ON FINSLER BUNDLES AND FINSLER CONNECTIONS

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Abstract

The Finsler bundles associated with two vector bundles over the same base and the corresponding various Finsler connections are defined in the paper. The constructions of Matsumoto are obtained as particular cases. In the case of a single vector bundle, linear cases considered by A. Bejancu and D. Opriş are obtained. For a Lagrange metric on a subbundle of a tangent bundle, a construction of V. Oproiu and N. Papaghiuc is obtained.

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Let $\xi' = (E', \pi_1, M)$ and $\xi'' = (E'', \pi_2, M)$ be two vector bundles over the same base M which have the fibres $I\!R^{k_1}$ and $I\!R^{k_2}$, respectively. Let $L(\xi') = (L(E'), M, GL(k_1, I\!R))$ and $L(\xi'') = (L(E''), M, GL(k_2, I\!R))$ be the principal frames bundles associated with the above vector bundles, and let π' and π'' be their canonical projections. Let

$$L\xi'(\xi'') = (LE'(E'') = \pi_1^* L(E''), E', GL(k_2, I\!R)),$$

and

$$L\xi''(\xi') = (LE''(E') = \pi_2^* L(E'), E'', GL(k_1, I\!R)),$$

be the induced principal bundles, and let $p_1$and p_2 be their canonical projections. Thus, the total spaces are:

$$LE'(E'') = \{(e', z'') \in E' \times L(E'') \mid \pi_1(e') = \pi''(z'')\}$$

$$LE''(E') = \{(e'', z') \in E'' \times L(E') \mid \pi_2(e'') = \pi'(z')\}.$$

We denote by $q_1 : LE'(E'') \to L(E'')$ and $q_2 : LE''(E') \to L(E')$ the canonical projection on the second factors.

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Let $\varphi : E' \to E''$ be a vector bundle morphism. Then a differentiable map of differentiable manifolds:

$$\Phi : LE'(E'') \to L(E'') \times I\!R^{k_2}, \Phi(e', z'') = \left(z'', (z'')^{-1}(\varphi(e'))\right).$$

is induced.

It is easy to prove that the map Φ is an injective immersion, or a surjective submersion, or a diffeomorphism, if φ respectively is. For example:

1. If ξ' is a subbundle of ξ'' and φ is the inclusion morphism, then Φ is an injective immersion. If there exists a splitting of the inclusion: $C : \xi'' \to \xi', C \circ \varphi = id_{E'}$, then it is easy to see that the map $\Phi_1 : L(E'') \times I\!R^{k_2} \to LE'(E'')$, $\Phi_1(z'', v) = (C \circ z''(v), z'')$ is a left inverse of Φ.

2. If $\xi' = \xi$, let $\varphi = id_{E'}$, then $\Phi : LE'(E') \to L(E') \times I\!R^{k_1}$ is a diffeomorphism. This case was studied in [5]. An important particular case, studied by Matsumoto, is obtained when the principal bundle is the Finsler principal bundle $F = L\tau M(\tau M)$. The diffeomorphism Φ defines, for every vector $\bar{v} \in I\!R^{k_1}$, a vector field $Y(\bar{v}) \in \mathcal{X}(LE'(E'))$, called an *induced-fundamental* vector field. We denote also as $\varepsilon : LE'(E') \to I\!R^{k_1}$ the map obtaind from Φ, projecting on the second factor.

Returning to the general case, we define now some distributions on the manifolds $LE'(E'')$ and $LE''(E')$. Since the constructions are symmetric, we define the vertical distribution for the bundle $L\xi'(\xi'')$.

- The *vertical distribution* $\ker p_{1*} \overset{not}{=} L^V\xi'(\xi'')$ has $L^V E'(E'')$ as total space. If no confusion arise, we denote by $\mathcal{F}^V$ the vertical distribution and $L\xi'(\xi'')$ as $\mathcal{F}$.

- The *quasi-vertical* distribution $\ker(\pi_1 \circ p_1)_* \overset{not}{=} L^q\xi'(\xi'')$ which has as total space $L^q E'(E'')$. It is easy to see that $L^V E'(E'') \subset L^q E'(E'')$. If no confusion arise, we denote by $\mathcal{F}^q$ the quasi-vertical distribution.

- The *vertical induced distribution* $\ker q_{1*} \overset{not}{=} L^i\xi'(\xi'')$ which has as total space $L^i E'(E'')$. If no confusion arise, we denote by $\mathcal{F}^i$ the vertical induced distribution.

Observation 1 If $X_u \in T_u\mathcal{F}$, then $X_u = 0$ iff $p_{1*}(X_u) = 0$ and $q_{1*}(X_u) = 0$, since p_1 and q_1 are the canonical projections of the fibered product $\mathcal{F}$.

