Classification of Semi-Riemannian Almost Product Structures

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Abstract

A.M. Naveira classified ([2]) the Riemannian almost product structures, in 64 classes (invariant under some action of the orthogonal group). We extended ([6]) this classification to the semi-Riemannian almost product manifolds, where the invariance is considered with respect to the pseudo-orthogonal group. A key tool is to study symmetries of the covariant derivative of the almost product tensor fiels, using the Levi-Civita connection.

In this paper, we describe how similar classifications arrise when, instead of the Levi- Civita connection, we use only metric connections (eventually with torsion).

Particularization for the Lorentzian framework is also provided.

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1 Introduction

The human sense of order (fighting against entropy) produces an endless need for classification. Usually, the geometrical structures are too weak to produce strongly enough invariants, in order to lead toward final classifications (up to an homeomorphism, diffeomorphism, isometry, etc). So, the classification processus gets a broader and less ambitious goal: to define (and discover examples of) more and more specific families of manifolds, as new living species waiting for a further "Linné classification".

In [2], Naveira classified the Riemannian almost product manifolds in 64 classes, using the symmetries of a tensor associated to the almost product structural field, covariantly derived with respect to the Levi-Civita connection. With some minor changes in proof, we extended this classification for arbitrary signatures [6].

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For the sake of completeness, we give here (§2) the sketch of that proof, with emphasis on the differences produced by the arbitrary signature.

Then, we refine the respective result: first, we remark that we can extend the framework to arbitrary metric connections, instead of torsion-less (Levi-Civita's) ones. The classification result is the same. This shows the classification is invariant not only with respect to a canonical product of semi-orthogonal groups, but also within the class of metric connections.

Finally, we particularize the previous context to Lorentzian almost product manifolds (§3). Here, we prove also a new existence result: *on any non-compact 4-manifold, there exists a Lorentzian almost product structure belonging to the Naveira class (F,F).*

2 Naveira-like classification of semi-Riemannian almost product manifolds

Let (M,g) be an n-dimensional semi-Riemannian manifold, of index ν, and P an almost product tensor field on M, compatible with g; so, P is a $(1,1)$-tensor field, satisfying $P^2 = Id$ and $g(PX,PY) = g(X,Y)$, for every $X,Y \in \mathcal{X}(M)$. The triple (M,g,P) is called a semi-Riemannian almost product manifold. We denote by $\mathcal{D}$ and $\mathcal{D}^\perp$ the global, orthogonal, complementary distributions associated to the P-eigenvalues 1 and (-1), respectively. We denote the corresponding projectors $V = (Id-P)/2$ and $V^\perp = (Id+P)/2$. We have the obvious relations

$$\mathcal{D} = KerV^\perp \quad , \quad \mathcal{D}^\perp = KerV$$

$$V \circ V^\perp = V^\perp \circ V = 0 \quad , \quad V + V^\perp = Id$$

In each point $p \in M$, the subspaces $\mathcal{D}_p$ and $\mathcal{D}^\perp$ are nondegenerated with respect to the induced metric. Denote their signatures by ν_1 and ν_2, respectively. Denote $n_1 = dim\mathcal{D}_p$ and $n_2 = dim\mathcal{D}_p^\perp$. We have $n_1 + n_2 = n$ and $\nu_1 + \nu_2 = \nu$.

Remark 2.1 *(i) The almost product structure invariates the causal character of vectors (transforms spacelike vectors into spacelike vectors, etc). The structural group of the manifold is $O_{\nu_1}(n_1) \times O_{\nu_2}(n_2)$, where the factors are pseudo- orthogonal groups.*

(ii) The $(0,2)$-tensor field $\Phi(X,Y) = g(PX,Y)$ has the symmetry properties

$$(\nabla_X\Phi)(Y,Z) = g((\nabla_X P)(Y,Z)$$

$$(\nabla_X\Phi)(Y,Z) = (\nabla_X\Phi)(Z,Y) \quad , \quad (\nabla_X\Phi)(PY,PZ) = (\nabla_X\Phi)(Y,Z)$$

(iii) In a fixed point $p \in M$, the tangent space T_pM is endowed with a scalar product of signature ν. The properties of $\nabla\Phi$ are those of a 3-times covariant tensor, with the previous symmetries. The set of all these tensors admits an invariant decomposition under the action of $O_{\nu_1}(n_1) \times O_{\nu_2}(n_2)$, which will be exposed next, in a purely algebraic framework ([2], [6]).

