 0,1,2,3$, has the form:

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} + \check{F}^i{}_j(x,y) \frac{\partial}{\partial y^j}. \tag{3.13}$$

The basis $\left(dx^i, \delta y^i\right)$, which is dual to the basis $\left(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i}\right)$, has the local covector fields δy^i given by:

$$\delta y^i = dy^i - \check{F}^i{}_j(x,y)\, dx^j. \tag{3.14}$$

The integrability tensor $R^i{}_{jk}$ of the non-linear connection N, defined by the equation (2.6), is determined by the following proposition:

Proposition 3.1. *The tensor $R^i{}_{jk}$ of the non-linear connection (3.12) has the expression:*

$$R^i{}_{jk} = \frac{\delta \check{F}^i{}_k}{\delta x^j} - \frac{\delta \check{F}^i{}_j}{\delta x^k}. \tag{3.15}$$

4 The Canonical Metric N-Connection of the Space GL^n

The generalized Lagrange space $GL^n = (M, g_{ij})$ studied in this paper has the fundamental tensor $g_{ij}(x,y) = g_{ij}(y)$:

$$g_{ij}(y) = \check{\eta}_{ij}\big(E(y)\big), \qquad E(y) = \eta_{ij}\, y^i y^j. \tag{4.1}$$

We consider then the contravariant components:

$$g^{ij}(y) = \check{\eta}^{ij}\big(E(y)\big), \tag{4.2}$$

where the matrix $\check{\eta}^{ij}$ is of the form:

$$\check{\eta}^{ij}\left(E(y)\right)=\begin{pmatrix} b_0{}^{-2}(E)/c^2 & 0 & 0 & 0\\ 0 & -b_1{}^{-2}(E) & 0 & 0\\ 0 & 0 & -b_2{}^{-2}(E) & 0\\ 0 & 0 & 0 & -b_3{}^{-2}(E)\end{pmatrix}. \tag{4.3}$$

The derivation operators $\dfrac{\delta}{\delta x^i}$ and $\dfrac{\partial}{\partial y^i}$ applied to the components $g^{ij}(y)$ give the results:

$$\frac{\delta g_{ij}}{\delta x^k}=\frac{\partial\check{\eta}_{ij}}{\partial x^k}+\check{F}^r{}_k\,\frac{\partial\check{\eta}_{ij}}{\partial y^r}=\check{F}^r{}_k\,\frac{\partial\check{\eta}_{ij}}{\partial E}\,\frac{\partial E}{\partial y^r}, \tag{4.4}$$

$$\frac{\partial g_{ij}}{\partial y^k}=\frac{\partial\check{\eta}_{ij}}{\partial E}\,\frac{\partial E}{\partial y^k}. \tag{4.5}$$

Then, the coefficients of the canonical metric connection $C\Gamma(N)$, defined by (2.14), can be written under the form:

$$\begin{cases} L^i{}_{jk}=\dfrac{1}{2}\,\check{\eta}^{is}\,\dfrac{\partial E}{\partial y^r}\left(\dfrac{\partial\check{\eta}_{sj}}{\partial E}\,\check{F}^r_k+\dfrac{\partial\check{\eta}_{sk}}{\partial E}\,\check{F}^r_j-\dfrac{\partial\check{\eta}_{jk}}{\partial E}\,\check{F}^r_s\right),\\[2ex] C^i{}_{jk}=\dfrac{1}{2}\,\check{\eta}^{is}\,\dfrac{\partial E}{\partial y^r}\left(\dfrac{\partial\check{\eta}_{sj}}{\partial E}\,\delta^r_k+\dfrac{\partial\check{\eta}_{sk}}{\partial E}\,\delta^r_j-\dfrac{\partial\check{\eta}_{jk}}{\partial E}\,\delta^r_s\right),\end{cases} \tag{4.6}$$

As we can see, the vanishing of the electromagnetic field tensor, i.e. $F^i{}_j=0$, implies $L^i{}_{jk}=0$. In addition, we used the expression

$$\frac{\partial E}{\partial y^s}=2\eta_{sj}\,y^j, \tag{4.7}$$

in order to obtain the last equalities in (4.4) and (4.5). Of course, we have $g_{ij|k}=0$, $g_{ij}\big|_k=0, T^i_{jk}=0$, $S^i_{jk}=0$.

