Zeta regularization and Noncommutative Geometry

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Abstract

Arguments from noncommutative geometry are useful to the study of infinite dimensional geometry. For example, applying such arguments together with ζ-regularization, we can define Grassmann algebra with $\infty - p$-forms. In this paper, we apply noncommutative geometric arguments and ζ-regularization to the calculus of $(\infty - p)$-forms. We show exactness of exterior differentiable $(\infty - p)$-forms and try to justify physists' answer of infinite dimensional Gaussian integral by using Ray-Singer determinant.

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1 Introduction

Noncommutative geometry is a powerful tool not only for physics but also for infinite dimensional geometry (cf. [1],[3], [10], [11],[15]). For example, a mapping space $Map(X, M)$ can be viewed as a Sobolev manifold modeled by $H = W^k(X)$. Here $W^k(X)$ is a Sobolev space over X. If X is a compact spin manifold, with suitable modification to $Map(X, M)$, we may regard $W^k(X)$ to be the Sobolev space of spinor fields on X. In this case, the Dirac operator $\not{D}$ of X induces a polarization $\epsilon = P_+ - P_-$ of H. Here $P_\pm$ are the positive and negative peoper spaces of $\not{D}$, respectively. The principle of noncommutative geometry asserts $\{H, \epsilon\}$ gave geometric information of X. For example, if G is a linear Lie group, $Map(X, G)$ is contained in the restricted general linear group $GL_p = \{T \in GL(H) | [\epsilon, T] \in I_p\}$, $p > d/2$, where $GL(H)$ is the group of all inversible bounded linear opertors of H, I_p is the p-th Schatten ideal, and d is the dimension of X. The topological structure of a GL_p-bundle $\{g_{UV}\}$ is completely determined by the noncommutative connection $\{\kappa_U\}$,

$$\kappa_U : U \mapsto I_p, \quad (\epsilon + \kappa_U) g_{UV} = g_{UV}(\epsilon + \kappa_V),$$

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([1]). To get more precise information than topology, we use the pair $\{H, G\}$, where G is a nondegenerate Schatten class operator such that its ζ-function $\zeta(G, s)$ is holomorphic at $s = 0$ ([2], [3]). Considering such pairing is closely related to Connes' spectral triple ([9]). Our approach is narrow than Connes' approach but more concrete. If $H = W^k(X)$, we take G to be the Green operator of a nondegenerate selfadjoint elliptic (pseudo) differential operator on X. For simple, we assume positivity of G in this paper. In abstract setting, we introduce Sobolev norm $\|x\|_k$ by $\|G^{-k}x\|$. The Sobolev space by the norm $\|x\|_k$ is denoted by W^k. The complete orthonormal basis $\{e_n\}$ of H is taken by proper vectors of G; $Ge_n = \mu_n e_n$. Here we arrange $\{\mu_n\}$ to be $\mu_1 \geq \mu_2 \geq \ldots > 0$. Then the complete orthonormal basis of W^k is given by $\{e_{n,k}\}$, $e_{n,k} = \mu_n^{-k} e_n$. The coordinate of $x \in W^k$ is fixed to be $(x_1, x_2, \ldots)$, $x = \sum x_n e_{n,k}$.

We say $\nu = \zeta(G, 0)$ to be the regularized dimension of H (and W^k). Other invariants of the pair $\{H, G\}$ are the position d of the first pole of $\zeta(G, s)$ and $detG = \exp(\zeta'(G, 0))$. By using these invariants, we have defined $(\infty - p)$-forms on W^k and investigated their calculus including exactness of exterior differentiable $(\infty - p)$-forms ([2],[3]). In this paper, we reinvestigate these definitions and results. Regularization of integrals of ∞-forms by using fractional calculus (cf.[13]) and ζ-regularization has been defined in [4]. Corresponding regularization procedure of exterior differential of $(\infty - p)$-forms is introduced and related to the regularization of differential operators on H ([3],[6]). Then as an example of regularized integral of ∞-forms, an attempt of mathematical justification of the formula

$$\int \mathrm{e}^{-2\pi(x, Dx)} \mathcal{D}x = \frac{1}{\sqrt{detD}},$$

where $detD$ is the Ray-Singer determinant of D (cf. [8], [14],[16]), is given. We also try to compute regularized volume of "sphere" in H. The answer is not yet obtained. But our calculation sugests the regularized volume might be $\dfrac{2\pi^{\nu/2}}{\Gamma(\frac{\nu}{2})}$ as expected.

2 Grassmann algebra with $(\infty - p)$-forms

We introduce the Sobolev duality between W^k and W^{-k} by

$$\langle x, \xi \rangle = \langle G^{-k}x, G^k \xi \rangle, \quad x \in W^k, \xi \in W^{-k}. \tag{1}$$

By definition, W^k is contained in W^l if $k > l$. We set

$$W^{k+0} = \bigcup_{l>k} W^l, \quad W^{k-0} = \bigcap_{l<k} W^l. \tag{2}$$

If $k = 0$, we denote $H^{\pm}$ instead of $W^{\pm 0}$. In W^{k-0}, we set

$$e_{\infty,k} = \sum_{n=1}^{\infty} \mu_n^{d/2} e_n, k. \tag{3}$$

e_∞, k depends on the choice of $\{e_n\}$. But we do not specify $\{e_n\}$ for simple.

