GEOMETRY OF BERWALD–CARTAN SPACES

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Abstract

The geometry of regular Hamiltonians as smooth functions on the cotangent bundle is mainly due to R. Miron and it is now systematically described in the monograph [4]. A manifold endowed with a regular Hamiltonian which is 2–homogeneous in momenta was called a *Cartan space*. An interesting particular class of Cartan spaces is given by the so–called Berwald–Cartan spaces. In this paper some new properties of the Berwald–Cartan spaces are proved.

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Introduction

Analytical Mechanics and some theories in Physics brought into discussion regular Lagrangians and their geometry, [5]. A regular Lagrangian which is 2-homogeneous in velocities is nothing but the square of a fundamental Finsler function and its geometry is Finsler geometry. This geometry was developed since 1918 by P. Finsler, E. Cartan, L. Berwald and many others, see [2] and the most recent graduate text [1]. But in Mechanics and Physics there exists also regular Hamiltonians whose geometry is also useful. This geometry is mainly due to R. Miron ,[3], and it is now systematically presented in the monograph [4]. A manifold endowed with a regular Hamiltonian which is 2-homogeneous in momenta was called a Cartan space. The notion of Cartan space was introduced by R. Miron in [3]. A particular and interesting class of Cartan spaces is given by the so–called Berwald–Cartan spaces, shortly BC-spaces. The geometry of the BC-spaces can be found in [4], Chs. 6-7. Our purpose is to prove some new properties of these spaces. A Cartan space is a pair (M, K) for M a smooth manifold and K a regular Hamiltonian which is 2-homogeneous in momenta. A BC space is defined as a Cartan space whose Chern–Rund connection coefficients of the canonical metrical connection do not depend on momenta, that is, $H^i_{jk}(x,p) = H^i_{jk}(x)$. For a Cartan space the pair $(T^*_xM, K(x,p))$ for any fixed $x \in M$ is a Minkowski space. We prove (Theorem 3.2) that for BC spaces the Minkowski spaces $(T^*_xM, K(x,p))$

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are all linearly isometric to each other. Noticing that the functions $H^i_{jk}(x)$ defines a symmetric linear connection ∇ on M we prove (Theorem 3.3) that ∇ is metrizable, that is, there exists a Riemannian metric on M whose Levi–Civita connection is ∇. These proofs are presented in Section 3. Some preliminaries from the geometry of cotangent bundle are given in Section 1, and Section 2 contains necessary facts from the geometry of Cartan spaces.

1 Preliminaries

Let M be an n-dimensional C^∞ manifold and $\tau^* : T^*M \to M$ its cotangent bundle. If (x^i) are local coordinates on M, then (x^i, p_i) will be taken as local coordinates on T^*M with the momenta (p_i) provided by $p = p_i dx^i$ where $p \in T^*_x M$, $x = (x^i)$ and (dx^i) is the natural basis of $T^*_x M$. The indices $i, j, k...$ will run from 1 to n and the Einstein convention on summation will be used. A change of coordinates $(x^i, p_i) \to (\widetilde{x}^i, \widetilde{p}_i)$ on T^*M has the form

$$\begin{aligned} &\widetilde{x}^i = \widetilde{x}^i(x^1, ..., x^n), \quad \operatorname{rank}\left(\frac{\partial \widetilde{x}^i}{\partial x^j}\right) = n \\ &\widetilde{p}_i = \frac{\partial x^j}{\partial \widetilde{x}^i}(\widetilde{x}) p_j, \end{aligned} \tag{1.1}$$

where $\left(\dfrac{\partial x^j}{\partial \widetilde{x}^i}\right)$ is the inverse of the Jacobian matrix $\left(\dfrac{\partial \widetilde{x}^j}{\partial x^k}\right)$.