Observation 2 If $\xi' = \xi'' = \xi$,then an induced-fundamental vector field $Y(\bar{v})$ is tangent to the vertical induced distribution and these vector fields generate the vector fields tangent to this distribution.

Proposition 1 *For every $u \in \mathcal{F}$ we have $\mathcal{F}^q_u = \mathcal{F}^V_u \oplus \mathcal{F}^i_u$.*

The group $GL(k_2, I\!R)$ acts on $LE'(E'')$, as the total space of a principal bundle, by the natural action:

$$((e', z''), g) \to (e', z''g),\ (\forall)(e', z'') \in LE'(E''), g \in I\!R^{k_2}$$

Proposition 2 *Considering the natural right action of $GL(k_2, I\!R)$ on $LE'(E'')$, the distributions $\mathcal{F}^V_u$, $\mathcal{F}^i_u$ and $\mathcal{F}^q_u$ are invariated by this action.*

1 Usual connections in Finsler bundles

A connection Γ on the principal bundle $L\xi'(\xi'')$ can be defined, as usual:

by the horizontal distribution $\mathcal{H}$ on $LE'(E'')$, invariated by the natural right action of $GL(k_2,I\!R)$ and complementary to the vertical distribution, or

by a 1-differential form ω on $LE'(E'')$, which takes the values in $gl(k_2,I\!R)$, and having the property:

$$\omega(\widetilde{A}) = A,\ \omega \circ R_g = ad(g^{-1})\omega, \tag{1}$$

$(\forall)A \in gl(k_2,I\!R), g \in GL(k_2,I\!R)$, and where $\widetilde{A}$ is the fundamental vertical field associated with A.

Notice that $\mathcal{H}$ is connected to ω by $\mathcal{H} = \ker\omega$.

Consider now the following adapted coordinates to the bundles structures, taken on the following manifolds

on M: $(x^i), i = \overline{1,m}$;

on E': (x^i, y^α), on E'': (x^i, t^u), $\alpha = \overline{1,k_1}$, $u = \overline{1,k_2}$;

on $L(E')$: $(x^i, (w^\alpha_\beta))$, on $L(E'')$: $(x^i, (z^u_v))$;

on $LE'(E'')$: $(x^i, y^\alpha, (z^u_v))$, on $LE''(E')$: $(x^i, t^u, (w^\alpha_\beta))$.

Let $\{E^u_v\}_{u,v=1,k_2}$ be a base in the Lie algebra $gl(k_2,I\!R)$. Using this base, the connection form ω is written in the form:

$$\omega = \sum_{u,v=1}^{k_2} \omega^u_v E^v_u.$$

Considering the above adapted coordinates, we have the following local expression for the forms ω^u_v:

$$\omega^u_v = \omega^u_{vi}dx^i + \omega^u_{v\alpha}dy^\alpha + \omega^{us}_{vt}dz^t_s.$$

Taking into account the second relation (1), it follows that

$$\Gamma^s_{ti} = z^t_u\omega^u_{vi}(z^{-1})^v_t,\ C^s_{t\alpha} = z^t_u\omega^u_{v\alpha}(z^{-1})^v_t$$

are local functions which depend only on the coordinates (x^i) and (y^α), thus

$$\omega^u_v = (z^{-1})^u_s(dg^s_v + z^t_v(\Gamma^s_{ti}(x^i,y^\alpha)dx^i + C^s_{t\alpha}(x^i,y^\alpha)dy^\alpha)). \tag{2}$$

Proposition 3 *The systems of local fields*

$$\left\{\frac{\partial}{\partial x^i} - \Gamma^s_{ti}z^t_r\frac{\partial}{\partial z^s_r}\ ,\ \frac{\partial}{\partial y^\alpha} - C^s_{t\alpha}z^t_r\frac{\partial}{\partial z^s_r}\right\}\ \begin{matrix} i = \overline{1,m} \\ \alpha = \overline{1,k_1}\end{matrix}$$

are local bases for the horizontal fields.