Definition. We set

$$W^{k-0}(0) = \{\sum x_n e_{n,k} \in W^{k-0} | \lim_{n\to\infty} \mu_n^{-d/2} x_n = 0\}, \tag{4}$$

and define the space $W^{k-0}(finite)$ by

$$W^{k-0}(finite) = W^{k-0}(0) \oplus \mathbb{R}e_{\infty,k}, \quad or\ W^{k-0}(0) \oplus \mathbb{C}e_{\infty,k}, \tag{5}$$

according to W^k is a real vactor space or a complex vector space.

We consider $W^{k-0}(0)$ to be a subspace of W^{k-0}. But $W^{k-0}(finite)$ is considered to be a product space of $W^{k-0}(0)$ and $\mathbb{R}$ or $\mathbb{C}$ as a topological space.

Since $W^{k-0}(0)$ is dense in W^{k-0}, the dual space of $W^{k-0}(0)$ is W^{-k+0}. We define the dual element $e^\dagger_{\infty,k}$ of $e_{\infty,k}$ by

$$\langle e^\dagger_{\infty,k}, x\rangle = \lim_{s\to+0} \frac{s}{c} \langle \sum \mu_n^{d/2+s} e_{n,-k}, x\rangle, \quad x \in W^{k-0}, \tag{6}$$

where $c = \mathrm{Res}_{s=0}\zeta(G, s)$. But since $e_{\infty,k}$ and $e^\dagger_{\infty,k}$ are not symmetric each other, we introduce $\mathbf{e}_k$ by

$$\langle \mathbf{e}_k, x\rangle = \lim_{s\to+0} \sqrt{\frac{s}{c}} \langle \sum \mu_n^{(d+s)/2} e_{n,k}, x\rangle, \quad x \in W^{k-0}. \tag{7}$$

Since $c = \lim_{s\to d+0}(s-d)\zeta(G, s)$, c is positive, so $\mathbf{e}_k$ is well defined although W^k is a real vector space.. By definition, we may write

$$\mathbf{e}_k = G^{2k}\mathbf{e}_{-k}, \quad or \quad \mathbf{e}_k = *\mathbf{e}_{-k},$$

where $*$ is the Hodge operator ([2]). By definition, we also have

$$\langle \mathbf{e}_k, \mathbf{e}{-k}\rangle = 1, \quad \langle \mathbf{e}_k, e_{n,-k}\rangle = \langle e_{n,k}, \mathbf{e}_{-k}\rangle = 0.$$

Hence we may set

$$(W^{-k+0} \oplus \mathbb{K}\mathbf{e}_{-k})^\dagger = W^{k-0} \oplus \mathbb{K}\mathbf{e}_k, \tag{8}$$

where $\mathbb{K}$ is either of $\mathbb{R}$ or $\mathbb{C}$.

Since $Map(X, M)$ is a Sobolev manifold modeled by $W^k(X)$, where k is larger than $dimX/2$, differential forms of $Map(X, M)$ take the values in $Gr(W^{-k}(X)$, the Grassmann algebra over $W^{-k}(X)$. So we treat $Gr(W^{-k+0})$ and denote the generators of this algebra corresponding to $e_{n,-k}$ by dx_n. We also introduce $d^\infty x$ as the element corresponding to $\mathbf{e}_k$ and regard it as the infinite product $dx_1 \wedge dx_2 \wedge \ldots$. We denote

Gr if forget multiplicative structure of Gr and regard only as a module. We give the left $Gr(W^{-k+0})$-module structure to $\mathrm{Gr}(W^{k-0}) \otimes d^\infty x$ by

$$\begin{aligned}
& (dx_{i_1} \wedge \ldots \wedge dx_{i_p}) \wedge (d\xi_{j_1} \wedge \ldots \wedge d\xi_{j_q}) \otimes d^\infty x = 0, \\
& \qquad \{i_1, \ldots, i_p\} \not\subset \{j_1, \ldots, j_q\}, \qquad (9) \\
& (dx_{i_1} \wedge \ldots \wedge dx_{i_p}) \wedge \\
& \qquad \wedge \big((d\xi_{i_1} \wedge \ldots \wedge d\xi_{i_p}) \wedge (d\xi_{j_1} \wedge \ldots \wedge d\xi_{j_q})\big) \otimes d^\infty x \\
= \; & (-1)^{(i_1-1)+\cdots+(i_p-p)} (d\xi_{j_1} \wedge \ldots \wedge d\xi_{j_q}) \otimes d^\infty x, \\
& \qquad \{i_1, \ldots, i_p\} \cap \{j_1, \ldots, j_q\} = \emptyset. \qquad (10)
\end{aligned}$$

In the rest, we denote

$$d^{\infty - \{i_1, \ldots, i_p\}} x = (dx_{i_1} \wedge \ldots \wedge dx_{i_p}) \otimes d^\infty x. \qquad (11)$$

We thought $d^{\infty - \{i_1, \ldots, i_p\}} x$ to be

$$dx_1 \wedge \ldots \wedge dx_{i_1 - 1} \wedge dx_{i_1 + 1} \wedge \ldots \wedge dx_{i_p - 1} \wedge dx_{i_p + 1} \wedge \ldots .$$

In $\mathrm{Gr}(W^{k-0}) \otimes d^\infty x$, elements written as $\sum_I f_I d^{\infty - I} x$, $I = \{i_1, \ldots, i_p\}$ are said to be $(\infty - p)$-forms and denoted by $\phi^{\infty - p}$, *etc.*. Then we define wedge product of p-form or $(\infty - p)$-form and $(\infty - q)$-form by (9), (10) and

$$\phi^{\infty - p} \wedge \psi^{\infty - q} = 0, \qquad (12)$$

$$\phi^p \wedge \psi^{\infty - q} = (-1)^{p(\nu - q)} \psi^{\infty - q} \wedge \phi^p, \quad -1 = \mathrm{e}^{\pi i}, \qquad (13)$$

$$\psi^{\infty - q} \wedge \phi^p = (-1)^{(\nu - q)p} \phi^p \wedge \psi^{\infty - q}, \quad -1 = \mathrm{e}^{-\pi i}, \qquad (14)$$

when W^k is a complex vector space. Here ν is arbitrary and need not assume integrity. *While if* W^k *is a real vector space, we need to assume integrity of* ν(cf. [5]).