Let $\left(\partial_i := \dfrac{\partial}{\partial x^i},\ \partial^i := \dfrac{\partial}{\partial p_i}\right)$ be the natural basis in $T_{(x,p)}T^*M$. The change of coordinates (1.1) produces

$$\begin{aligned} &\partial_i = (\partial_i \widetilde{x}^j)\widetilde{\partial}_j + (\partial_i \widetilde{p}_j)\widetilde{\partial}^j, \\ &\widetilde{\partial}^i = (\partial_j \widetilde{x}^i)\partial^j. \end{aligned} \tag{1.2}$$

The natural cobasis (dx^i, dp_i) from $T^*_{(x,p)}T^*M$ transforms as follows.

$$d\widetilde{x}^i = (\partial_j \widetilde{x}^i) dx^j,\ d\widetilde{p}_i = \frac{\partial x^j}{\partial \widetilde{x}^i}\, dp_j + \frac{\partial^2 x^j}{\partial \widetilde{x}^i \partial \widetilde{x}^k}\, p_j\, dx^k. \tag{1.3}$$

The kernel $V_{(x,p)}$ of the differential $d\tau^* : T_{(x,p)}T^*M \to T_xM$ is called the *vertical* subspace of $T_{(x,p)}T^*M$ and the mapping $(x,p) \to V_{(x,p)}$ is a regular distribution on T^*M called the *vertical distribution.* This is integrable with the leaves T^*_xM, $x \in M$ and is locally spanned by (∂^i). The vector field $C^* = p_i \partial^i$ is called the Liouville vector field and $\omega = p_i dx^i$ is called the Liouville 1-form on T^*M. Then $d\omega$ is the canonical symplectic structure on T^*M. For an easier handling of the geometrical objects on T^*M it is usual to consider a supplementary distribution to the vertical distribution, $(x,p) \to N_{(x,p)}$, called the *horizontal distribution* and to report all geometrical objects on T^*M to the decomposition

$$T_{(x,p)}T^*M = N_{(x,p)} \oplus V_{(x,p)}. \tag{1.4}$$

The pieces produced by the decomposition (1.4) are called d–geometrical objects (d is for distinguished) since their local components behave like geometrical objects on M, although they depend on $x = (x^i)$ and momenta $p = (p_i)$.

The horizontal distribution is taken as being locally spanned by the local vector fields

$$\delta_i := \partial_i + N_{ij}(x,p)\partial^j, \tag{1.5}$$

and for a change of coordinates (1.1), the condition

$$\delta_i = (\partial_i \widetilde{x}^j)\widetilde{\delta}_j \text{ for } \widetilde{\delta}_j := \widetilde{\partial}_j + \widetilde{N}_{jk}(\widetilde{x},\widetilde{p})\widetilde{\partial}^k, \tag{1.6}$$

is equivalent with

$$\widetilde{N}_{ij}(\widetilde{x},\widetilde{p}) = \frac{\partial x^s}{\partial \widetilde{x}^i}\,\frac{\partial x^r}{\partial \widetilde{x}^j}\,N_{sr}(x,p) + \frac{\partial^2 x^r}{\partial \widetilde{x}^i \partial \widetilde{x}^r}\,p_r. \tag{1.7}$$

The horizontal distribution is called also a *nonlinear connection* on T^*M and the functions (N_{ij}) are called the local coefficients of this nonlinear connection. It is important to note that any regular hamiltonian on T^*M determines a nonlinear connection whose local coefficients verify $N_{ij} = N_{ji}$. The basis (δ_i, ∂^i) is adapted to the decomposition (1.4). The dual of it is $(dx^i, \delta p_i)$, for $\delta p_i = dp_i - N_{ji}dx^j$ and then $\delta\widetilde{p}_i = \dfrac{\partial x^j}{\partial \widetilde{x}^i}\,\delta p_j$.

2 Cartan spaces

A *Cartan structure* on M is a function $K : T^*M \to [0,\infty)$ with the following properties:

1. K is C^∞ on $T^*M \setminus 0$ for $0 = \{(x,0),\ x \in M\}$,
2. $K(x,\lambda p) = \lambda K(x,p)$ for all $\lambda > 0$,
3. The $n \times n$ matrix (g^{ij}), where $g^{ij}(x,p) = \dfrac{1}{2}\,\partial^i\partial^j K^2(x,p)$, is positive–definite at all points of $T^*M \setminus 0$.

We notice that in fact $K(x,p) > 0$, whenever $p \neq 0$.

Definition 2.1. The pair (M,K) is called a *Cartan space.*

Example. Let $(\gamma_{ij}(x))$ be the matrix of the local coefficients of a Riemannian metric on M and $(\gamma^{ij}(x))$ its inverse. Then $K(x,p) = \sqrt{\gamma^{ij}(x)p_ip_j}$ gives a Cartan structure. Thus any Riemannian manifold can be regarded as a Cartan space. More examples can be found in Ch. 6 of [4].