Let $\mathcal{T}$ be a semi-euclidean vector space, endowed with a scalar product $<,>$ of signature ν. Let P be an involution of $\mathcal{T}$, i.e. $P^2 = Id$. Suppose $dim T = n$; V and H two orthogonal subspaces of dimensions n_1 and n_2, corresponding to the eigenvalues (-1) and 1 respectively, such that $T = V \oplus H$. The restrictions of $<,>$ on V and H have signatures ν_1 and ν_2 respectively. (In these conditions, it follows that V and H are non-degenerate subspaces with respect to $<,>$).

We denote by $a, b, c, \ldots \in \{1,2,\ldots,n_1\}$; $u, v, w, \ldots \in \{1,2,\ldots,n_2\}$; $i, j, k, \ldots \in \{1,2,\ldots,n\}$; $A, B, C, \ldots$"vertical" vectors in V; $X, Y, Z, \ldots$ "horizontal" vectors in H; $K, L, N, ..$ arbitrary vectors in T. We denote T^* the dual space of T and

$$W = \{\alpha \in (T^*)^3 \mid \alpha(K,L,N) = \alpha(K,N,L) = -\alpha(K,PL,PN)\}$$

the subspace of 3-linear applications with the same symmetries as $\nabla\Phi$. The relation $\alpha(K,L,N) = -\alpha(K,PL,PN)$ is equivalent to

$$\alpha(K,A,B) = \alpha(K,X,Y) = 0 \tag{1}$$

Let $\{e_i\}_{i=\overline{1,n}} = \{e_1,\ldots,e_{n_1},f_1,\ldots,f_{n_2}\}$ (so $f_u = e_{n_1+u}$, for $u = \overline{1,n_2}$), an orthonormal basis for T, where $\{e_a\}_a$ is an orthonormal basis for V and $\{f_u\}_u$ is an orthonormal basis for H, with respect to the induced scalar products. We define the applications

$$(.)^v, (.)^h, \underline{(.)} : W \to T^* \quad , \quad \alpha^v(K) = \sum_{a=1}^{n_1} \alpha(e_a,e_a,K)$$

$$\alpha^h(K) = \sum_{u=1}^{n_2} \alpha(f_u,f_u,K) \quad , \quad \underline{\alpha} = \alpha^v + \alpha^h$$

We have $\alpha^v(A) = \alpha^h(X) = 0$. On W one defines a (positively defined) scalar product

$$(\alpha,\beta) = \sum_{i,j,k=1}^{n} \alpha(e_i,e_j,e_k)\beta(e_i,e_j,e_k) \tag{2}$$

Lemma 2.1 *The space W decomposes as orthogonal sum $W = W^v + W^h$, where*

$$W^v = W_1 \oplus W_2 \oplus W_3 \quad , \quad W^h = W_4 \oplus W_5 \oplus W_6$$

and

$$W_1 = \{\alpha \in W \mid \alpha(A,A,X) = \alpha(X,Y,A) = 0\}$$

$$W_2 = \{\alpha \in W \mid \alpha(A,B,X) = \alpha(B,A,X) \ , \ \alpha^v = 0 \ , \ \alpha(X,Y,A) = 0\}$$

$$W_3 = \{\alpha \in W \mid \alpha(A,B,X) = \frac{1}{n_1 - 2\nu_1} < A,B > \alpha^v(X) \ , \ \alpha(X,Y,A) = 0\}$$

$$W_4 = \{\alpha \in W \mid \alpha(X,X,A) = \alpha(A,B,X) = 0\}$$

$$W_5 = \{\alpha \in W \mid \alpha(X,Y,A) = \alpha(Y,X,A) \ , \ \alpha^h = 0 \ , \ \alpha(A,B,X) = 0\}$$

$$W_6 = \{\alpha \in W \mid \alpha(X,Y,A) = \frac{1}{n_2 - 2\nu_2} < X,Y > \alpha^h(A) \ , \ \alpha(A,B,X) = 0\}$$

Moreover, for n_1, $n_2 > 1$, these subspaces are invariant and irreducible under the action of $O_{\nu_1}(n_1) \times O_{\nu_2}(n_2)$.

Remark 2.2 *(i) By convention, when $n_1 = 2\nu_1$, the expression $\frac{1}{n_1 - 2\nu_1}$ =0, and the same is true for index ν_2.*

(ii) The group $O_{\nu_1}(n_1) \times O_{\nu_2}(n_2)$ acts on $V \oplus H$ and this action extends canonically on $V^* \oplus H^*$, then on W.