Definition. The algebra $Gr(W^{-k+0}) \oplus \mathrm{Gr}(W^{k-0}) \otimes d^\infty x$ with the wedge product defined by the rules (9), (10) and (12) - (14) is said to be the Grassmann algebra with ∞ -forms and denoted by $Gr^\infty(W^{-k+0})$.

Note. Commutaion relations (13) and (14) are same those of generators of noncommutative torus (or matrices algebra) when ν is not a rational number (or a rational number) (cf. [12]). We ask are there any relation between Grassman algebra with ∞-forms (or Clliford algebra with ∞-spinors, which is defined by the same way ([5])) and noncommutative torus (or matrices algebra) (cf. [7]).

3 Exterior differential of $(\infty - p)$-forms

Similar to the finite degree forms exterior differential of an $(\infty - p)$-form $\sum f_I d^{\infty - I} x$ is defined by

$$d\Big(\sum_I f_I d^{\infty - I} x\Big) = \sum_I df_I \wedge d^{\infty - I} x, \quad df = \sum_{n=1}^{\infty} \frac{\partial f}{\partial x_n} dx_n. \qquad (15)$$

But since

$$d(\sum_{i_1,\dots,i_{p+1}} f_{i_1,\dots,i_{p+1}} d^{\infty-\{i_1,\dots,i_{p+1}\}}x)$$

$$= \sum_{i_1,\dots,i_p}(\sum_{k=0}^{p+1}\sum_{i_k<j<i_{k+1}}(-1)^{j-k}\frac{\partial f_{i_1,\dots,i_k,j,i_{k+1},\dots,i_p}}{\partial x_j})d^{\infty-\{i_1,\dots,i_p\}}x,$$

where $i_0 < j < i_1$ and $i_p < j < i_{p+1}$ mean $j < i_1$ and $i_p < j$, respectively, $d\phi^{\infty-p}$ diverges in general. We say $\phi^{\infty-p}$ is exterior differentiable if $d\phi^{\infty-p}$ converges.

Note. $\phi^{\infty-p}$ is expressed as alternative function $f(x) = f(x, x_1, \dots, x_p) : W^k \to W^k$. Denoting Fréchet differential of f by $\hat{d}f$, df is given by

$$df(x, x_1, \dots, x_{p-1}) = (-1)^{p-1} tr\hat{d}f(x, x_1, \dots, x_{p-1}, x).$$

So to define df, we need to assume $\hat{d}f$ to be a trace class operator. This is a coordiante free definition of exterior differentiable form ([3]).

Theorem 1. *An exterior differentiable* $(\infty - p)$*-form is exact.*

Proof. Since Theorem is true if $p = 0$, first we prove Theorem for $(\infty - 1)$-form $\phi = \sum f_n d^{\infty-\{n\}}x$. First we note that if ϕ is exterior differentiable, then there exists a constant $M > 0$ such that

$$|\sum_{n=1}^{N}(-1)^{n-1}\frac{\partial f_n}{\partial x_n}| \le M, \qquad (16)$$

for all N. The equation $\phi = d\psi$, $\psi = \sum_n g_{n,n+1} d^{\infty-\{n,n+1\}}x$ is equivalent to the system

$$\frac{\partial g_{1,2}}{\partial x_2} = f_1, \quad (-1)^{n-2}(\frac{\partial g_{n-1,n}}{\partial x_{n-1}} - \frac{\partial g_{n,n+1}}{\partial x_{n+1}}) = f_n, \quad n \ge 2. \qquad (17)$$

A solution of this system is given by

$$g_{1,2} = \int_0^{x_1} f_1 dt, \quad g_{n,n+1} = \int_0^{x_{n+1}}((-1)^{n-1}f_n + \frac{\partial g_{n-1,n}}{\partial x_{n+1}})dt.$$

Since

$$g_{2,3} = \int_0^{x_2}(-f_2 + \frac{\partial}{\partial x_1}\int_0^{x_2} f_1 d\tau)dt = \int_0^{x_3}(-f_2 + \int_0^{x_2}\frac{\partial f_1}{\partial x_1}d\tau)dt,$$

we get

$$\frac{\partial g_{2,3}}{\partial x_2} = \int_0^{x_3}(-\frac{\partial f_2}{\partial x_2} + \frac{\partial f_1}{\partial x_1})dt.$$

We assume

$$\frac{\partial g_{n-1,n}}{\partial x_{n-1}} = \int_0^{x_n}(\sum_{i=1}^{n-1}(-1)^{i-1}\frac{\partial f_i}{\partial x_i})dt. \qquad (18)$$

