A REMARKABLE CLASS OF NATURAL METRIC QUARTER-SYMMETRIC CONNECTION ON A HYPERBOLIC KAHLERIAN SPACE

Jovanka Nikić and Nevena Pušić

Abstract

There is a wide class of natural metric quarter-symmetric connections on a hyperbolic Kaehlerian space, depending on their generators. But just few of them satisfy the most common curvature conditions, namely their curvature tensors are skew-symmetric in first pair of indices, invariant under changing places of first and second pair of indices and satisfy the first Bianchi identity. Under some conditions, this narrow class of connections can be "nearly F".

AMS Subject Classification: 53A30.
Key words: Isotropic vector field, harmonic vector field, Weyl's conformal curvature tensor.

1. On a hyperbolic Kaehlerian manifold

A hyperbolic Kaehlerian space (manifold) is an even-dimensional pseudo-Riemannian manifold with an F-structure satisfying

$$F_k{}^i F_j{}^k = \delta^i_j, \tag{1}$$

$$F_{ij} = F_k{}^k g_{kj} = -F_{ji}, \tag{2}$$

$$\overset{\circ}{\nabla}_k \, F_i{}^j = 0, \tag{3}$$

where $\overset{\circ}{\nabla}$ stands for the Levi-Civita covariant differentiation operator in the underlying Riemannian manifold. By (1.2), the isomorphism F sends any tangent vector into

Editor Gr.Tsagas *Proceedings of The Conference of Applied Differential Geometry - General Relativity and The Workshop on Global Analysis, Differential Geometry and Lie Algebras, 2002,* 96-101

an orthogonal one; as the structure has n linearly independent eigenvectors, they are null or isotropic. That is the main reason for being so geometrically different from (elliptic) Kaehlerian spaces, even if differences may formally be very small.

There are tangent vectors of positive scalar square, negative scalar square and null or isotropic vectors. Null vectors may not be eigenvectors for the structure. There are several kinds of special bases of tangent space, which we can use if we find it convenient.

2. A quarter-symmetric metric connection

According to Yano and Imai ([6]), if there is given a quarter-symmetric connection ∇ with torsion tensor

$$T^i_{jk} = p_j A_k{}^i - p_k A_j{}^i \tag{4}$$

(p_i being components of a 1-form and $A_k{}^i$ components of any (1,1) tensor) and if it should be a metric one

$$\nabla_k g_{ij} = 0, \tag{5}$$

then its components will have the following form

$$\Lambda^i_{jk} = \{^{\,i}_{jk}\} - p_k U_j{}^i + p_j V^i{}_k - p^i V_{jk}, \tag{6}$$

where

$$U_{ij} = \frac{1}{2}(A_{ij} - A_{ji});\ V_{ij} = \frac{1}{2}(A_{ij} + A_{ji}) \tag{7}$$

and, consequently

$$A_{ij} = U_{ij} + V_{ij}. \tag{8}$$

On a hyperbolic Kaehlerian space there are two fundamental tensors; one of them is symmetric and the other one is skew-symmetric. It is very natural to construct a quarter-symmetric metric connection over them. If $V_{ij} = g_{ij}$ and $U_{ij} = F_{ij}$, then

$$\Lambda^i_{jk} = \{^{\,i}_{jk}\} - p_k F_j{}^i + p_j \delta^i_k - p^i g_{jk}, \tag{9}$$

and these are components of *natural quarter symmetric metric connection*, which has been introduced in [5].

3. A special case of natural metric quarter-symmetric connection

We shall need some results from curvature theory of natural metric quarter-symmetric connections. All of them have been proved in [5]. The main symbol R denotes the curvature tensor, the Ricci tensor and scalar curvature of the natural metric quarter-symmetric connection. The main symbol K denotes the same quantities depending of Levi-Civita connection on the underlying pseudo-Riemannian space. Also symbols ∇ and $\overset{\circ}{\nabla}$ denote covariant differentiation towards these two connections respectively.

$$R_{ijkl} = K_{ijkl} - g_{il}p_{kj} + g_{ik}p_{lj} - g_{kj}p_{li} + g_{lj}p_{ki} + p_j p_l F_{ik} + p_i p_k F_{jl} - p_j p_k F_{il} - p_i p_l F_{jk}, \tag{10}$$

where p_{kj} stands for

$$\overset{\circ}{\nabla}_k\, p_j - p_k p_j + p_k q_j + \frac{1}{2} p_s p^s g_{kj}.$$