(iii) When $n_1 = 1$ and $n_2 > 1$, there are only 4 invariant subspaces

$$W_1 = W_2 = W_3 \ , \ W_4 \ , \ W_5 \ , \ W_6$$

A similar situation occurs when $n_1 > 1$ and $n_2 = 1$. If $n_1 = n_2 = 1$, there are only two invariant subspaces.

(iv) In the Riemannian case, the lemma was partially proved in [2]. We give here a complete proof in the general semi-Riemannian case.

Proof of the Lemma 2.1. Due to the property $T = V \oplus H$ and to the relation (1), we obtain the isomorphisms

$$W \cong T^* \otimes V^* \otimes H^* = V^* \otimes V^* \otimes H^* + H^* \otimes V^* \otimes H^* \tag{3}$$

We denote the first and the second term with W^v and W^h respectively. So, $W = W^v + W^h$. One knows ([1], p.45) the following property: if E is an r-dimensional vector space, with scalar product of signature μ, then $E \times E$ decomposes into irreducible components under the action of $O_\mu(r)$, as

$$E \otimes E = \Lambda^2 E \oplus SO^2 E \oplus A_E$$

We denoted $\Lambda^2 E$ the subspace of skew- symmetric tensors, $SO^2 E$ the subspace of trace-free symmetric tensors and A_E the line spanned by

$$\sum_{i=1}^{n} e_i \otimes e_i$$

where $\{e_i\}_{i=1,r}$ is an orthonormal basis in E. If we identify the elements in $E \times E$ with the bilinear forms on $E \times E$, then $\alpha \in A_E$ if and only if

$$\alpha(K, L) = \frac{1}{r - 2\mu} < K, L > Tr\alpha$$

using the convention from the previous Remark 3,(i). Back in (3), we obtain

$$V^* \otimes V^* \otimes H^* = \Lambda V^* \otimes H^* + SO^2 V^* \otimes H^* + A_{V^*} \otimes H^*$$

$$H^* \otimes V^* \otimes H^* = \Lambda^2 H^* \otimes V^* + SO^2 H^* \otimes V^* + A_{H^*} \otimes V^*$$

The wanted decomposition of W follows from the notations

$$W_1 = \Lambda^2 V^* \otimes H^* \ , \ W_2 = SO^2 V^* \otimes H^* \ , \ W_3 = A_{V^*} \oplus H^*$$

$$W_4 = \Lambda^2 H^* \otimes V^* \ , \ W_5 = SO^2 H^* \otimes V^* \ , \ W_6 = A_{H^*} \oplus V^*$$

The dimensions of the subspaces are

$$dimW_1 = \frac{1}{2} n_1 n_2 (n_1 - 1) \ , \ dimW_2 = \frac{1}{2}(n_1^2 + n_1 - 2) n_2 \ , \ dimW_3 = n_2$$

(4)

$$dimW_4 = \frac{1}{2} n_1 n_2 (n_2 - 1) \ , \ dimW_5 = \frac{1}{2}(n_2^2 + n_2 - 2) n_1 \ , \ dimW_6 = n_1$$

and $dimW = n_1 n_2 (n_1 + n_2)$. The subspaces W_i , $i = \overline{1,6}$ are mutually orthogonal, with respect to the scalar product (2) induced on W. For example, if $\alpha \in W_1$ and $\beta \in W_2$, we calculate

$$(\alpha, \beta) = \sum_{i,j=1}^{n} \sum_{a=1}^{n_1} \alpha(e_i, e_j, e_a) \beta(e_i, e_j, e_a) = 0$$

due to

$$\alpha(X, Y, A) = \beta(X, Y, A)$$

and for $i \neq j$

$$\alpha(e_i, e_j, e_a) = 0 \ , \ \beta(e_i, e_j, e_a) \alpha(e_i, e_j, e_a) = -\beta(e_j, e_i, e_a) \alpha(e_j, e_i, e_a)$$

A simple check establishes the identity between the classes in (4) and those from the lemma assertion. Rests to prove that the decomposition is irreducible and invariant under the action of $G = O_{\nu_1}(n_1) \times O_{\nu_2}(n_2)$ on T. The invariance is evident. The irreducibility will follow from the next three assertions:

(i) The vector space of real homogeneous G-invariant polynomials is spanned by products of traces ([1]).

(ii) If W is a G-invariant subspace of T, then every real homogeneous G-invariant polynomial on W is a linear combination of products of traces.