Then, since

$$\frac{\partial g_{n,n+1}}{\partial x_{n+1}} = (-1)^{n-1} f_n + \frac{\partial g_{n-1,n}}{\partial x_{n-1}} = (-1)^{n-1} f_n + \int_0^{x_n} (\sum_{i=1}^{n-1} (-1)^{i-1} \frac{\partial f_i}{\partial x_i}) dt,$$

we obtain

$$\begin{aligned} \frac{\partial g_{n+1,n+2}}{\partial x_{n+2}} &= (-1)^n f_{n+1} + \frac{\partial g_{n,n+1}}{\partial x_n} = \\ &= (-1)^n f_{n+1} + \frac{\partial}{\partial x_n} \int_0^{x_{n+1}} \big((-1)^{n+1} f_n + \int_0^{x_n} \sum_{i=1}^{n-1} (-1)^{i-1} \frac{\partial f_i}{\partial x_i} d\tau\big) dt \\ &= (-1)^n f_{n+1} + \int_0^{x_{n+1}} (\sum_{i=1}^{n} (-1)^{i-1} \frac{\partial f_i}{\partial x_i}) dt. \end{aligned}$$

Hence we get $\frac{\partial g_{n,n+1}}{\partial x_n} = \int_0^{x_{n+1}} (\sum_{i=1}^{n} (-1)^{i-1} \frac{\partial f_i}{\partial x_i}) dt$. Therefore (18) is hold for any $n \geq 1$.

If ϕ is exterior differentiable, we have

$$|\frac{\partial g_{n,N+1}}{\partial x_n}| \leq |x_{n+1}| M, \tag{19}$$

by (16). Since $\sum |x_n|^2 < \infty$, $\sum g_{n,n+1} d^{\infty - \{n,n+1\}} x$ converges by (19). Hence THeorrem holds if $p = 1$.

Let $p \geq 2$ and $J = \{j_1, \ldots, j_p\}$, $j_1 < \cdots < j_p$ be a set of natural numbers. We give the lexicographic linear order to the set J. Let J' be the set $\{j_1, \ldots, j_{p-1}\}$ and write an $\infty - p$-form ϕ as follows:

$$\phi = \sum_{J'} \sum_{i > j_{p-1}} f_{\{J',i\}} d^{\infty - \{J',i\}} x. \tag{20}$$

Then formally $d\phi$ is given by

$$\begin{aligned} d\phi &= \sum_{J'} (\sum_{k \in J'} \frac{\partial f_{\{J',j\}}}{\partial x_k} dx_k \wedge d^{\infty - \{J',j\}} x) + \\ &\quad + \sum_{j > j_{p-1}} (-1)^{j+p} \frac{\partial f_{\{J',j\}}}{\partial x_j} d^{\infty - \{J',j\}} x. \end{aligned}$$

Hence if ϕ is exterior differentiable, there exist constants $M_{J'} > 0$ such that

$$|\sum_{j \leq N} (-1)^j \frac{\partial f_{\{J',j\}}}{\partial x_j}| < M_{J'}, \tag{21}$$

for all $N > j_{p-1}$. The following sum also converges.

$$\sum_{J'} \sum (-1)^j \frac{\partial f_{\{J',j\}}}{\partial x_j}. \tag{22}$$

Let ψ be an $(\infty - p - 1)$-form such that $d\psi = \phi$ and

$$\psi = \sum_{J'} \sum_{i > j_p - 1} g_{\{J',i,i+1\}} d^{\infty - \{J',i,i+1\}} x.$$

Then, since

$$\begin{aligned} d\psi &= \sum_{J'} ((\sum_{k < j_{p-1}, k \notin J'} (\pm \frac{\partial g_{\{J',k,j_{p-1}+1\}}}{\partial x_k} + \\ &\quad +(-1)^{j_{p-1}-p+1} \frac{\partial g_{\{J',j_{p-1}+1,j_{p-1}+2\}}}{\partial x_{j_{p-1}+2}}) d^{\infty - \{J',j_{p-1}+1\}} x) + \\ &\quad + \sum_{j > j_{p-1}+1} (-1)^{j+p} (\frac{\partial g_{\{J',j+1,j+2\}}}{\partial x_{j+2}} - \frac{\partial g_{\{J',j,j+1\}}}{\partial x_{j+2}})) d^{\infty - \{J',j+1\}} x, \end{aligned}$$

it must be

$$\begin{aligned} f_{\{J',j_{p-1}+1\}} &= (\sum_{k < j_{p-1}, k \notin J'} \pm \frac{\partial g_{\{J',k,j_{p-1}+1\}}}{\partial x_k}) + \\ &\quad +(-1)^{j_{p-1}-p+1} \frac{\partial g_{\{J',j_{p-1}+1,j_{p-2}+2\}}}{\partial x_{j_{p-1}+2}}, \end{aligned} \tag{23}$$

$$f_{\{J',j\}} = (-1)^{j+p} (\frac{\partial g_{\{J',j+1,j+2\}}}{\partial x_{j+2}} - \frac{\partial g_{\{J',j,j+1\}}}{\partial x_j}), \tag{24}$$

where $j > j_{p-1} + 1$, in (24). Since the right hand side of (23) is a finite sum, we set

$$f_{\{J',j_{p-1}+1\}} = f_{\{J',j_{p-1}+1\}} - \sum_{k < j_{p-1}, k \notin J'} \pm \frac{\partial g_{\{J',k_{p-1}+1\}}}{\partial x_k}.$$

Similar to the case $p = 1$, $g_{\{J',j,j+1\}}$, $j > j_{p-1} + 1$ are determined by

$$\begin{aligned} g_{\{J',j_{p-1}+1,j_{p-2}+2\}} &= \int_0^{x_{j_{p-1}+1}} f_{\{J',j_{p-1}+1\}} dt, \\ g_{\{J',j,j+1\}} &= \int_0^{x_j+1} (-1)^{j+p} (f_{\{J',,j\}} - \frac{\partial g_{\{J',j-1,j\}}}{\partial x_j}) dt, \ j > j_{p-1} + 1. \end{aligned}$$

Then by (21) and convegence of (22), ψ converges if ϕ is exterior differentiable. Hence we have Theorem.