We put $p^i = \frac{1}{2}\ \partial^i K^2$ and $C^{ijk} = -\frac{1}{4}\ \partial^i\partial^j\partial^k K^2$. The properties of K imply

$$
\begin{aligned}
&p^i = g^{ij}p_j,\ p_i = g_{ij}p^j,\ K^2 = g^{ij}p_ip_j = p_ip^j,\\
&C^{ijk}p_k = C^{ikj}p_k = C^{kij}p_k = 0.
\end{aligned}
\tag{2.1}
$$

One considers the *formal Christoffel symbols*

$$
\gamma^i_{jk}(x,p) := \frac{1}{2}\ g^{is}(\partial_k g_{js} + \partial_j g_{sk} - \partial_s g_{jk})
\tag{2.2}
$$

and the contractions $\gamma^\circ_{jk}(x,p) := \gamma^i_{jk}(x,p)p_i$, $\gamma^\circ_{j\circ} := \gamma^i_{jk}p_ip^k$. Then the functions

$$
N_{ij}(x,p) = \gamma^\circ_{ij}(x,p) - \frac{1}{2}\ \gamma^\circ_{h\circ}(x,p)\partial^h g_{ij}(x,p),
\tag{2.3}
$$

verify (1.7). In other words, these functions define a nonlinear connection on T^*M. This nonlinear connection was discovered by R. Miron, [3]. Thus a decomposition (1.4) holds. From now on we shall use only the nonlinear connection given by (2.3).

A linear connection D on T^*M is said to be an *N–linear connection* if

1° D preserves by parallelism the distributions N and V,

2° $D\theta = 0$, for $\theta = \delta p_i \wedge dx^i$.

One proves that an N-linear connection can be represented in the adapted basis (δ_i, ∂^i) in the form

$$
\begin{aligned}
&D_{\delta_j}\delta_i = H^k_{ij}\delta_j, \quad D_{\delta_j}\partial^i = -H^i_{kj}\partial^k,\\
&D_{\partial^j}\delta_i = V^{kj}_i\delta_k, \quad D_{\partial^j}\partial^i = -V^{ij}_k\delta^k,
\end{aligned}
\tag{2.4}
$$

where V^{kj}_i is a d–tensor field and $H^k_{ij}(x,p)$ behave like the coefficients of a linear connection on M. The functions H^k_{ij} and V^{kj}_i define operators of h–covariant and v-covariant derivatives in the algebra of d-tensor fields, denoted by ${}_{|k}$ and $\big|^k$, respectively. For g^{ij} these are given by

$$
\begin{aligned}
&g^{ij}{}_{|k} = \delta_k g^{ij} + g^{sj}H^i_{sk} + g^{is}H^j_{sk},\\
&g^{ij}\big|^k = \partial^k g^{ij} + g^{sj}V^{ik}_s + g^{is}V^{jk}_s.
\end{aligned}
\tag{2.5}
$$

An N-linear connection given in the adapted basis (δ_i, ∂^j) as $D\Gamma(N) = (H^i_{jk}, V^{ik}_j)$ is called *metrical* if

$$
g^{ij}{}_{|k} = 0, \quad g^{ij}\big|^k = 0.
\tag{2.6}
$$

One verifies that the N-linear connection $C\Gamma(N) = (H^i_{jk}, C^{jk}_i)$ with

$$
\begin{aligned}
&H^i_{jk} = \frac{1}{2}\ g^{is}(\delta_j g_{sk} + \delta_k g_{js} - \delta_s g_{jk}),\\
&C^{jk}_i = -\frac{1}{2}\ g_{is}(\partial^j g^{sk} + \partial^k g^{sj} - \partial^s g^{jk}) = g_{is}C^{sjk},
\end{aligned}
\tag{2.7}
$$

is metrical and its h-torsion $T^i_{jk} := H^i_{jk} - H^i_{kj} = 0$, v-torsion $S^{jk}_i := C^{jk}_i - C^{kj}_i = 0$ and the deflection tensor $\Delta_{ij} = N_{ij} - p_k H^k_{ij} = 0$. Moreover, it is unique with these properties. This is called the canonical metrical connection of the Cartan space (M, K). It has also the following properties:

$$
\begin{aligned}
&K_{|j} = 0, \quad K\big|^j = \frac{p^j}{K}, \quad K^2{}_{|j} = 0, \quad K^2\big|^j = 2p^j, \\
&p_{i|j} = 0, \quad p_i\big|^j = \delta^j_i, \quad p^i{}_{|i} = 0, \quad p^i\big|^j = g^{ij}.
\end{aligned}
\tag{2.8}
$$

Besides $C\Gamma(N)$ one may consider on T^*M three other important N-linear connection which are partially or not at all metrical: Chern–Rund connection $CR\Gamma(N) = (H^i_{jk}, 0)$, the Hashiguchi connection $H\Gamma(N) = (\partial^i N_{jk}, C^{kj}_i)$ and the Berwald connection $B\Gamma(N) = (\partial^i N_{jk}, 0)$.