(iii) The vector space of G-invariant quadratic forms on W is spanned by $||\alpha_1||^2$, $A_1(\alpha)$, $||\alpha^v||^2$, $||\alpha_2||^2$, $A_2(\alpha)$ and $||\alpha^h||^2$, where α_1 and α_2 are the components of α in W^v and W^h, respectively, and

$$A_1(\alpha) = \sum_{a_1,a_2=1}^{n_1} \sum_{u_1=n_1+1}^{n} \alpha(e_{a_1}, e_{a_2}, e_{u_1})\alpha(e_{a_2}, e_{a_1}, e_{u_1})$$

$$A_2(\alpha) = \sum_{a_1=1}^{n_1} \sum_{u_1,u_2=n_1+1}^{n} \alpha(e_{u_1}, e_{u_2}, e_{a_1})\alpha(e_{u_2}, e_{u_1}, e_{a_1})$$

This space has dimension 6 if $n_1, n_2 > 1$; dimension 4 if $n_1 = 1$ and $n_2 > 1$, or $n_1 > 1$ and $n_2 = 1$; dimension 2 if $n_1 = n_2 = 1$.

The assertion (ii) is a consequence of assertion (i). Due to this second assertion, the only non-vanishing products of traces on W are

$$\sum \alpha(e_{a_1}, e_{a_2}, e_{u_1})\alpha(e_{a_1}, e_{a_2}, e_{u_1}) = \frac{1}{2}||\alpha_1||^2$$

$$\sum \alpha(e_{a_1}, e_{a_2}, e_{u_1})\alpha(e_{a_2}, e_{a_1}, e_{u_1}) = A_1(\alpha)$$

$$\sum \alpha(e_{u_1}, e_{a_1}, e_{u_1})\alpha(e_{a_2}, e_{a_2}, e_{u_1}) = ||\alpha^v||^2$$

$$\sum \alpha(e_{u_1}, e_{u_2}, e_{a_1})\alpha(e_{u_1}, e_{u_2}, e_{a_1}) = \frac{1}{2}||\alpha_2||^2$$

$$\sum \alpha(e_{u_1}, e_{u_2}, e_{a_1})\alpha(e_{u_2}, e_{u_1}, e_{a_1}) = A_2(\alpha)$$

$$\sum \alpha(e_{u_1}, e_{u_1}, e_{u_1})\alpha(e_{u_2}, e_{u_2}, e_{a_1}) = ||\alpha^h||^2$$

where a_1, a_2 sum from 1 to n_1 and u_1, u_2 sum from (n_1+1) to n. Assertion (iii) is proved.

On another hand, to each irreducible component of the G-representation on W corresponds a symmetric invariant bilinear form, which vanishes on the remaining irreducible components. So, the number of irreducible components of the G-representation on W is lesser or equal the dimension of the space of quadratic forms which are invariant under this representation. For the "general position" (n_1, $n_2 > 1$), this dimension is 6. We conclude that, in this case, the subspaces W_1, ..., W_6 are irreducible under the G- action. □

Remark 2.3 *(i) We return to the initial setting from the beginning of the paragraph. We say that a semi-Riemannian almost product manifold belongs to the class W_i if, in each point, $\nabla\phi$ belongs to the subspace W_i. In the same manner we define the belonging to the classes $W_i \oplus W_j$, etc. "Degenerate" cases correspond to the subspaces $\{0\}$ (when the almost product structure is parallel with respect to the Levi- Civita connection) and W (when the structure is in the "general position"). We remark that this classification (which we call "Naveira-like") do not contain disjoint classes only (for example, $W_i \subseteq W_i \oplus W_j$). Moreover, the "generic" class W contains structures with poor information content.*

On another hand, almost product structures exists, which in different points belong to different classes; these structures are not under the incidence of the present classification.

(ii) Consider the couple of distributions $(\mathcal{D}, \mathcal{D}^\perp)$ characterizing the almost product structure. We shall describe the 64 classes of almost product semi-Riemannian manifolds in terms of the properties of this couple. First, we need the following

Definition 2.1 *([2]) The distribution $\mathcal{D}$ belongs to the class*

(F) (foliation) if it is completely integrable.

(D_1) if $(\nabla_A P)A = 0$, for any $A \in \mathcal{D}$.

(D_2) if $\alpha^v(X) = 0$, for any $X \in \mathcal{D}$.

(D_3) if $g((\nabla_A P)B + (\nabla_B P)A = \frac{2}{n_1 - 2\nu_1} g(A, B)\alpha^v(X)$.

(F_i) if $\mathcal{D}$ has the property (F) and (D_i), $i = \overline{1,3}$.

$(\emptyset)$ if there are no retrictions for ∇P.

In the same manner one defines corresponding properties for $\mathcal{D}^\perp$.