Note. Theorem 1 shows $d^2 \neq 0$ on the space of $(\infty - p)$-forms. For example, let ψ be $\sum (1 - 1/2^n) x_n x_{n+1} d^{\infty - \{n,n+1\}} x$, then

$$d\psi = \sum (-1)^n \frac{x_n}{2^n} d^{\infty - \{n\}} x, \quad d^2\psi = -d^\infty x \neq 0.$$

Since we have $d(f\phi) = df \wedge \phi + fd\phi$, where f is a smooth function, we obtain

$$d^2(f\phi) = d^2f \wedge \phi - df \wedge d\phi + df \wedge d\phi + fd^2\phi = fd^2\phi.$$

Hence by induction, we get

$$d^{2n}(f\phi) = fd^{2n}\phi, \tag{25}$$
$$d^{2n+1}(f\phi) = df \wedge d^{2n}\phi + fd^{2n+1}\phi. \tag{26}$$

If $\phi = d\psi$, then ψ is exterior differentiable. Hence $\psi = dv$ for some v. That is $\phi = d^2 v$. By (25), by using smooth partition of unity, v exists globally. Hence if an $\infty - p$-form ϕ is exterior differentiable, then ϕ *is globally exact*.

4 Regularized exterior differential

We have defined the action G^s to the spaces W^k, *etc.* ([6]). We also define

$$G^{sG^t} : G^{sG^t} e_n = \mu_n^{s\mu_n^t} e_n. \tag{27}$$

G^{sG^t} acts on the space of infinite differential forms if s and t large. Explicitely, we ahve

$$G^{sG^t} d^{\infty - \{i_1,\ldots,i_p\}} x = \mu_{i_1}^{s\mu_{i_1}^t} \cdots \mu_{i_p}^{s\mu_{i_p}^t} \prod_{j=1}^{\infty} \mu_j^{s\mu_j^t} d^{\infty - \{i_1,\ldots,i_p\}} x. \tag{28}$$

Since Ray-Singer detrminant $detG$ is the analytic continuation of $\prod_{n=1}^{\infty} \mu_n^{\mu_n^t}$ to $t = 0$, we have by (28)

$$G^{sG^t} d^{\infty - \{i_1,\ldots,i_p\}} x|_{t=0} = \mu_{i_1}^s \cdots \mu_{i_p}^s (detG)^{-s} d^{\infty - \{i_1,\ldots,i_p\}} x. \tag{29}$$

Here $|_{t=0}$ means analytic continuation to $t = 0$. For simple, hereafter we use the notation

$$G^{s,*}\phi = \sum_I f_I G^{sG^t} d^{\infty - I} x|_{t=0}, \quad \phi = \sum_I f_I d^{\infty - I} x. \tag{30}$$

Definition. We define the regularized exterior differential $: d : \phi$ by

$$: d : \phi = d(G^{s,*}\phi)|_{s=0}. \tag{31}$$

Note 1. We may ignore the factor $(detG)^s$ in the definition of $G^{s,*}\phi$, because we are working on a flat space. If we work on a curved space, this factor might have meanings.

Note 2. We may define regularized exterior differential for finite degree forms. But in this case, we have $: d : \alpha = d\alpha$.

Example. Let ω be $\sum(-1)^{n-1}d^{\infty-\{n\}}x$. Then $d\omega$ diverges. But, since

$$G^{s,*}(\omega)=\sum_{n=1}^{\infty}(-1)^{n-1}\mu_n^s(detG)^{-s}x_nd^{\infty-\{n\}}x,$$

we have

$$:d:\omega=\zeta(G,s)(detG)^sd^{\infty}x|_{s=0}=\nu d^{\infty}x.$$

Similarly, we obtain

$$:d:(r^a\omega)=(a+\nu)d^{\infty}x,\quad r=\sqrt{\sum x_n^2}. \tag{32}$$

Especially, $:d:(r^{-\nu}\omega)$ is equal to 0 as expected.

For simple, we denote $G^{s,*}\omega=\omega(s)$. $\omega(s)$ is exterior differentiable if $s>d$. Formally, we have

$$\omega(s)=d\psi(s),\quad \psi(s)=\sum(-1)^n(\sum_{i=1}^{n}\mu_i^s)x_nx_{n+1}d^{\infty-\{n,n+1\}}x.$$

$\psi(s)$ converges if $s>d/2$. Therefore $\omega(s)$*,* $d\geq s>d/2$*, is not exterior differentiable, but exact.* In other word, the space of exact $(\infty-p)$-forms is wider than the space of exterior differentiable $(\infty-p)$-forms.

By definition, we have $G^{s,*}(G^{t,*}\phi)=G^{s+t,*}\phi$, we have

$$:d:(:d:\phi)=d^2G^{s+t,*}\phi|_{s=0,t=0}.$$

Hence to define $:d^m:\phi$ by $d^mG^{s,*}\phi|_{s=0}$, we have

$$:d^m:=(:d:)^m. \tag{33}$$

In [3], we defined formal adjoint δ of d by

$$\delta u^p=(-1)^p*^{-1}d*u^p,\quad \delta\phi^{\infty-p}=(-1)^p*^{\nu-p}d*\phi^{\infty-p}, \tag{34}$$

where $*$ is the Hodge operator defined in [2]. By (34), we define regularized formal adjoint of d by

$$:\delta:u^p=(-1)^p:d:*u^p,\quad :\delta:\phi^{\infty-p}=(-1)^p*^{\nu-p}:d:*\phi^{\infty-p}. \tag{35}$$

Then we have

$$:\triangle:=:d::\delta:+:\delta::d:, \tag{36}$$

where $:\triangle:$ is the regularized Laplacian defined in [6].