3 Berwald–Cartan spaces

Let $C^n = (M, K)$ be a Cartan space with the canonical metrical connection $C\Gamma(N) = (H^i_{jk}, C^{jk}_i)$ given by (2.7).

Definition 3.1. The Cartan space C^n is called a *Berwald–Cartan space*, shortly a BC space, if the connection coefficients H^i_{jk} do not depend on momenta, that is, $H^i_{jk}(x,p) = H^i_{jk}(x)$.

In [4], by direct methods or using the duality between Finsler and Cartan spaces given by the Legendre map, one proves

Theorem 3.1. *The following assertions are equivalent:*

1° *The Cartan space C^n is a BC space,*

2° *The coefficients $B^i_{jk} = \partial^i N_{jk}$ of the Berwald connection are functions of position only, that is $B^i_{jk}(x,p) = B^i_{jk}(x)$,*

3° *The curvature $P_j{}^i{}_k{}^h := \dot{\partial}^h B^i_{jk}$ of the Berwald connection vanishes.*

4° $C^{ijk}{}_{|h} = 0$.

For the Cartan space $C^n = (M, K)$, the function $K_x := K(x, \cdot) : T^*_x M \to \mathbb{R}$ is a *Minkowski norm* for every $x \in M$. Thus we have the Minkowski spaces $(T^*_x M, K_x)$, $x \in M$. For BC spaces, the following theorem holds.

Theorem 3.2. *Let (M, K) be a BC space. Whenever M is connected the Minkowski spaces $(T^*_x M, K_x)$ are all linearly isometric to each other.*

Proof. Let $\omega = \omega_i dx^i$ an 1-form and $v = v^j \partial_j$ a vector field on M. Using the connection coefficients $H^i_{jk}(x)$ we may define a covariant derivative of ω in the direction of v as follows: $\nabla_v \omega = v^k(\partial_k \omega_i - H^j_{ik}\omega_j)dx^i$.

We restrict ω to a curve $c : t \to x(t)$, $t \in \mathbb{R}$, on M, define the covariant derivative of ω along c by $\dfrac{\nabla\omega}{dt} = \left[\dfrac{d\omega_i}{dt} - H^j_{ik}\omega_j \dfrac{dx^k}{dt}\right] dx^i$ and we say that ω is parallel along c if $\dfrac{\nabla\omega}{dt} = 0$. Let us estimate $\dfrac{dK^2(x(t),\omega(t))}{dt}$. We write the equality $K^2(x,p) = g^{ij}(x,p)p_jp_j$ for $(x(t),\omega(t))$ and we obtain that along the curve c: $\dfrac{dK^2}{dt} = \dfrac{dg^{ij}}{dt}\omega_i\omega_j + 2g^{ij}\omega_i \dfrac{d\omega_j}{dt}$. But $\dfrac{d}{dt}(g^{ij}) = (\delta_k g^{ij})\dfrac{dx^k}{dt} + (\partial^k g^{ij})\dfrac{\delta p_k}{dt}$ and using $g^{ij}{}_{|k} = 0$ as well as the last equation (2.1) one gets:

$$\frac{dK^2}{dt} = 2g^{ij}\omega_i\left(\frac{d\omega_j}{dt} - H^s_{jk}\omega_s \frac{dx^k}{dt}\right).$$

From here we read

Lemma 3.1. *If the 1-form ω is parallel along the curve $c : t \to x(t)$, then the function $K(t) := K(x(t),\omega(t))$ is constant along the curve c.*