Theorem 2.1 *The 64 classes of semi- Riemannian almost product manifolds may be characterized through the properties of the couple $(\mathcal{D}, \mathcal{D}^\perp)$, as follows: to each class corresponds one and only one of the couples (P_1, P_2), where $\mathcal{D}$ has the property P_1, $\mathcal{D}^\perp$ has the property P_2 and P_1, $P_2 \in \{(\emptyset), (D_1), (D_2), (D_3), (F), (F_1), (F_2), (F_3)\}$.*

The proof is similar to that in the Riemannian case ([2]) and will be skipped.

Remark 2.4 *(i) The previous classification contains 64 distinct classes. In the Riemannian case ([2]), due the "symmetry" betweem D and $\mathcal{D}^\perp$), the pairs number may be reduced to 36. For arbitrary signatures, this reduction is not possible.*

(ii) All the previous constructions and results do not depend on the torsion of ∇, so they may be extended without any change to arbitrary metric connections on (M, g). The new Naveira-like classifications will be the same as the previous one; this fact proves that classification is invariant, not only with respect to the pseudo-orthogonal group, but also with respect to a change of the linear metric connection.

In the next section, the Theorem 2.1 will be particularized for the Lorentzian almost product structures, in order to obtain a classification of the latter.

3 Refinement for the classification of the Lorentzian almost product structures

Consider (M, G) an n- dimensional Lorentzian time-oriented manifold. Let X a timelike, unitary, future oriented vector field on M (its existence is ensured by the time-orientation). Denote by $\mathcal{D}$ the timelike distribution spanned by X, and by $\mathcal{D}^\perp$ its orthogonal complementary distribution; then $\mathcal{D}^\perp$ is a spacelike distribution of rank 3. To the pair of complementary distributions $\mathcal{D}$ and $\mathcal{D}^\perp$, with corresponding projectors V, resp. $V^\perp$, we associate a Lorentzian almost product structure on M, as a particular case of the construction in §2. This structure will be called the canonical Lorentzian almost product structure associated to the triple (M, g, X). With §2 notations, $n_1 = 1$, $n_2 = n - 1$, $\nu_1 = \nu = 1$, $\nu_2 = 0$.

We are now able to exploit the results from the previous paragraph.

Proposition 3.1 *[6] For the canonical Lorentzian almost product structures, the Naveira-like classification contains the classes $(\mathcal{D}, \mathcal{D}^\perp)$, with the properties:*

$$(F_1, F_1) \ , \ (F_1, F) \ , \ (F, F_1) \ , \ (F, F) \quad if \ \ n_2 = 1$$

$$\mathcal{D} \in \{(F), (F_1)\} \ , \ \mathcal{D}^\perp \in \{(F), (D_i), (F_j), (\emptyset); i, j = \overline{1,3}\} \quad if \ \ n_2 > 1$$

Proof. When $n_1 = 1$, from the Remark 3, *iii* follows that $\mathcal{D}$ belong to two classes at most. Being 1-dimensional, $\mathcal{D}$ is completely integrable, so it has the property (F), or (F_1) if it is also parallel. When $n_2 = 1$, similar situation occurs for the distribution $\mathcal{D}^\perp$.

If $n_2 > 1$, Theorem 6 implies that $\mathcal{D}^\perp$ may have any of the ennounced properties (a total of 8 possibilities). □

Example 3.1 *(i) Minkowski spacetime, with the natural Lorentzian almost product structure belongs to the class (F_1, F_1).*

(ii) A Robertson-Walker spacetime ([3]), with nonconstant warping function, belongs to the class (F_1, F).

(iii) A Schwarzschild spacetime ([3]) belongs to the class (F, F). This fact admits as generalization the following

Theorem 3.1 *[6] On every 4-dimensional noncompact differentiable manifold, there exists a Lorentzian almost product structure of class (F, F).*

Proof. We know there exists on M a global nowhere vanishing vector field X. Denote by D the 1-dimensional distribution spanned by X. Consider D' a complementary distribution of D (eventually, one may construct D' by taking an arbitrary Riemannian metric g and defining D' as the orthogonal distribution of D, with respect to g).

Under these hypothesis, from [4] we know there exists a 3-dimensional foliation D" on M (even homotopical with D').

Let E be a complementary distribution of D". The couple $(E, D")$ admits a canonically associated almost product structure P and (many) Lorentzian metrics $\tilde{g}$ such that E and D" be orthogonal with respect to $\tilde{g}$. Then, the manifold $(M, \tilde{g}, P)$ is a Lorentzian almost product manifold of class (F,F). □

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