# ON THE PARALLEL PLANES IN VECTOR BUNDLES

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**Abstract**

In this paper are found the necessary and sufficient conditions of local existence of parallel $r$-plane or which is equivalent with existence of parallel $r$-forms. The existence of such parallel $r$-planes ($r$-forms) depend on the initial conditions, which in this paper reduce to a homogeneous system of linear equations. If these integrability conditions are satisfied, then the parallel $r$-form is represented as a convergent functional series which contains inside only covariant derivatives.

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**Key words:** parallel planes, parallel forms, vector bundles

## 1 Introduction

This paper is a continuation of the papers [1–8] which are due to the solving of ordinary and partial differential equations and their application in the differential geometry.

In the paper [1] it was found a formula for the $k$-th covariant derivative. Further that formula was generalized for $k \in R$. Specially, if $k = -1$ it yields to a general solution for a system of linear differential equations [2]. In the paper [3] two main results are proven. It is found the general solution of an arbitrary system of linear differential equations of order $k$, and it is found the general solution of an arbitrary non-linear system of differential equations of the first order. The previous results are applied to the system of Frenet equations of curves [4]. In [5] it is considered linear and non-linear systems of partial differential equations. Indeed the compatibility conditions are found, and if they are satisfied, the solutions are found. All the solutions for both ordinary and partial differential equations are given as functional series. The results in [5] are used in [6] for studying the non-linear connections. In the paper [7] are found the compatibility conditions for the existence of parallel vector field in a space with linear, non-linear and $d$-connection, and if they are satisfied, the solution

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is found. Note that the compatibility conditions depend on the choice of the initial conditions for the parallel vector field. In [8] are made some further generalizations of the results in [7].

In this paper we consider the existence of parallel $r$-plane with given initial plane at one point. The existence of such plane depends on the initial plane and if such field of planes exists, in this paper it is found.

## 2 On the parallel $r$-planes

In this section we give some preliminaries concerning the $r$-planes. Let us consider the space $R^n$ and we denote by $\mathbf{e}_1, \cdots, \mathbf{e}_n$ the standard basis of $R^n$. The space of $r$-forms consists of terms of form

$$\omega = \sum_{i_1,\cdots,i_r} a_{i_1 i_2 \cdots i_r} \mathbf{e}_{i_1} \wedge \mathbf{e}_{i_2} \wedge \cdots \wedge \mathbf{e}_{i_r}$$

which is a vector space of dimension $\binom{n}{r}$, with the basis $\{\mathbf{e}_{i_1} \wedge \mathbf{e}_{i_2} \wedge \cdots \wedge \mathbf{e}_{i_r}\}$ for $1 \leq i_1 < i_2 < \cdots < i_r \leq n$. Note that any $r$-dimensional subspace $\Pi$ generates unique $r$-form $w$ up to a scalar multiplier as follows. If $Y_{(1)}, \cdots, Y_{(r)}$ is a basis of $\Pi$, then $w = Y_{(1)} \wedge \cdots \wedge Y_{(r)}$ is uniquely determined up to scalar multiplier. The converse is not true, and thus in this paper we will consider only those $r$-forms which can be obtained in the previous way. Indeed, for such an $r$-form $w$ we determine an $r$-dimensional vector subspace ($r$-plane) $< \omega >$ by $X \in < \omega >$ if and only if $\omega \wedge X = 0$. Although in section 3 the theorem 3.1 considers this special type of $r$-forms, it easily can be generalized for arbitrary $r$-forms.

If $Y_{(i)} = \sum\limits_{j=1}^{n} Y^j_{(i)} \mathbf{e}_j$, then

$$\omega = \lambda \cdot \sum_{i_1,\cdots,i_r=1}^{n} Y^{i_1}_{(1)} Y^{i_2}_{(2)} \cdots Y^{i_r}_{(r)} \mathbf{e}_{i_1} \wedge \mathbf{e}_{i_2} \wedge \cdots \wedge \mathbf{e}_{i_r}.$$