Now, we can calculate the deflection tensors $D^i{}_j$ and $d^i{}_j$ defined by the equations (2.15):

$$\begin{cases} D^i{}_j=y^i{}_{|j}=\dfrac{\delta y^i}{\delta x^j}+y^s\,L^i{}_{sj}=\check{F}^i_j+y^s\,L^i_{sj},\\ d^i{}_j=y^i\big|_j=\delta^i_j+y^s\,C^i_{sj}.\end{cases} \tag{4.8}$$

The covariant components of these tensors are:

$$\begin{cases} D_{ij}=g_{ir}\,D^r{}_j=\check{\eta}_{ir}\left(\check{F}^r_j+y^s\,L^r_{sj}\right)=\\ \qquad=F_{ij}(x)+\dfrac{1}{2}\,y^s\,\dfrac{\partial E}{\partial y^p}\left(\dfrac{\partial\check{\eta}_{is}}{\partial E}\,\check{F}^p{}_j+\dfrac{\partial\check{\eta}_{ij}}{\partial E}\,\check{F}^p{}_s-\dfrac{\partial\check{\eta}_{sj}}{\partial E}\,\check{F}^p{}_i\right),\\[2ex] d_{ij}=g_{ir}\,d^r{}_j=\check{\eta}_{ij}+\dfrac{1}{2}\,y^s\,\dfrac{\partial E}{\partial y^p}\left(\dfrac{\partial\check{\eta}_{is}}{\partial E}\,\delta^p_j+\dfrac{\partial\check{\eta}_{ij}}{\partial E}\,\delta^p_s-\dfrac{\partial\check{\eta}_{sj}}{\partial E}\,\delta^p_i\right).\end{cases} \tag{4.9}$$

Let $\check{F}_{ij}$ be the covariant components of the electromagnetic tensor with respect to the energy-dependent metric $\check{\eta}_{ir}$:

$$\check{F}_{ij} = \check{\eta}_{ir}\, F^r_j = \check{\eta}_{ir}\, \eta^{rs}\, F_{sj}, \tag{4.10}$$

where

$$(\check{\eta}_{ir}\, \eta^{rs}) = \begin{pmatrix} {b_0}^2 & 0 & 0 & 0 \\ 0 & {b_1}^2 & 0 & 0 \\ 0 & 0 & b_2^2 & 0 \\ 0 & 0 & 0 & {b_3}^2 \end{pmatrix} = \left({b_i}^2 \delta^i_s\right). \tag{4.11}$$

Therefore, we have

$$\check{F}_{0j} = {b_0}^2 F_{0j}, \quad \check{F}_{1j} = {b_1}^2 F_{1j}, \quad \check{F}_{2j} = {b_2}^2 F_{2j}, \quad \check{F}_{3j} = {b_3}^2 F_{3j},$$

or, equivalently

$$\check{F}_{ij} = b_i^2 F_{ij}, \qquad i = 0, 1, 2, 3. \tag{4.12}$$

As a consequence, the tensor $\check{F}_{ij}$ is not antisymmetric, that is

$$\check{F}_{ij} \neq -\check{F}_{ji}. \tag{4.13}$$

We define now the h-electromagnetic internal tensor on the space GL^n by the relation:

$$\mathcal{F}_{ij} = \frac{1}{2}\left(D_{ij} - D_{ji}\right) = F_{ij}(x) + \frac{1}{2}\, y^s\, \frac{\partial E}{\partial y^p}\left(\frac{\partial \check{\eta}_{is}}{\partial E}\, \check{F}^p_j - \frac{\partial \check{\eta}_{js}}{\partial E}\, \check{F}^p_i\right). \tag{4.14}$$