(p_j) is a gradient and (q_j) is ${F_j}^s p_s$. Further, in [5] it has been proved that

$$p_j p_l = \frac{1}{(n-2)(n-4)}[R_{jk} - (n-3)R_{kj}]{F_l}^k - \frac{1}{n-4} p_s p^s g_{jl}, \tag{11}$$

$$p_j q_k = \frac{1}{(n-2)(n-4)}[R_{jk} - (n-3)R_{kj}] + \frac{1}{n-4} p_s p^s F_{jk}, \tag{12}$$

$$p_{kj} = \frac{1}{n-2}[K_{jk} - R_{jk} - g_{jk}\frac{K-R}{2(n-1)}] - \frac{1}{(n-2)^2}(R_{kj} - R_{jk}) - \frac{1}{n-4} p_s p^s F_{jk}, \tag{13}$$

$$R_{jk}F^{jk} = 2(n-2)p_s p^s, \tag{14}$$

$$R_{kj}F^{jk} = -2(n-2)p_s p^s. \tag{15}$$

Here we shall consider such a special case of natural metric quarter-symmetric connection when $q_k = p_k$. Then $p_s p^s = 0$ and

$$p_{kj} = \overset{\circ}{\nabla}_k\, p_j, \tag{16}$$

$$R_{jk} = (n-2)p_j p_k, \tag{17}$$

$$R = 0, \tag{18}$$

$$R_{jk}F^{jk} = R_{kj}F^{jk} = 0. \tag{19}$$

We call such a connection a *remarkable* natural metric quarter-symmetric connection.

Any natural metric quarter-symmetric connection with an isotropic generator has a curvature-like tensor invariant ([5]). Such a tensor equals to Weyl's conformal curvature tensor of the underlying pseudo-Riemannian space:

$$R_{ijkl} - \frac{1}{n-2}[g_{il}R_{jk} - g_{ik}R_{jl} + g_{jk}R_{il} - g_{jl}R_{ik} - \frac{R}{n-1}(g_{il}g_{jk} - g_{ik}g_{lj})] - \frac{1}{(n-2)^2}[g_{il}(R_{kj} - R_{jk}) - g_{ik}(R_{lj} - R_{jl}) + \tag{20}$$

$$+g_{kj}(R_{li}-R_{il})-g_{lj}(R_{ki}-R_{ik})]-$$
$$-\frac{1}{(n-2)(n-4)}\{[R_{js}-(n-3)R_{sj}]F_l{}^sF_{ik}$$
$$+[R_{is}-(n-3)R_{si}]F_k{}^sF_{jl}-[R_{js}-(n-3)R_{si}]F_k{}^sF_{il}$$
$$-[R_{is}-(n-3)R_{si}]F_l{}^sF_{jk}\}.$$

By (3.8), we have proved

Theorem 1. *The Ricci tensor of a remarkable natural metric quarter-symmetric connection is symmetric.*

By (3.8), we have

$$R_{js}F_l{}^s=(n-2)p_jp_sF_l{}^s=(n-2)p_jp_l=R_{jl}. \tag{21}$$

Then, we have proved

Theorem 2. *The invariant (3.11) in case of remarkable connection has the form:*

$$R_{ijkl}-\frac{1}{n-2}[R_{jk}(g_{il}-F_{il})-R_{jl}(g_{ik}-F_{ik}) \tag{22}$$
$$+R_{il}(g_{jk}-F_{jk})-R_{ik}(g_{jl}-F_{jl})].$$

The tensor (3.13) equals to the Weyl's conformal curvature tensor of the underlying pseudo-Riemannian space.

4. A nearly F-connection

It is easy to see that a natural metric quarter symmetric connection is not an F-connection. Namely,

$$\nabla_kF_{ij}=-p_iF_{kj}-p_jg_{ik}-p_jF_{ik}+p_ig_{jk}. \tag{23}$$

Now, we want to find conditions for a remarkable connection to be a nearly F- connection, that means

$$\nabla_l\nabla_kF_{ij}-\nabla_k\nabla_lF_{ij}=0. \tag{24}$$

(4.2) is equivalent to

$$-R^s{}_{ikl}F_{sj}-R^s{}_{jkl}F_{is}=T^s_{kl}\nabla_sF_{ij}. \tag{25}$$

The right-hand side of (4.3)equals to zero by a straightforward calculation. Then, using (3.1), we obtain