Note. Theorem 1 shows we can not expect to get de Rham theory by using $(\infty-p)$-forms. Precisely, denoting the spaces of $(\infty-p)$-forms, exterior differentiable

$(\infty-p)$-forms and closed $(\infty-p)$-forms on U, an open set of $W^{k-0}(finite)$, by $\mathcal{C}^{\infty-p}(U)$, $\mathcal{E}^{\infty-p}(U)$ and $\mathcal{B}^{\infty-p}(U)$, respectively, we have

$$\mathcal{C}^{\infty-p}(U) \supset d\mathcal{E}^{\infty-(p+1)}(U) \supset \mathcal{E}^{\infty-p}(U) \supset \mathcal{B}^{\infty-p}(U), \tag{37}$$

$$d\mathcal{E}^{\infty-(p+1)}(U) \cong \mathcal{E}^{\infty-(p+1)}(U)/\mathcal{B}^{\infty-(p+1)}(U). \tag{38}$$

We also denote $\mathcal{E}_k^{\infty-p}(U)$, $1 \le k \le p$, the space of $(\infty-p)$-forms on U such that d^k is defined. Then we have

$$\mathcal{E}^{\infty-p}(U) = \mathcal{E}_1^{\infty-p}(U) = d\mathcal{E}_2^{\infty-(p+1)}(U),$$
$$d\mathcal{E}^{\infty-(p+1)}(U)/\mathcal{E}^{\infty-p}(U) = d\mathcal{E}_1^{\infty-(p+1)}(U)/\mathcal{E}_2^{\infty-(p+1)}(U).$$

In general, since $\mathcal{B}^{\infty-p}(U) \subset \mathcal{E}_k^{\infty-p}(U)$ for all k, we get

$$d\mathcal{E}_k^{\infty-(p+1)}(U)/d\mathcal{E}_{k+1}^{\infty-(p+1)}(U) = \mathcal{E}_k^{\infty-(p+1)}/\mathcal{E}_{k+1}^{\infty-(p+1)}(U).$$

On the other hand, we have $\mathcal{E}_k^{\infty-q}(U) = d\mathcal{E}_{k-1}^{\infty-(q+1)}(U)$, $k \ge 2$. Hence to denote $d\mathcal{E}^{\infty-(p+1)}(U)/\mathcal{E}^{\infty-p}(U)$ by $\mathrm{F}^{\infty-p}(U)$, we obtain the descent formula

$$\mathrm{F}^{\infty-p}(U) \cong \mathcal{E}_k^{\infty-(p+k)}/\mathcal{E}_{k+1}^{\infty-(p+k)}(U). \tag{39}$$

We also introduce the kernel space $\mathcal{B}_k^{\infty-p}(U)$ of d^k. Then by the map $\phi \to d^k\phi$, we have

$$\mathcal{B}_{m-k}^{\infty-p+k}(U) \cong \mathcal{B}_m^{\infty-p}(U)/\mathcal{B}_k^{\infty-p}(U). \tag{40}$$

(39) and (40) may have relation to de Rham complexes with $d^N = 0$ (cf.[7]).

By using regularized exterior differential $: d :$, we define the spaces $\mathcal{E}_{reg}^{\infty-p}(U)$, $\mathcal{B}_{reg}^{\infty-p}(U)$ and $\mathcal{B}_{k,reg}^{\infty-p}(U)$, similarly. By definitions, $\mathcal{E}_{reg}^{\infty-p}(U)$ contains $\mathcal{B}_{reg}^{\infty-p}(U)$ and

$$: d :: \mathcal{E}_{reg}^{\infty-(p+1)}(U) \cong \mathcal{E}_{reg}^{\infty-(p+1)}(U)/\mathcal{B}_{reg}^{\infty-(p+1)}(U), \tag{41}$$

$$: d^k :: \mathcal{B}_{m-k,reg}^{\infty-p+k}(U) \cong \mathcal{B}_{m,reg}^{\infty-p}(U)/\mathcal{B}_{k,reg}^{\infty-p}(U). \tag{42}$$

But the relation between $: d : \mathcal{E}_{reg}^{\infty-(p+1)}(U)$ and $\mathcal{E}_{reg}^{\infty-p}(U)$ is not known.