Analogous, the v-electromagnetic internal tensor on the space GL^n has the components

$$f_{ij} = \frac{1}{2}\left(d_{ij} - d_{ji}\right) = \frac{1}{2}\, y^s\, \frac{\partial E}{\partial y^p}\left(\frac{\partial \check{\eta}_{is}}{\partial E}\, \delta^p_j - \frac{\partial \check{\eta}_{js}}{\partial E}\, \delta^p_i\right). \tag{4.15}$$

If we use (4.12), then the components of the tensor $\mathcal{F}$ (horizontal) and f (vertical) defined in the equations (4.14) and (4.15) become:

$$\begin{cases} \mathcal{F}_{ij} = \dfrac{1}{2}\left(b_i^2 + b_j^2\right) F_{ij} + \left(\dfrac{\partial \check{\eta}_{ik}}{\partial E}\, F_{rj} - \dfrac{\partial \check{\eta}_{jk}}{\partial E}\, F_{ri}\right) y^r y^k, \\[2ex] f_{ij} = \left(\dfrac{\partial \check{\eta}_{ik}}{\partial E}\, \eta_{rj} - \dfrac{\partial \check{\eta}_{jk}}{\partial E}\, \eta_{ri}\right) y^r y^k. \end{cases} \tag{4.16}$$

Having these components, we can write the corresponding Maxwell equations. First of all, let us observe that applying the Ricci identities to the Liouville vector field y^i and taking into account the expressions of the deflection tensors $D^i_j = {y^i}_{|j}$, ${d^i}_j = {y^i}\big|_j$, we obtain [7]:

$$\begin{aligned} {D^i}_{j|k} - {D^i}_{k|j} &= y^m {R_m}^i{}_{jk} - d^i_m {R^m}_{jk}, \\ {D^i}_j\big|_k - {d^i}_{k|j} &= y^m {P_m}^i{}_{jk} - D^i_m C^m_{jk} - d^i_m {P^m}_{jk}, \\ {d^i}_j\big|_k - {d^i}_k\big|_j &= y^m {S_m}^i{}_{jk}. \end{aligned}$$

If we multiply these equations by g_{hi} and take the sum over i (contraction), then we have:

$$\begin{cases} D_{ij|k} - D_{ik|j} = y^m R_{mijk} - d_{im} R^m{}_{jk}, \\ D_{ij}\big|_k - d_{ik|j} = y^m P_{mijk} - D_{im}\, C^m{}_{jk} - d_{im}\, P^m{}_{jk}, \\ d_{ij}\big|_k - d_{ik}\big|_j = y^m S_{mijk}. \end{cases} \tag{4.17}$$

Considering the cyclic permutations of the first equation (4.17) and adding the results, we deduce:

$$\begin{aligned} 2\left(\mathcal{F}_{ij|k} + \mathcal{F}_{jk|i} + \mathcal{F}_{ki|j}\right) &= y^m\left(R_{mijk} + R_{mjki} + R_{mkij}\right) - \\ &- \left(d_{im} R^m{}_{jk} + d_{jm} R^m{}_{ki} + d_{km} R^m{}_{ij}\right). \end{aligned} \tag{4.18}$$

Analogous, from the second and third equations (4.17), we obtain

$$\begin{aligned} &2\big(\mathcal{F}_{ij}\big|_k + \mathcal{F}_{jk}\big|_i + \mathcal{F}_{ki}\big|_j\big) - 2\left(f_{ij|k} + f_{kj|i} + f_{ji|k}\right) = \\ &= y^m\left(P_{mijk} - P_{mjik} + \text{c.p.}\right) - \left(D_{im} C^m_{jk} - D_{jm} C^m_{ik} + \text{c.p.}\right) - \\ &- \left(d_{im} P^m_{jk} - d_{jm} P^m_{ik} + \text{c.p.}\right), \end{aligned} \tag{4.19}$$

and, respectively:

$$2\big(f_{ij}\big|_k + f_{jk}\big|_i + f_{ki}\big|_j\big) = y^m\left(S_{mijk} + S_{mjki} + S_{mkij}\right). \tag{4.20}$$

The equations (4.18), (4.19) and (4.20) are the *generalized Maxwell equations* satisfied by the internal electromagnetic fields $\mathcal{F}_{ij}$ and f_{ij}.