Thus we convenient to denote the $r$-form $Y_{(1)} \wedge \cdots \wedge Y_{(r)}$ as antisymmetric contravariant tensor $Y^{i_1 i_2 \cdots i_r}$ such that

$$\frac{1}{r!} \sum_{i_1,\cdots,i_r=1}^{n} Y^{i_1 \cdots i_r} \mathbf{e}_{i_1} \wedge \mathbf{e}_{i_2} \wedge \cdots \wedge \mathbf{e}_{i_r} = Y_{(1)} \wedge \cdots \wedge Y_{(r)}.$$

Let $\xi = (\mathcal{E}, \pi, M)$ be a vector bundle of class $C^\omega$ of rank $n$ on a $k$-dimensional differentiable manifold $M$ of class $C^\omega$. Suppose that the vector bundle is endowed with a linear connection $\Gamma$ of class $C^\omega$. In order the covariant derivation of a mixed tensor $X^i_j (i \in \{1, \cdots, n\}, j \in \{1, \cdots, k\})$ to be defined, we assume that the base manifold $M$ is endowed with a linear connection $\Gamma^i_{jr}$ $(1 \leq i, j, r \leq k)$, of class $C^\omega$. We will consider the existence of parallel $r$-plane in the bundle. Suppose that the $r$-plane is generated by the following $r$ vector fields $Y_{(1)}, Y_{(2)}, \cdots, Y_{(r)}$, i.e. the $r$-plane is generated by the following $r$-form $Y_{(1)} \wedge Y_{(2)} \wedge \cdots \wedge Y_{(r)}$.

**Definition.** The field of $r$-planes $\Pi$ (or simply the $r$-plane) on the considered vector bundle is *parallel* if the parallel displacement of any vector from the vector bundle along any curve on the manifold preserves the incidence relation of a vector in the field of $r$-planes.

Note that an $r$-form is parallel if it is parallel as antisymmetric tensor field. Now we prove the following theorem.

**Theorem 2.1.** *An $r$-plane $\Pi$ is parallel, if and only if there exists a parallel $r$-form which generates the $r$-plane $\Pi$.*

*Proof.* Suppose that $Y_{(1)} \wedge \cdots \wedge Y_{(r)}$ is a parallel $r$-form which generates the $r$-plane $\Pi$. In order to prove that the $r$-plane $\Pi$ is parallel, we should prove that if $X$ is a parallel vector field over a curve $C$ over the manifold and $(Y_{(1)} \wedge \cdots \wedge Y_{(r)}) \wedge X = 0$ at one point of the curve, then $Y_{(1)} \wedge \cdots \wedge Y_{(r)} \wedge X = 0$ along the curve. Let $U$ be the tangent vector of the curve $C$. Then

$$\nabla_U(Y_{(1)} \wedge \cdots \wedge Y_{(r)} \wedge X) = 0$$

because $\nabla_U(Y_{(1)} \wedge \cdots \wedge Y_{(r)}) = 0$ and $\nabla_U(X) = 0$. Thus $Y_{(1)} \wedge \cdots \wedge Y_{(r)} \wedge X$ is a zero $(r+1)$-form along the curve $C$ and hence the $r$-plane $\Pi$ is also parallel.

Conversely, suppose that an $r$-plane $\Pi$ is parallel. Locally, there exist $r$ vector fields $Y_{(1)}, \cdots, Y_{(r)}$ such that $Y_{(1)} \wedge \cdots \wedge Y_{(r)}$ generates the $r$-plane $\Pi$. Since $\Pi$ is parallel, the covariant derivative of any vector $Y_{(i)}$ in any direction must be linear combination of $Y_{(1)}, \cdots, Y_{(r)}$. Thus for any direction $U$ there exists a scalar $\lambda$ such that

$$\nabla_U(Y_{(1)} \wedge \cdots \wedge Y_{(r)}) = \lambda \cdot Y_{(1)} \wedge \cdots \wedge Y_{(r)}.$$

We should prove that locally there exists a scalar function $\lambda$, such that

$$\lambda \cdot Y_{(1)} \wedge \cdots \wedge Y_{(r)}$$

is a parallel $r$-form. Let us choose a local coordinate system $(x^1, \cdots, x^k)$ on the manifold and let $U_i = \partial/\partial x^i$. If $\lambda \cdot Y_{(1)} \wedge \cdots \wedge Y_{(r)}$ is a parallel tensor field, then

$$(\nabla_{U_i}\lambda) \cdot (Y_{(1)} \wedge \cdots \wedge Y_{(r)}) + \lambda \cdot \nabla_{U_i}(Y_{(1)} \wedge \cdots \wedge Y_{(r)}) = 0.$$