$$-R^s{}_{ikl}F_{sj}-R^s{}_{jkl}F_{is}=g_{ki}(\overset{\circ}{\nabla}_l\,p_j+p_lp_j) \tag{26}$$
$$-g_{li}(\overset{\circ}{\nabla}_k\,p_j+p_kp_j)+g_{kj}(\overset{\circ}{\nabla}_l\,p_i+p_lp_i)$$

$$-g_{lj}(\overset{\circ}{\nabla}_k\, p_i + p_k p_i) + F_{ki}(\overset{\circ}{\nabla}_l\, p_j + p_l p_j) - F_{li}(\overset{\circ}{\nabla}_k\, p_j + p_k p_j)$$
$$+F_{jk}(\overset{\circ}{\nabla}_l\, p_i + p_l p_i) - F_{jl}(\overset{\circ}{\nabla}_k\, p_i + p_k p_i) = 0.$$

As the generator p_i is supposed to be a gradient, then the tensor $M_{li} = \overset{\circ}{\nabla}_l\, p_i + p_i p_l$ is symmetric. Transvecting (4.4) by F^{kj} and using the fact that p_i is an eigenvector for the structure, we obtain

$$(n-2)(\overset{\circ}{\nabla}_l\, p_j + p_j p_l) = 0,$$

or

$$\overset{\circ}{\nabla}_l\, p_j = -p_j p_l. \tag{27}$$

Then $\overset{\circ}{\nabla}_a\, p^a = 0$ and, using (3.4), $K = 0$.. Using (3.4) again, we obtain

$$K_{ij} = 0. \tag{28}$$

So, we have proved

Theorem 3. *If a remarkable natural metric quarter-symmetric connection is nearly F, then the underlying pseudo-Riemannian space is Ricci-flat.*

In such a case, Weyl's conformal curvature tensor of the underlying Riemannian space reduces to its curvature tensor and we have, by (3.13)

$$R_{ijkl} - \frac{1}{n-2}[R_{jk}(g_{il} - F_{il}) - R_{jl}(g_{ik} - F_{ik}) \tag{29}$$
$$+R_{il}(g_{jk} - F_{jk}) - R_{ik}(g_{jl} - F_{jl})] = K_{ijkl}.$$

Taking into account (3.8), we can prove by a straightforward calculation:

Theorem 4. *The curvature tensor of a remarkable natural metric quarter-symmetric connection which is nearly F is skew-symmetric in first pair of indices, invariant under changing places of first and second pairs of indices and satisfy the first Bianchi identity.*

Also, there holds

Theorem 5. *The curvature tensor of a remarkable metric quarter-symmetric connection which is nearly F, never vanishes.*

In the case of the curvature tensor vanishing, the Ricci tensor would also vanish and so would the generator by (3.8).

If the generator is not an eigenvector for the structure ([5]), the Ricci tensor of such a natural metric quarter-symmetric connection can never been symmetric and its curvature tensor can never satisfy the most common curvature conditions, given in Theorem 5.

Finally, there holds

Theorem 6. *A remarkable natural metric quarter-symmetric connection on a hyperbolic Kaehlerian space can be a nearly F connection if there exists an eigenvector for the structure which is a gradient, with covariant derivative of the form (4.5), for the Levi-Civita connection.*

References

[1] S. Gołąb, *On semi-symmetric and quarter-symmetric linear connections*, Tensor, N. S., 29 (1975) 249-254

[2] R. S. Mishra and S. N. Pandey, *On quarter-symmetric metric F-connections*, Tensor, N. S., 34 (1980) 1-7

[3] M. Prvanović and N. Pušić, *On manifolds admitting some semi-symmetric connections*, Indian Journal of Mathematics, 37, 1 (1995) 37-67

[4] M. Prvanović and N. Pušić, *On Kaehler manifolds with a kind of semi-symmetric F-connection*, Indian Journal of Mathematics (to appear)

[5] N. Pušić, *On quarter-symmetric metric connections on a hyperbolic Kaehlerian space*, Publ. de l' Inst. Math. (to appear)

[6] K. Yano and T. Imai, *Quarter-symmetric connections and their curvature tensors*, Tensor, N. S., 38 (1982) 13-18

Authors' addresses:

Jovanka Nikić
Faculty of Technical Science,
University of Novi Sad,
Novi Sad, Yugoslavia,
E-mail: nikic@uns.ns.ac.yu

Nevena Pušić
Faculty of Science,
University of Novi Sad,
Novi Sad, Yugoslavia,
E-mail: nevena@im.ns.ac.yu