Let x, y be points of M joined by a curve $c : [0,1] \to M$ such that $c(0) = x$, $c(1) = y$. Let be $\alpha \in T^*_xM$. We consider the unique solution $\omega = (\omega_i)$ of the system of linear ordinary differential equations $\dfrac{d\omega_i}{dt} - H^j_{ik}\omega_j \dfrac{dx^k}{dt} = 0$ with the initial condition $\omega(0) = \alpha$ and we associate to α the element $\alpha' = \omega(1)$ of T^*_yM. The mapping $T^*_xM \to T^*_yM$ given by $\alpha \to \alpha'$ is a linear isomorphism. By Lemma 3.1, $K(x(t),\omega(t))$ has the same values at $t = 0$. Hence $K_x(\alpha) = K_y(\alpha')$. This means that the Minkowski spaces (T^*_xM, K_x) and (T^*_yM, K_y) are linearly isometric for every $x, y \in M$, $\square$

Another interesting property of BC spaces is as follows.

The connection coefficients $H^i_{jk}(x,p) = H^i_{jk}(x)$ define a symmetric linear connection ∇ on M and it happens that this is *metrizable*, that is, there exists on M a Riemannian metric h such that ∇ is the Levi–Civita connection associated to it. This h is not unique.

We prove this fact by adapting an idea of Z.I. Szabó [6]. The duality with Finsler spaces is not used.

Theorem 3.3. *Let $C^n = (M,K)$ be a BC space with M connected and ∇ the symmetric linear connection on M of local coefficients $H^i_{jk}(x,p) = H^i_{jk}(x)$. Then there exists a Riemannian metric h on M such that ∇ is the Levi–Civita connection of it.*

Proof. Let be the Minkowski space $(T^*_{x_0}M, K_{x_0})$ for a fixed $x_0 \in M$. Then $S_{x_0} = \{\omega \mid K_{x_0}(\omega) = 1\}$ is a compact subset of $T^*_{x_0}M$. Let G be the group of all linear isomorphisms of $T^*_{x_0}M$ that preserve S_{x_0}. This G is a compact Lie group. It contains as a subgroup the holonomy group H_{x_0} defined by $(H^i_{jk}(x))$ according to Lemma 3.1. In general, H_{x_0} is not compact.

Let $<,>$ be *any* inner product in $T^*_{x_0}M$. Define a new inner product on $T^*_{x_0}M$ by

$$h_{x_0}(\varphi,\omega) = \frac{1}{\mathrm{vol}(G)} \int_G < a\varphi, a\omega > \mu_G,\ \ \varphi,\omega \in T^*_{x_0}M, \tag{3.1}$$

for $a \in G$, where μ_G denotes the bi–invariant Haar measure on G. It results $h_{x_0}(b\varphi, b\omega) = h_{x_0}(\varphi,\omega)$ for every $b \in G$ (from the properties of μ_G), that is h_{x_0} is G-invariant. In particular, h_{x_0} is H_{x_0}-invariant.

Let now any $x \in M$ and a curve $c : t \to c(t)$ joining x with x_0, $c(0) = x$, $c(1) = x_0$. Denote by $P_c : T^*_xM \to T^*_{x_0}M$ the parallel transport of covectors defined by $H^i_{jk}(x)$. For every $\varphi \in T^*_xM$, $P_c(\varphi) = \omega(1) \in T^*_{x_0}M$, where $\omega = (\omega_i)$ is the unique solution of the system of linear differential equations

$$\frac{d\omega_i}{dt} - H^i_{jk}\omega_j\,\frac{dx^k}{dt} = 0,\ \text{ with } \omega(0) = \varphi. \tag{3.2}$$

In the proof of Theorem 3.2 we have seen that P_c is a linear isometry of Minkowski spaces. We define an inner product on T^*_xM by

$$h_x(\varphi,\psi) = h_{x_0}(P_c\varphi, P_c\psi),\ \ \varphi,\psi \in T^*_{x_0}M. \tag{3.3}$$

Lemma 3.2. *h_x does not depend on the curve c.*

Indeed, if $\widetilde{c}$ is another curve joining x and x_0, denote by c_- the reverse of c and consider the loop $\widetilde{c} \circ c_-$. Then $P_{\widetilde{c}\circ c_-} \in H_{x_0}$ and from the H_{x_0}-invariance of h_{x_0}, that is, $h_{x_0}(P_{\widetilde{c}\circ c_-}\varphi, P_{\widetilde{c}\circ c_-}\psi) = h_{x_0}(\varphi,\psi)$ we get $h_{x_0}(P_{\widetilde{c}}\varphi, P_{\widetilde{c}}\psi) = h_{x_0}(P_c\varphi, P_c\psi)$ as we claimed.