Hence $\lambda$ can be found as a solution of equation of the form

$$\frac{\partial}{\partial x^i}\lambda = F_i(x^1, \cdots, x^k, \lambda), \quad (1 \le i \le k),$$

such that

$$F_i(x^1, \cdots, x^k, \lambda)(Y_{(1)} \wedge \cdots \wedge Y_{(r)}) = -\lambda \nabla_{U_i}(Y_{(1)} \wedge \cdots \wedge Y_{(r)}). \tag{2.1}$$

Thus we have to verify only the compatibility conditions.

According to (2.1) we obtain

$$\begin{aligned}(\nabla_{U_j} F_i(x^1, \cdots, x^k, \lambda))(Y_{(1)} \wedge \cdots \wedge Y_{(r)}) + F_i(x^1, \cdots, x^k, \lambda)\nabla_{U_j}(Y_{(1)} \wedge \cdots \wedge Y_{(r)}) \\ = -(\nabla_{U_j}\lambda)(\nabla_{U_i}(Y_{(1)} \wedge \cdots \wedge Y_{(r)}) - \lambda\nabla_{U_j}\nabla_{U_i}(Y_{(1)} \wedge \cdots \wedge Y_{(r)}).\end{aligned}$$

Multiplying this equality by $\lambda$ and using (2.1), we obtain

$$\lambda(\nabla_{U_j}F_i(x^1,\cdots,x^k,\lambda))(Y_{(1)}\wedge\cdots\wedge Y_{(r)})$$
$$-F_i(x^1,\cdots,x^k,\lambda)F_j(x^1,\cdots,x^k,\lambda)(Y_{(1)}\wedge\cdots\wedge Y_{(r)})$$
$$=F_j(x^1,\cdots,x^k,\lambda)F_i(x^1,\cdots,x^k,\lambda)(Y_{(1)}\wedge\cdots\wedge Y_{(r)})-\lambda^2\nabla_{U_j}\nabla_{U_i}(Y_{(1)}\wedge\cdots\wedge Y_{(r)}).$$

By permuting the indices $i$ by $j$ we obtain

$$\lambda(\nabla_{U_i}F_j(x^1,\cdots,x^k,\lambda))(Y_{(1)}\wedge\cdots\wedge Y_{(r)})$$
$$-F_j(x^1,\cdots,x^k,\lambda)F_i(x^1,\cdots,x^k,\lambda)(Y_{(1)}\wedge\cdots\wedge Y_{(r)})$$
$$=F_i(x^1,\cdots,x^k,\lambda)F_j(x^1,\cdots,x^k,\lambda)(Y_{(1)}\wedge\cdots\wedge Y_{(r)})-\lambda^2\nabla_{U_i}\nabla_{U_j}(Y_{(1)}\wedge\cdots\wedge Y_{(r)}).$$

Using that $[U_i,U_j]=0$ and

$$R(U_i,U_j)(\lambda Y_{(1)}\wedge\cdots\wedge Y_{(r)})=0,\quad \text{i.e.}\quad R(U_i,U_j)(Y_{(1)}\wedge\cdots\wedge Y_{(r)})=0,$$

because the $r$-plane $\Pi$ is parallel, by subtracting the last two equalities we obtain that

$$\lambda\Big[\frac{\partial F_i(x^1,\cdots,x^k,\lambda)}{\partial x^j}-\frac{\partial F_j(x^1,\cdots,x^k,\lambda)}{\partial x^i}\Big](Y_{(1)}\wedge\cdots\wedge Y_{(r)})$$
$$=\lambda^2\Big(\nabla_{U_i}\nabla_{U_j}(Y_{(1)}\wedge\cdots\wedge Y_{(r)})-\nabla_{U_j}\nabla_{U_i}(Y_{(1)}\wedge\cdots\wedge Y_{(r)})\Big)$$
$$=R(U_i,U_j)(Y_{(1)}\wedge\cdots\wedge Y_{(r)})=0$$

and hence

$$\frac{\partial F_i(x^1,\cdots,x^k,\lambda)}{\partial x^j}=\frac{\partial F_j(x^1,\cdots,x^k,\lambda)}{\partial x^i}.$$