It follows that a horizontal vector field Z has the following local expression, using adapted coordinates:

$$Z = Z^i \frac{\partial}{\partial x^i} + Z^\alpha \frac{\partial}{\partial y^\alpha} + Z^t_s \frac{\partial}{\partial z^t_s}$$

It follows also that an horizontal field Z projects (locally) on the horizontal distribution by

$$hZ = Z^i \left(\frac{\partial}{\partial x^i} - \Gamma^s_{ti} z^t_r \frac{\partial}{\partial z^s_r} \right) + Z^\alpha \left(\frac{\partial}{\partial y^\alpha} - C^s_{t\alpha} z^t_r \frac{\partial}{\partial z^s_r} \right)$$

and on the vertical distribution by

$$vZ = \left(Z^i \Gamma^s_{ti} z^t_r + Z^\alpha C^s_{t\alpha} z^t_r + Z^t_s \right) \frac{\partial}{\partial z^t_s} \ .$$

For an arbitrary vector field $W \in \mathcal{X}(E')$ it can be defined the *horizontal lift* $W^h \in \mathcal{X}(LE'(E''))$, which is an horizontal field, defined by the isomorphism of the fibres of the bundles $\tau(E')$ and $LE'(E'')$ (the horizontal bundle associated to the connection). Using the above local bases, the local expression of this lift is, for

$$W = W^i \frac{\partial}{\partial x^i} + W^\alpha \frac{\partial}{\partial y^\alpha},$$

given by the formula:

$$W^h = W^i \left(\frac{\partial}{\partial x^i} - \Gamma^s_{ti} z^t_r \frac{\partial}{\partial z^s_r} \right) + W^\alpha \left(\frac{\partial}{\partial y^\alpha} - C^s_{t\alpha} z^t_r \frac{\partial}{\partial z^s_r} \right).$$

2 Special connections on Finsler bundles

Definition 1 An *horizontal connection* on $L(\xi'(\xi''))$ is a distribution $\Gamma^h = \Gamma^h \xi'(\xi'')$ on $\mathcal{F} = LE'(E'')$ which satisfies the condition

$$1) \mathcal{F}_u = \Gamma^h_u \oplus \mathcal{F}^q_u \ , \ 2) R_{g*} \Gamma^h_u = \Gamma^h_{ug}$$

for every $u \in \mathcal{F}$, where $\mathcal{F}^q = L^q E'(E'')$ is the quasi-vertical distribution, and R_g is the right action of $g \in GL(k_2, I\!R)$.

Observation 3 From Proposition 1 it follows that $(\forall) u \in \mathcal{F}$ we have

$$\mathcal{F}_u = \Gamma^h_u \oplus \mathcal{F}^i_u \oplus \mathcal{F}^V_u \, . \tag{3}$$

Using Proposition 2, it follows that the distribution $\mathcal{F}^i$ is invariated by the action of $GL(k_2, I\!R)$, thus the distribution $\Gamma^h \oplus \mathcal{F}^i$ defines a connection on the Finsler bundle $F\xi'(\xi'')$.

Definition 2 A *vertical connection* on $L\xi'(\xi'')$ is a distribution $\Gamma^V = \Gamma^V \xi'(\xi'')$ on $\mathcal{F} = LE'(E'')$ which satisfies the condition

$$1) \mathcal{F}^q_u = \Gamma^V_u \oplus \mathcal{F}^V_u \ , \ 2) R_{g*} \Gamma^V_u = \Gamma^V_{ug}$$

for every $u \in \mathcal{F}$, where $\mathcal{F}^V = L^V E'(E'')$ is the total space of the vertical distribution.

For example, according to Propositions 1 and 2, the induced vertical distribution $\mathcal{F}^i$ defines a vertical connection on $\mathcal{F}$.

Observation 4 As in Observation 3, if Γ^V is a vertical connection and Γ^h is an horizontal connection, then $\Gamma^h \oplus \Gamma^V$ is a connection on the Finsler bundle $\mathcal{F}$.

Lemma 1 *Let $\pi_1 : E_1 \to M$ and $\pi_2 : E_2 \to M$ be two surjective submersions, $E = E_1 \times_M E_2 = \{(e_1, e_2) \in E_1 \times E_2 \mid \pi_1(e_1) = \pi_2(e_2)\}$ be the fibered product of π_1 and π_2, and $p_1 : E \to E_1$, $p_2 : E \to E_2$ be the canonical projections.*
Then
$$p_{1*}(\ker p_{2*}) = \ker \pi_{1*}\ .$$

Returning to the setting before the above Lemma, we have:

Proposition 4 *The differential p_{1*} of the canonical projection $p_1 : LE'(E'') \to E'$ sends:*

the vertical induced distribution Γ^i onto the vertical distribution of E', and

the distribution Γ^h, which corresponds to an horizontal connection on $LE'(E'')$, onto the horizontal distribution of a non-linear connection on the vector bundle ξ'.

Definition 3 A *Finsler connection* associated with E' and E'' is a pair (Γ, N), where Γ is a connection in the Finsler bundle $LE'(E'')$, and N is a non-linear connection on the vector bundle E'.
If $E' = E''$, we say that a such connection is a Finsler connection on E'.