5 Regularized integral of $(\infty-p)$-forms

To define regularization of infinite dimensional integral on a qube domain

$$Q(\mathbf{a}) = \{\sum x_n e_{n,k} | 0 \le x_n \le a_n\}, \quad \mathbf{a} = (a_1, a_2, \ldots),$$

contained in $W^{k-0}(finite)$, we use fractional integral

$$\int_0^a f(x) d^c x = \frac{1}{\Gamma(c)} \int_0^a (a-x)^{c-1} f(x) dx,$$

(cf. [4], [13]), and introduce the following operation.

$$
\begin{aligned}
I^{\mathbf{c}}_{Q(\mathbf{a})}(f) &= \lim_{n\to\infty}\Gamma(1+c_1)\int_0^{a_n}\big(\Gamma(1+c_2)\int_0^{a_{n-1}}\cdots \\
&\quad \cdots(\Gamma(1+c_1)\int_0^{a_1} f d^{c_1}x)\cdots d^{c_{n-1}}x\big)d^{c_n}x, \qquad (43)
\end{aligned}
$$

where $\mathbf{c}=(c_1,c_2,\ldots)$. We denote $\zeta(G,s\,G^t)$ instead of $\mathbf{c}$ if $c_1=\mu_1^{s\mu_1^t}$, $c_2=\mu_2^{s\mu_2^t}$, and so on. Then in [4], the regularized integral $\int_{Q(\mathbf{a})} f d^\infty : x :$ was defined by

$$
\int_{Q(\mathbf{a})} f d^\infty : x := (I^{\zeta(G,s^{G^t})}_{Q(\mathbf{a})}(f)|_{t=0})|_{s=0}. \qquad (44)
$$

Here f is a function on $Q(\mathbf{a})$ with suitable regularity. For example, we have

$$
\int_{Q(\mathbf{a})} 1 d^\infty : x :=: \prod a_n :, \qquad (45)
$$

where $:\prod a_n:$ is the regualrized infinite product defined in [4].

Note. For simple, we set

$$
: I :^{\zeta(G,s)}_{Q(\mathbf{a})} (f) = I^{\zeta(G,s^{G^t})}_{Q(\mathbf{a})}(f)|_{t=0}. \qquad (46)
$$

Then we have

$$
\int_{Q(\mathbf{a})} f d^\infty : x :=: I :^{\zeta(G,s)}_{Q(\mathbf{a})} (f)|_{s=0}. \qquad (47)
$$

This was the definition of $\int_{Q(\mathbf{a})} f d^\infty : x :$ in [4].

We apply this regularization procedure to justify physicists' calculation of the pathintegral

$$
\int_H \mathrm{e}^{-2\pi i(x,Dx)}\mathcal{D}x = \frac{1}{\sqrt{detD}}. \qquad (48)
$$

Here D is the positive nondegenerate selfadjoint elliptic operator whose Green operator is G. The proper values of D are $\mu_1^{-1},\mu_2^{-1},\ldots$. Since $\lim_{n\to\infty}\mu_n=0$, we assume $1>\mu_1\ge\mu_2\ge\ldots>0$, for simple. Then we have

$$
\lim_{s\to\infty}\zeta(G,s)=0. \qquad (49)
$$

Since $\mathrm{e}^{-2\pi(x,Dx)}=\prod \mathrm{e}^{-\mu_n^{-1}2\pi x_n^2}$, to compute $: I :^{\zeta(G,s)}_{Q(\mathbf{a})} (f)$, we need to compute

$$
\begin{aligned}
&\frac{\Gamma(1+\mu_n^s)}{\Gamma(\mu_n^s)}\int_0^{a_n}(a_n-x_n)^{\mu_n^s-1}\mathrm{e}^{-\mu_n^{-1}2\pi x_n^2}dx_n \\
=\ &\mu_n^s(\sqrt{\mu_n})^{\mu_n^s}\int_0^{b_n}(b_n-\xi)^{\mu_n^s-1}\mathrm{e}^{-2\pi\xi^2}d\xi, \quad b_n=\sqrt{\mu_n^s}a_n.
\end{aligned}
$$

Since

$$\lim_{s\to\infty} \mu_n^s \int_0^{b_n} (b_n - \xi)^{\mu_n^s - 1} \mathrm{e}^{-2\pi\xi^2} d\xi = \mathrm{e}^{-2\pi b_n^2},$$

$\lim_{s\to\infty} : I :_{Q(\mathbf{a})}^{\zeta(G,s)} (\mathrm{e}^{-(x,Dx)})$ exists, if $\sum a_n e_n \in H^-$.

Let $detD$ be the Ray-Singer determinant $\mathrm{e}^{-\zeta'(D,0)}$ of D. Then, since $-\zeta'(D,s) = -\zeta'(G,s)$, we have

$$\prod_{n=1}^{\infty} (\sqrt{\mu_n})^{\mu_n^s}|_{s=0} = \frac{1}{\sqrt{detD}}. \tag{50}$$

Hence to derive (48), it is sufficient to show

$$\lim_{b_n\to\infty} \prod_{n-1}^{\infty} \mu_n^s (2\int_0^{b_n} (b_n - x_n)^{\mu_n^s - 1} \mathrm{e}^{-2\pi x_n^2} dx_n)|_{s=0} = 1. \tag{51}$$

b_n's may tend to ∞ independently. But for simple, we set $b_n = r\mu_n^c$. Then, since

$$\lim_{r\to\infty} \lim_{s\to 0} \mu_n^s 2 \int_0^{b_n} (b_n - x)^{\mu_n^s - 1} \mathrm{e}^{-2\pi x_n^2} dx = 1,$$

to get (51), we need to take $c > 0$. This shows to derive (48) according to the regularization procedure proposed in [4], *path integral should be taken on* $W^{-d/2-c}$, $c > 0$ *is arbitrary.*

Since $2\int_0^\infty \exp(-2\pi x_n^2)dx = 1$ and $\lim_{s\to 0} \mu_n^s (b_n - x)^{\mu_n^s - 1} = 1$, to show (51), we need to evaluate $1 - \mu_n^s (b_n - x)^{\mu_n^s - 1}$. We note that

$$\log\bigl((b_n - x)^{\mu_n^s - 1}\bigr) = (\mu_n^s - 1)\log(b_n - x), \quad \mu_n^s - 1 = \sum_{m=1}^{\infty} \frac{(\log \mu_n)^m}{m!} s^m$$