Thus the compatibility conditions of the system (2.1) are satisfied. ‖

*Remark.* Although this theorem is proven for linear connections, indeed it is true for any connection $\nabla$ such that

$$\nabla(X+Y)=\nabla X+\nabla Y, \tag{2.2}$$

because we used in the proof that

$$R(U_i,U_j)\lambda(Y_{(1)}\wedge\cdots\wedge Y_{(r)})=\lambda R(U_i,U_j)(Y_{(1)}\wedge\cdots\wedge Y_{(r)})$$

which is a consequence of the above linearity. Note that it does not mean that the connection must be linear, because it may happens that

$$\nabla_{X+Y}\neq\nabla_X+\nabla_Y,$$

and the theorem 2.1 is also true. Thus for the connections satisfying (2.2) the request for parallel $r$-planes reduces on finding parallel $r$-forms, while for the connections which do not satisfy (2.2) we can require only parallel $r$-forms.

# 3 Basic result

We use the same notations and assumptions as in the section 2. According to the theorem 2.1 we should consider the existence of parallel $r$-forms, if we want to consider the parallel $r$-planes in a vector bundle with linear connection. The main problem in this paper is to examine locally the existence of parallel $r$-form $\omega$, i.e.

$$\omega^{i_1\cdots i_r}_{;u} = 0 \quad \text{for} \quad (1 \le u \le k), \tag{3.1}$$

with the initial condition

$$\omega^{i_1\cdots i_r}(0,\cdots,0) = \omega_0^{i_1\cdots i_r}. \tag{3.2}$$

**Theorem 3.1.**

(i) *Locally there exists a parallel $r$-form $\omega$ with the given initial condition, i.e. (3.1) and (3.2) are satisfied if and only if the following system of linear equations*

$$\begin{gathered}
R^{i_1}_{juv}Y^{ji_2\cdots i_r} + R^{i_2}_{juv}Y^{i_1j\cdots i_r} + \cdots + R^{i_r}_{juv}Y^{i_1i_2\cdots j} = 0\\
R^{i_1}_{juv;w_1}Y^{ji_2\cdots i_r} + R^{i_2}_{juv;w_1}Y^{i_1j\cdots i_r} + \cdots + R^{i_r}_{juv;w_1}Y^{i_1i_2\cdots j} = 0\\
R^{i_1}_{juv;w_1;w_2}Y^{ji_2\cdots i_r} + R^{i_2}_{juv;w_1;w_2}Y^{i_1j\cdots i_r} + \cdots + R^{i_r}_{juv;w_1;w_2}Y^{i_1i_2\cdots j} = 0\\
\cdots\cdots\\
R^{i_1}_{juv;w_1;\cdots;w_{N-1}}Y^{ji_2\cdots i_r} + R^{i_2}_{juv;w_1;\cdots;w_{N-1}}Y^{i_1j\cdots i_r} + \cdots + R^{i_r}_{juv;w_1;\cdots;w_{N-1}}Y^{i_1i_2\cdots j} = 0
\end{gathered} \tag{3.3}$$

*for $i_1,\cdots,i_r \in \{1,\cdots,n\}$, $u,v,w_1,\cdots,w_{N-1} \in \{1,\cdots,k\}$, $N = \binom{n}{r}$ has an analytical solution $Y^{i_1i_2\cdots i_r}$, such that*

$$Y^{i_1i_2\cdots i_r}(0,\cdots,0) = \omega_0^{i_1i_2\cdots i_r}.$$

(ii) *Let the compatibility conditions from (i) be satisfied, and let $Y^{i_1i_2\cdots i_r}$ be an arbitrary analytical solution of the system* (3.3) *with the initial conditions* (3.2). *Then there exist functions*

$$Q^{i_1i_2\cdots i_r<v_1,\ldots,v_k>} \quad (1 \le i_1,\cdots,i_r \le n,\ v_1,\cdots,v_k \in \mathbf{N}_o)$$