We can write these equations in a more usual form if we consider the Bianchi identities satisfied by the canonical metric connection and take into account the vanishing of the tensors $T^i{}_{jk}$ and $S^i{}_{jk}$. The relevant Bianchi identities are:

$$\begin{cases} R_j{}^i{}_{kh} + \text{c.p.} - \left(R^m{}_{jk} C^i_{mh} + \text{c.p.}\right) = 0, \\ S_j{}^i{}_{kh} + \text{c.p.} = 0, \\ C_j{}^i{}_{s|k} - C_k{}^i{}_{s|j} + P^m_{js} C^i_{km} - P^m_{ks} C^i_{jm} = -\left(P_j{}^i{}_{ks} - P_k{}^i{}_{js}\right). \end{cases} \tag{4.21}$$

But, the propreties $g_{ij|k} = 0$, $g_{ij}\big|_k = 0$ and Ricci identities give the following relations:

$$R_{ijkh} + R_{jikh} = 0, \quad P_{ijkh} + P_{jikh} = 0, \quad S_{ijkh} + S_{jikh} = 0.$$

Then, we can transform the first equation in (4.21) as follows:

$$R_{jikh} + R_{kihj} + R_{hijk} = R^m{}_{jk} C_{mih} + R^m{}_{kh} C_{mij} + R^m{}_{hj} C_{mik}.$$

Contracting this equation by y^i we obtain:

$$-y^m\left(R_{mjkh} + R_{mkhj} + R_{mhjk}\right) = y^i\left(R^m{}_{jk} C_{mih} + R^m{}_{kh} C_{mij} + R^m{}_{hj} C_{mik}\right).$$

Finally, using this result, the first generalized Maxwell equation (4.18) becomes:

$$2\left(\mathcal{F}_{ij|k}+\mathcal{F}_{jk|i}+\mathcal{F}_{ki|j}\right)=y^m\left(R^s{}_{ij}C_{smk}+R^s{}_{jk}C_{smi}+R^s{}_{ki}C_{smj}\right), \tag{4.22}$$

where $R^i{}_{jk}=\eta^{is}\dfrac{\partial F_{jk}}{\partial x^s}$. The third generalized Maxwell equation (4.20) can be written under the form [using (4.21)]:

$$f_{ij}\Big|_k+f_{jk}\Big|_i+f_{ki}\Big|_j=0. \tag{4.23}$$

Finally, we transform the second generalized Maxwell equation (4.19) as follows. First, we write the last identity in (4.21) in the form:

$$C_{jis|k}-C_{kis|j}+P^m{}_{ks}C_{jim}=P_{ijks}-P_{ikjs}. \tag{4.24}$$

Then, contracting (4.24) with y^i we have:

$$y^i\left(C_{jis|k}-C_{kis|j}\right)+y^i\left(P^m_{js}C_{kim}-P^m_{ks}C_{jim}\right)=y^m\left(P_{mjks}-P_{mkjs}\right), \tag{4.25}$$

or, equivalently (changing i with m and s with i):

$$\begin{aligned}y^m\big(C_{jmi|k}-C_{kmi}\big|_j\big)+y^m\left(P^s_{ji}C_{kms}-P^s_{ki}C_{jms}\right)=\\ =y^m\left(P_{mjki}-P_{mkij}\right).\end{aligned} \tag{4.26}$$

Therefore, we obtain:

$$\begin{aligned}y^m\left(P_{mijk}-P_{mjik}+\text{c.p.}\right)=y^m\left(C_{jmi|k}-C_{imj|k}+\text{c.p.}\right)+\\ +y^m\left(P^s_{ji}C_{kms}-P^s_{ki}C_{jms}+\text{c.p.}\right).\end{aligned} \tag{4.27}$$