The mapping $x \to h_x : T^*_xM \times T^*_xM \to R$ is smooth since P_c smoothly depends on x, according to a general result regarding the dependence of solution of system of differential equations by initial data. Thus we have constructed a Riemannian metric h in the cotangent bundle of M.

The connection coefficients $(H^i_{jk}(x))$ define a linear connection ∇ in the cotangent bundle as follows:

$$\nabla : \mathcal{X}(M) \times \Gamma(T^*M) \to \Gamma(T^*M),\ (X,\omega) \to \nabla_X\omega = X^k\left(\frac{\partial\omega_i}{\partial x^k} - H^j_{ik}\omega_j\right)dx^i$$

and the operator ∇_X, $X \in \mathcal{X}(M)$, extends to the tensorial algebra of the cotangent bundle. For instance, if we regard h as a section in the vector bundle $L^s_2(T^*M, \mathbb{R})$, then we have

$$(\nabla_X h)(\varphi,\psi) = X(h(\varphi,\psi)) - h(\nabla_X\varphi,\psi) - h(\varphi,\nabla_X\psi). \tag{3.4}$$

Lemma 3.3. $\nabla_X h = 0,\ X \in \mathcal{X}(M)$.

Proof. We choose a basis $(\varphi_i(x))$ in $T^*_x M$. It suffices to show that $(\nabla_X h)(\varphi_i(x), \varphi_j(x)) = 0$. Let be the vector $X = \left.\frac{dc}{dt}\right|_\circ$ tangent to a curve c starting from $x \in M$ at $t = 0$. We parallel translate $\varphi_i(x)$ along c and we obtain a field of basis $\varphi_i(t)$ along c. The general formula

$$\frac{\nabla h}{dt}(\varphi, w) = \frac{dh(\varphi, \psi)}{dt} - h\left(\frac{\nabla \varphi}{dt}, \psi\right) - h\left(\varphi, \frac{\nabla \psi}{dt}\right),$$

gives

$$\frac{\nabla h}{dt}(\varphi_i(x), \varphi_j(x)) = \left.\frac{dh(\varphi_i, \varphi_j)}{dt}\right|_{t=0}$$

because of $\frac{\nabla \varphi_i}{dt} = 0$.

Now we show that $h(\varphi_i(t), \varphi_j(t))$ does not depend on t.

Indeed, $h_{c(t)}(\varphi_i(t), \varphi_j(t)) = h_{x_0}(P_{\varphi_i}, P_{\varphi_j})$, where P is the parallel translation from $T^*_{c(t)}M$ to $T_{x_0}M$. This P may be thought as the composition of a parallel translation P_2 from $T^*_{c(t)}M$ to $T^*_x M$ and of a parallel translation P_1 from $T^*_x M$ to $T^*_{x_0}M$. We have $h_{c(t)}(\varphi_i(t), \varphi_j(t)) = h_{x_0}((P_2 \circ P_1)\varphi_i, (P_2 \circ P_1)\varphi_j) = h_{x_0}(P_1\varphi_i, P_2\varphi_j) = h_x(\varphi_i(x), \varphi_j(x))$. Hence $h_{c(t)}(\varphi_i(t), \varphi_j(t))$ does not depend on t, as we claimed.

This fact ends the proof of Lemma 3.3.

To end the proof of Theorem, we take the covariant part of h as a section in the vector bundle $L^s_2(TM, \mathbb{R})$ and so we get a Riemannian metric on M, denoted with the same letter h. The operator ∇_X acts also on vector fields on M by the rule $\nabla_X Y = X^k\left(\frac{\partial Y^i}{\partial x^k} + H^i_{jk}Y^j\right)$ for $Y = Y^i\,\frac{\partial}{\partial x^i}$ and $(X, Y) \to \nabla_X Y$ gives a linear connection on M such that $\nabla_X h = 0$. As ∇ has no torsion, it coincides with the Levi–Civita connection of h, $\square$

Remark . An alternative way to prove Lemma 3.3 is to prove first that $\frac{\nabla h}{dt}(\varphi, \psi) = \lim_{t \to 0} \frac{h(P_c\varphi, P_c\psi) - h(\varphi, \psi)}{t}$, where P_c is the parallel translation from $c(0)$ to $c(t)$.

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