*such that*

$$Q^{i_1i_2\cdots i_r<0,\ldots,0>} = Y^{i_1i_2\cdots i_r} \qquad (1 \le i_1,i_2,\cdots,i_r \le n), \tag{3.4}$$

$$\begin{gathered}
Q^{i_1i_2\cdots i_r<v_1,\ldots,v_u+1,\ldots,v_k>} = Q^{i_1i_2\cdots i_r<v_1,\ldots,v_k>}_{;u} \equiv\\
\equiv \frac{\partial}{\partial x^u}Q^{i_1i_2\cdots i_r<v_1,\ldots,v_k>} + \sum_{s=1}^{r}\Gamma^{i_s}_{ju}Q^{i_1\cdots i_{s-1}ji_{s+1}\cdots i_r<v_1,\ldots,v_k>},\\
(1 \le i_1,\cdots,i_r \le n, \quad 1 \le u \le k, \quad v_1,\cdots,v_k \in \mathbf{N}_o)
\end{gathered} \tag{3.5}$$

*and the solution of* (3.1) *and* (3.2) *is given by*

$$\omega^{i_1\cdots i_r}=\sum_{v_1=0}^{\infty}\sum_{v_2=0}^{\infty}\cdots\sum_{v_k=0}^{\infty}\frac{(-x^1)^{v_1}}{v_1!}\frac{(-x^2)^{v_2}}{v_2!}\cdots\frac{(-x^k)^{v_k}}{v_k!}\times \tag{3.6}$$

$$\times Q^{i_1\cdots i_r<v_1,\dots,v_k>}\qquad (1\le i_1,\cdots,i_r\le n).$$

*Proof.* (i) If the system (3.1) with (3.2) is integrable, then each solution $Y^{i_1i_2\cdots i_r}$ must satisfy

$$R^{i_1}_{juv}Y^{ji_2\cdots i_r}+R^{i_2}_{juv}Y^{i_1j\cdots i_r}+\cdots+R^{i_r}_{juv}Y^{i_1i_2\cdots j}=0.$$

Since $Y^{i_1i_2\cdots i_r}_{;w}=0$ it holds

$$R^{i_1}_{juv;w_1}Y^{ji_2\cdots i_r}+R^{i_2}_{juv;w_1}Y^{i_1j\cdots i_r}+\cdots+R^{i_r}_{juv;w_1}Y^{i_1i_2\cdots j}=0,$$

$$R^{i_1}_{juv;w_1;w_2}Y^{ji_2\cdots i_r}+R^{i_2}_{juv;w_1;w_2}Y^{i_1j\cdots i_r}+\cdots+R^{i_r}_{juv;w_1;w_2}Y^{i_1i_2\cdots j}=0,$$

$$\cdots\cdots$$

and hence the system (3.3) is satisfied.

Conversely, suppose the that system (3.3) has at least one analytical solution $Y^{i_1i_2\cdots i_r}$ such that $Y^{i_1i_2\cdots i_r}(0,\cdots,0)=\omega_0^{i_1i_2\cdots i_r}$. We will prove that the system (3.3) is equivalent to the following system of infinitely many equations

$$R^{i_1}_{juv}Y^{ji_2\cdots i_r}+R^{i_2}_{juv}Y^{i_1j\cdots i_r}+\cdots+R^{i_r}_{juv}Y^{i_1i_2\cdots j}=0.$$

$$R^{i_1}_{juv;w_1}Y^{ji_2\cdots i_r}+R^{i_2}_{juv;w_1}Y^{i_1j\cdots i_r}+\cdots+R^{i_r}_{juv;w_1}Y^{i_1i_2\cdots j}=0, \tag{3.7}$$

$$R^{i_1}_{juv;w_1;w_2}Y^{ji_2\cdots i_r}+R^{i_2}_{juv;w_1;w_2}Y^{i_1j\cdots i_r}+\cdots+R^{i_r}_{juv;w_1;w_2}Y^{i_1i_2\cdots j}=0,$$

$$\cdots\cdots$$

Since the rank$(\xi)=n$ and the bundle of $r$-forms on $\xi$ can be considered as vector bundle of dimension $N=\binom{n}{r}$, there exists number $s\le N-1$ such that if the first $s+1$ equalities of (3.7) are satisfied, then the $(s+2)$-nd equality is also true, i.e.