Theorem 1 *There is an one-to-one correspondence between Finsler connections (Γ, N), associated with the vector bundles E' and E'', and the set of pairs (Γ^h, Γ^V), where Γ^h is an horizontal connection and Γ^V is a vertical connection on $\mathcal{F} = LE'(E'')$.*

The converse association is performed using Observation 4 and Proposition 4.

3 Induced Finsler connection on subbundles

Let $\xi = (E, \pi, M)$ be a vector subbundle of the vector bundle $\xi' = (E', \pi_1, M)$, and $i : E \to E'$ be the inclusion morphism of ξ in ξ'. Consider now the i-morphism of the tangent bundles $i_* : TE \to TE'$. It induce the morphism, denoted in the same way:

$$i_* : TE \to i^*(TE') \tag{4}$$

which is an injective morphism of vector bundles on the base E. It enables us to consider τE as a vector subbundle of $i^*(\tau E')$. Notice that the fibres of the vector subbundle $V\xi$ (of τE) are carried by i_* in the fibres of the vector subbundle $i^*(V\xi')$ (of $\tau E'$). It is induced an injective morphism

$$i'_* : VE \to i^*(VE'). \tag{5}$$

The well-known canonical isomorphisms

$$r : VE \to \pi^* E,\ r' : VE' \to \pi_1^* E'$$

enable us to write the morphism (5) as

$$i'_* : \pi^* E \to i^*(\pi_1^* E'),$$

modulo the corresponding identifications.

But $i^*(\pi_1^* E') = (\pi_1 \circ i)^* E' = \pi^* E'$, thus the morphism (5) can be written, modulo canonical isomorphisms, as

$$i'_* : \pi^* E \to \pi^* E' \tag{6}$$

having the form $(e'_1, e'_2) \to (e'_1, e'_2)$.

Definition 4 We say that a splitting of the injective morphism (5) or, equivalently, (6), is a *Finsler splitting* of the vector subbundle ξ of ξ'.

Examples

1. Consider a metric tensor on the vector bundle $V\xi'$ (i.e. a $(0,2)$ tensor which is symmetric and (strict) positive defined). It induces on the vector bundle $i^*(V\xi')$ also a metric tensor, thus the subbundle $V\xi^\perp$ of $i^*(V\xi')$, orthogonal to the subbundle $V\xi$ can be considered. It induces a Finsler splitting of the inclusion $i : \xi \to \xi'$.

2. A particular case of the above example is given as follows. Consider a submanifold M of the manifold M', given by the imbedding $\varphi : M \to M'$, and take $i = \varphi_* : TM \to \varphi^*(TM)'$, $\xi = \tau M$ and $\xi' = \varphi^*(\tau M')$. Then a Finsler or, more general, a Lagrange metric on M' carries a metric tensor on the vector bundle $V\tau M \cong \pi^* \tau M$ (where $\pi : \tau M \to M$ is the canonical projection), which is the so-called Finsler bundle of M. Thus $(V\tau M)^\perp$ is the orthogonal bundle of $V\tau M$ in the vector bundle

   $$i^*(V(\varphi^*(\tau M'))) \cong i^*(\pi_0^*(\varphi^*(\tau M'))) \cong (\varphi \circ \pi_0 \circ i)^*(\tau M') = (\varphi \circ \pi_1)^*(\tau M')$$

   where $\pi_0 : \varphi^*(TM') \to M$ is the canonical projection of the vector bundle $\varphi^*(\tau M')$.

   This construction is used in the study of the Finsler subbundles in [1, 2, 4, 6].

Theorem 2 *Let $\xi = (E, \pi, M)$ be a vector subbundle of the vector bundle $\xi' = (E', \pi', M)$, $i : \xi \to \xi$ be the inclusion morphism and (Γ', N') be a Finsler connection on the Finsler bundle $L\xi'(\xi')$ of ξ'.*

Then every Finsler splitting of the inclusion i induces a Finsler connection on the Finsler Bundle $L\xi(\xi)$ of ξ.

Corollary 1 *Let M be a submanifold of the manifold M and (Γ', N') be a Finsler connection on $\tau M'$.*

Then every Finsler splitting of the inclusion of τM in $i^* \tau M$ induces a Finsler connection (Γ, N) on τM, where $i : M \to M'$ is the inclusion.

In the case of a Finsler or Lagrange metric on M', a Finsler splitting can be defined on the submanifold M, as in the Example 2) from above. It can be shown that in this case the Finsler connection induced on M is the same as that induced in [1, 2], in the linear case.

In the case when (M, L) is Lagrange space on the manifold M, $\xi' = \tau M$ and ξ is a vector subbundle of τM, the induced Finsler connection on ξ from [6] is obtained.

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