Now, introducing (4.27) in (4.19) we have:

$$\begin{aligned}&2\big(\mathcal{F}_{ij}\big|_k+\mathcal{F}_{jk}\big|_i+\mathcal{F}_{ki}\big|_j\big)-2\left(f_{ij|k}+f_{jk|i}+f_{ki|j}\right)=\\ &=y^m\left(C_{jmi|k}-C_{imj|k}+\text{c.p.}\right)+y^m\left(P^s_{ji}C_{kms}-P^s_{ij}C_{kms}+\text{c.p.}\right)-\\ &-\left(D_{im}C^m_{jk}-D_{jm}C^m_{ik}+\text{c.p.}\right)-\left(d_{im}P^m_{jk}-d_{jm}P^m_{ik}+\text{c.p.}\right).\end{aligned} \tag{4.28}$$

On the other hand, using the expressions:

$$\begin{cases}P^i_{jk}=\dfrac{\partial N^i_j}{\partial y^k}-L^i_{kj}=-L^i_{kj}=-L_j{}^i{}_k,\\ C_{imj}=C_{jmi},\quad C_j{}^i{}_k=C^i_{kj},\end{cases} \tag{4.29}$$

we obtain:

$$\begin{cases}D_{im}C^m_{jk}-D_{jm}C^m_{ik}+\text{c.p.}=0,\\ d_{im}P^m_{jk}-d_{jm}P^m_{ik}+\text{c.p.}=0,\\ C_{jmi|k}-C_{imj|k}+\text{c.p.}=0,\\ P^s_{ji}C_{kms}-P^s_{ij}C_{kms}+\text{c.p.}=0,\end{cases} \tag{4.30}$$

Introducing (4.30) in (4.28) we can write the second generalized Maxwell equation in the form:

$$\mathcal{F}_{ij}\Big|_k + \mathcal{F}_{jk}\Big|_i + \mathcal{F}_{ki}\Big|_j = f_{ij|k} + f_{jk|i} + f_{ki|j}. \tag{4.31}$$

Different authors [8, 9] provided a geometric description of the interactions between particles or fields, in the sense that each interaction "produces" its own metric. In particular, using the geometrical propreties of the generalized Lagrange spaces endowed with energy-dependent metrics, we can study the electromagnetic interaction and, possible, other types of interactions. We can also obtain solutions for the generalized Maxwell equations and establish a geometrical significance of the energy.

5 An example of energy-dependent metric

We consider, as a simple example, the metric

$$ds^2 = a(E)dt^2 - [(dx^1)^2 + (dx^2)^2 + (dx^3)^2], \tag{5.1}$$

where $a(E)$ is an arbitrary function of the energy E, and the units $c = 1$ are understood. Then, the non-vanishing coefficients C^i_{jk} of the canonical metric connection $C\Gamma(N)$ defined in (4.6) are:

$$\begin{aligned} &C^0_{00} = \frac{a'}{a}y^0,\ C^0_{01} = -\frac{a'}{a}y^1,\ C^0_{02} = -\frac{a'}{a}y^2,\ C^0_{03} = -\frac{a'}{a}y^3, \\ &C^1_{00} = -a'y^1,\ C^2_{00} = -a'y^2,\ C^3_{00} = -a'y^3, \end{aligned} \tag{5.2}$$

with $a' = \dfrac{da}{dE}$. We suppose also that there is no electromagnetic field in the model we considered.

The independent Einstein's equations corresponding to the metric (5.1) have the form:

$$a' = 0, \tag{5.3}$$

$$2aa'' - a'^2 = 0. \tag{5.4}$$

The equation (5.3) has the solution a = const. which reproduces the Minkows-ki metric, without energy-dependence. The equation (5.4) has the solution

$$a(E) = \frac{1}{4}\left(\alpha_0 + \frac{E}{E_0}\right)^2, \tag{5.5}$$

where α_0 and E_0 are two constants of integration.

It is important to remark that for $\alpha_0 = 0$ this solution coincides with those given in Ref. 5 for the strong interactions: $a(E) = \dfrac{1}{4}\left(\dfrac{E}{E_0}\right)^2$. On the other hand, if we choose $\alpha_0 = 1$, then we obtain

$$a(E) = \frac{1}{4}\left(1 + \frac{E}{E_0}\right)^2, \tag{5.6}$$

which is the solution given in Ref. 5 for the gravitational interaction.

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