$$R^{i_1}_{juv;w_1;\cdots;w_{s+1}}Y^{ji_2\cdots i_r}+R^{i_2}_{juv;w_1;\cdots;w_{s+1}}Y^{i_1j\cdots i_r}+\cdots+R^{i_r}_{juv;w_1;\cdots;w_{s+1}}Y^{i_1i_2\cdots j}=0.$$

Using this implication for $Y^{i_1i_2\cdots i_r}_{;w_{s+2}}$, it is easy to see that if the first $s+1$ equalities of (3.7) are satisfied, then the $(s+3)$-rd equality is satisfied, i.e.

$$R^{i_1}_{juv;w_1;\cdots;w_{s+2}}Y^{ji_2\cdots i_r}+R^{i_2}_{juv;w_1;\cdots;w_{s+2}}Y^{i_1j\cdots i_r}+\cdots+R^{i_r}_{juv;w_1;\cdots;w_{s+2}}Y^{i_1i_2\cdots j}=0.$$

Continuing this process we obtain that the systems (3.3) and (3.7) are equivalent.

In order to prove that the system (3.1) is integrable, we prove (ii).

(ii) In order to prove that there exist functions

$$Q^{i_1i_2\cdots i_r<v_1,\dots,v_k>}\quad (1\le i_1,\cdots,i_r\le n,\ v_1,\cdots,v_k\in\mathbf{N}_o)$$

such that (3.4) and (3.5) are satisfied, we define $Q^{i_1\cdots i_r<0,\ldots,0>}$ by (3.4), and it is sufficient to prove that

$$Q^{i_1\cdots i_r<v_1,\ldots,v_k>}_{;u;v} = Q^{i_1\cdots i_r<v_1,\ldots,v_k>}_{;v;u}$$

for each $i_1, \cdots, i_r \in \{1, \cdots, n\}, u, v \in \{1, \cdots, k\}$ and $v_1, \cdots, v_k \in \mathbf{N}_o$. Since

$$Q^{i_1\cdots i_r<v_1,\ldots,v_k>}_{;v;u} - Q^{i_1\cdots i_r<v_1,\ldots,v_k>}_{;u;v} = \sum_{p=1}^{r} R^{i_p}_{juv} Q^{i_1\cdots i_{p-1} j i_{p+1}\cdots i_r<v_1,\ldots,v_k>},$$

we should prove that

$$\sum_{p=1}^{r} R^{i_p}_{juv} Q^{i_1\cdots i_{p-1} j i_{p+1}\cdots i_r<v_1,\ldots,v_k>} \equiv 0. \tag{3.8}$$

Indeed, we prove by induction of $v_1, \cdots, v_k$ that the following system

$$\sum_{p=1}^{r} R^{i_p}_{juv} Q^{i_1\cdots i_{p-1} j i_{p+1}\cdots i_r<v_1,\ldots,v_k>} = 0$$

$$\sum_{p=1}^{r} R^{i_p}_{juv;w_1} Q^{i_1\cdots i_{p-1} j i_{p+1}\cdots i_r<v_1,\ldots,v_k>} = 0 \tag{3.9}$$

$$\sum_{p=1}^{r} R^{i_p}_{juv;w_1;w_2} Q^{i_1\cdots i_{p-1} j i_{p+1}\cdots i_r<v_1,\ldots,v_k>} = 0$$

$$\cdots\cdots$$

is satisfied. If $v_1 = \cdots = v_k = 0$, then (3.9) is satisfied according to our assumption. Further if (3.9) is satisfied, then we should prove that

$$\sum_{p=1}^{r} R^{i_p}_{juv} Q^{i_1\cdots i_{p-1} j i_{p+1}\cdots i_r<v_1,\ldots,v_k>}_{;s} = 0$$

$$\sum_{p=1}^{r} R^{i_p}_{juv;w_1} Q^{i_1\cdots i_{p-1} j i_{p+1}\cdots i_r<v_1,\ldots,v_k>}_{;s} = 0 \tag{3.10}$$

$$\sum_{p=1}^{r} R^{i_p}_{juv;w_1;w_2} Q^{i_1\cdots i_{p-1} j i_{p+1}\cdots i_r<v_1,\ldots,v_k>}_{;s} = 0$$

$$\cdots\cdots$$

It is easy to verify that (3.10) is a consequence of (3.9). Since $s$ is an arbitrary element of $\{1, \cdots, k\}$, the proof of (3.9) and hence of (3.8) is ready.