Hence $(b_n - x)^{\mu_n^s - 1} - 1$ is a power series $\sum_{m\geq 1} c_m (s\log\mu_n)^m$, where c_m is a polynomial of $\log(b_n - x)$. If $b_n = r\mu_n^c$, then changing $\xi = x/\mu_n^c$, we may set

$$c_m(\log(b_n - x)) = \mu_n^c c_m(\log(r - \xi) + c\log\mu_n).$$

Precisely saying, our regularization procedure is consisted by the following two schemes

$$1 = \mu_n^s|_{s=0}, \quad \mu_n^s = \mu_n^{s\mu_n^t}|_{t=0}.$$

Acording to these schemes, we replace $\prod(c_n)$ by $\prod \mu_n^s(c_n)$ and rewrite

$$\prod_{n=1}^{\infty} \mu_n^s c_n = \prod_{n=1}^{\infty} \bigl(\mu_n^s - (\mu_n^s - \mu_n^s c_n)\bigr).$$

To show the convergence of this infinite product, it is sufficient to show the convergence of $\sum \mu_n^s (1 - c_n)$. Then, since $\zeta^{(k)}(G,s) = \sum (\log\mu_n)^k \mu_n^s$, we have

$$\sum_{n=1}^{\infty} \mu_n^s (b_n - x_n)^{\mu_n^s - 1} = \sum_{m=1}^{\infty} \sum_{k=1}^{m} c_{m,k} (\log r\bigl(s^k \zeta^{(m)}(s + c)\bigr)) + O(\frac{1}{\sqrt{r}}), \tag{52}$$

if $x_n < \sqrt{r}$. Since

$$\int_s qrtr^r (r-x)^{c-1} \mathrm{e}^{-2\pi x^2} dx < \frac{1}{c} r^{c+1} \mathrm{e}^{-r},$$

these estimates on $x_n, n = 1, 2, \ldots$ are sufficient to derive (50). Hence we can apply analytic continuation of $\zeta(G, s)$ and may conclude (51).

Note. Regularized integral can be dfined for $(\infty - p)$-forms. For example, let S^∞ be the sphere (or ellipsoid) in $W^{k-0}(finite)$ given by

$$\sum_{n=1}^{\infty} (\mu_n^{-d/2} x_n)^2 = 1, \quad \sum x_n e_{n,k} \in W^{k-0}(finite). \tag{53}$$

We consider regularized integral of $\omega = \sum (-1)^{n-1} x_n d^{\infty - \{n\}} x$ on S^∞. For this purpose, we set

$$r_N(x) = \sqrt{\sum_{n>N} (\mu_n^{-d/2} x_n)^2}, \quad N = 1, 2, \ldots.$$

Then we have

$$\omega = x_1 d^{\infty - \{1\}} x + \sum_{n \geq 2} \frac{\mu_n^{-d}}{\mu_1^{-d} x_1} d^{\infty - \{1\}} x = \frac{\mu_1^d}{x_1} d^{\infty - \{1\}} x,$$

on S^∞. Because $\sum \mu_n^{-d} x_n dx_n = 0$ on S^∞.

If $(x_1, x_2, \ldots) \in S^\infty$, then they satisfy

$$-\mu_1^{d/2} \sqrt{1 - r_1(x)^2} \leq x_1 \leq \mu_1^{d/2} \sqrt{1 - r_1(x)^2},$$
$$-\mu_2^{d/2} \sqrt{1 - r_2(x)^2} \leq x_2 \leq \mu_2^{d/2} \sqrt{1 - r_2(x)^2}, \ldots.$$

Hence calculation of regularized integral of ω on S^∞ is reduced tothe calculation of

$$\lim_{N \to \infty} \prod_{n \leq N} \Gamma(1 + \mu_n^s) \int_0^{\mu_N^{d/2} r_N(x)} \cdots \int_0^{\mu_1^{d/2} r_1(x)} \frac{2\mu_1^d}{x_1} d^{\mu_1^s} x_1 \cdots 2 d^{\mu_N^s} x_N. \tag{54}$$

Since we get

$$\begin{aligned}
& \int_0^{\mu_n^{d/2} r_n(x)} r_{n-1}(x)^c d^a x \\
= \; & \int_0^{\mu_n^{d/2 r_n(x)}} r_n(x)^c \Big(\sum (-1)^n \frac{c(c-1)\cdots(c-m+1)\mu_n^{-dm} x_m^2}{m! r_m(x)^m} \Big) d^a x \\
= \; & \sum (-1)^m \frac{c(c-1)\cdots(c-m+1)(2m)!}{m! \Gamma(2m+a+1)} \mu_n^{(d/2)a} r_n(x)^a,
\end{aligned}$$

by binary expansion. Hence computation of (54) is reduced to the computation of

$$\Gamma(1 + \mu_n^s) \int_0^{\mu_n^{d/2} r_n(x)} r_{n-1}(x)^{-1 + \mu_1^s + \cdots + \mu_{n-1}^s} d^{\mu_n^s} x. \tag{55}$$

Since we have

$$\begin{aligned}&\sum_{m=0}^{\infty}(-1)^m\frac{c(c-1)\cdots(c-m+1)(2m)!}{m!\Gamma(2m+a+1)}\\ = &\frac{1}{\Gamma(a)}\int_0^1(1-t)^{a-1}(1-t^2)^c dt,\end{aligned}$$

computation of the integral (55) is reduced to the computation of this last integral.

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