Finally we should prove that the system (3.1) and the initial conditions (3.2) are satisfied. According to (3.6) we obtain

$$\partial\omega^{i_1\cdots i_r}/\partial x^u=\sum_{v_1=0}^{\infty}\cdots\sum_{v_u=1}^{\infty}\cdots\sum_{v_k=0}^{\infty}\frac{(-x^1)^{v_1}}{v_1!}\cdots(-1)\frac{(-x^u)^{v_u-1}}{(v_u-1)!}\cdots\frac{(-x^k)^{v_k}}{v_k!}\times$$

$$\times Q^{i_1\cdots i_r<v_1,\dots,v_k>}+\sum_{v_1=0}^{\infty}\cdots\sum_{v_k=0}^{\infty}\frac{(-x^1)^{v_1}}{v_1!}\cdots\frac{(-x^k)^{v_k}}{v_k!}\cdot\frac{\partial}{\partial x^u}Q^{i_1\cdots i_r<v_1,\dots,v_k>}$$

$$=-\sum_{v_1=0}^{\infty}\cdots\sum_{v_u=0}^{\infty}\cdots\sum_{v_k=0}^{\infty}\frac{(-x^1)^{v_1}}{v_1!}\cdots\frac{(-x^u)^{v_u}}{v_u!}\cdots\frac{(-x^k)^{v_k}}{v_k!}\times$$

$$\times\Big[Q^{i_1\cdots i_r<v_1,\dots,v_u+1,\dots,v_k>}-\frac{\partial}{\partial x^u}Q^{i_1\cdots i_r<v_1,\dots,v_k>}\Big]$$

$$=-\sum_{v_1=0}^{\infty}\cdots\sum_{v_k=0}^{\infty}\frac{(-x^1)^{v_1}}{v_1!}\cdots\frac{(-x^k)^{v_k}}{v_k!}\cdot\sum_{s=1}^{r}\Gamma^{i_s}_{ju}Q^{i_1\cdots i_{s-1}ji_{s+1}\cdots i_r<v_1,\dots,v_k>}$$

$$=-\sum_{s=1}^{r}\Gamma^{i_s}_{ju}\Big[\sum_{v_1=0}^{\infty}\cdots\sum_{v_k=0}^{\infty}\frac{(-x^1)^{v_1}}{v_1!}\cdots\frac{(-x^k)^{v_k}}{v_k!}Q^{i_1\cdots i_{s-1}ji_{s+1}\cdots i_r<v_1,\dots,v_k>}\Big]$$

$$=-\sum_{s=1}^{r}\Gamma^{i_s}_{ju}\omega^{i_1\cdots i_{s-1}ji_{s+1}\cdots i_r},$$

and $\omega^{i_1\cdots i_r}(0,\cdots,0)=Q^{i_1\cdots i_r<0,\cdots,0>}(0,\cdots,0)=Y^{i_1\cdots i_r}(0,\cdots,0)=\omega_0^{i_1\cdots i_r}$, for $i_1,\cdots,i_r\in\{1,\cdots,n\}$.

The convergence of the right side of (3.6) can be proven analogously to the proof of the convergence given in [5]. ‖

*Remark 1.* We note that the theorem 3.1 can be generalized for non-linear and $d$ - connections also, like in [7] and [8].

*Remark 2.* Note that we can choose an arbitrary (analytical) solution $Y^{i_1i_2\cdots i_r}$ of the linear homogeneous system (3.3) such that $Y^{i_1i_2\cdots i_r}(0,\cdots,0)=w_0^{i_1i_2\cdots i_r}$, and then the solution (3.6) will not depend on this choice. Hence we have the following corollary.

**Corollary 3.2.** *The number of linearly independent parallel r-forms in a neighborhood of a considered point is equal to* $\binom{n}{r}-R$ *where* $R$ *is the rank of the homogeneous linear system (3.3) where the coefficients are the components of the curvature tensor and its derivatives up to* $\binom{n}{r}-1$ *order.*

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