DETERMINATION OF CHARACTERISTICALLY NILPOTENT LIE ALGEBRAS

Gr.Tsagas, Th.Tapanidis, S.Kortesis,
A.Synefaki and Ch.Christophoridou

Abstract

The aim of this paper is to determine the characteristically Nilpotent Lie algebras of the Nilpotent Lie algebras of dimension eight with maximal abelian ideal of dimension four.

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1 Introduction

Let g be a Nilpotent Lie algebra of dimension eight over a field K of characteristic zero with maximal abelian ideal of dimension four. The purpose of this paper is to find out which of these Lie algebras are characteristically Nilpotent Lie algebras. We refer to the classification of the Nilpotent Lie algebras in [4].

The whole paper contains three sections each of them is analyzed as follows.

The first section is the introduction.

Basic elements of Lie algebras are given in the second one.

In the third section we determine the characteristically Nilpotent Lie algebras using a combination of a computer program and the Mathematica software package and the corresponding representation matrices of the derivations. In this case we have used a special theory of Lie algebras.

2 Basic elements of Lie algebras

We shall give some basic notions used in this paper. Let g be a Lie algebra over the field K of characteristic zero of dimension n. It is known that from this algebra we

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can form the following sequence of ideals of g:

$$C^0 g = g, \quad C^1 g = [g, g], ..., \quad C^q g = [g, C^{q-1} g], ..., \tag{1}$$

which is called *descending central sequence.*

If there exists an integer $q \geq 2$ such that $C^q g = \{0\}$, then the Lie algebra g is called *Nilpotent* of nilpotency q.

A linear mapping f on g is called *derivation*, if it satisfies the relation:

$$f[x, y] = [fx, y] + [x, fy], \quad \forall x, y \in g.$$

The set of all derivations f on g is denoted by $D(g)$ and is a Lie algebra.

The Lie algebra g is called *characteristically Nilpotent*, if the Lie algebra $D(g)$ is Nilpotent.

Characteristically Nilpotent Lie algebras were defined in 1957 by Dixmier and Lister in [2].

A derivation f on a Lie algebra g of dimension eight is a linear mapping such that its representation, with respect to the base $\{e_1, ...e_8\}$ of g, has the form:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} & a_{17} & a_{18} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} & a_{27} & a_{28} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} & a_{37} & a_{38} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} & a_{47} & a_{48} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & a_{56} & a_{57} & a_{58} \\ a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & a_{66} & a_{67} & a_{68} \\ a_{71} & a_{72} & a_{73} & a_{74} & a_{75} & a_{76} & a_{77} & a_{78} \\ a_{81} & a_{82} & a_{83} & a_{84} & a_{85} & a_{86} & a_{87} & a_{88} \end{bmatrix} \tag{2}$$

The vectors $\{e_1, ...e_8\}$ are represented by the matrices:

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, ..., \quad e_8 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \tag{3}$$

If the matrix A is Nilpotent, then the Lie algebra g is characteristically Nilpotent.

If we apply the derivation f on the Lie brackets of g, after some calculations, we obtain a linear homogeneous system with the unknowns a_{ij}, $1 \leq i, j \leq 8$.

3 Representation matrices of derivations

In order to find the representation matrices of the derivations corresponding to the different Lie algebras we develop a computer program and then combine it with the Mathematica software package.

The program has been built taking under consideration all properties of derivations of Lie algebras and creates the systems whose solutions give the representation matrices of the corresponding derivations.

We have used the Mathematica software package in order to solve the systems and to create the representation matrices of the derivations.

The Lie algebras which we examine are the following:

$$g_1: \begin{array}{ll} [e_1,e_2]=k_{13}e_7 & [e_1,e_3]=k_{42}e_5+k_{22}e_6 \\ [e_1,e_4]=k_{52}e_5+k_{42}e_6+k_{33}e_7 & [e_1,e_6]=e_8 \\ [e_2,e_3]=k_{52}e_5+k_{42}e_6+k_{43}e_7 & [e_2,e_4]=k_{51}e_5+k_{52}e_6 \\ [e_2,e_5]=e_8 & [e_3,e_4]=k_{63}e_7+k_{64}e_8 \\ [e_3,e_6]=e_7 & [e_4,e_5]=e_7 \end{array} \tag{4}$$

$$g_2: \begin{array}{ll} [e_1,e_2]=e_3 & [e_1,e_3]=k_{12}e_6-2k_{52}e_7 \\ [e_1,e_4]=-k_{52}e_5+k_{22}e_6+k_{23}e_7 & [e_1,e_6]=e_8 \\ [e_2,e_3]=k_{31}e_5+k_{41}e_7 & [e_2,e_4]=k_{41}e_5+k_{43}e_7 \\ [e_2,e_5]=e_6 & [e_3,e_4]=k_{52}e_6 \\ [e_3,e_5]=e_8 & [e_4,e_7]=e_8 \end{array} \tag{5}$$

$$g_3: \begin{array}{ll} [e_1,e_2]=e_3 & [e_1,e_3]=k_{12}e_6-2k_{52}e_7 \\ [e_1,e_4]=-k_{52}e_5+k_{22}e_6+k_{23}e_7 & [e_1,e_6]=e_8 \\ [e_2,e_3]=k_{41}e_7 & [e_2,e_4]=k_{41}e_5+k_{43}e_7 \\ [e_2,e_5]=e_6 & [e_3,e_4]=k_{52}e_6 \\ [e_3,e_5]=e_8 & [e_4,e_5]=e_7 \\ [e_4,e_7]=e_8 & \end{array} \tag{6}$$

$$g_4: \begin{array}{lll} [e_1,e_2]=e_3 & [e_3,e_4]=k_{54}e_8 & [e_2,e_5]=e_6 \\ [e_1,e_4]=-k_{54}e_6+k_{24}e_8 & [e_4,e_5]=e_8 & [e_2,e_6]=e_8 \\ [e_2,e_3]=k_{34}e_8 & [e_1,e_3]=k_{14}e_8 & [e_3,e_5]=e_7 \\ [e_2,e_4]=k_{43}e_7+k_{44}e_8 & [e_1,e_6]=e_7 & \end{array} \tag{7}$$

$$g_5: \begin{array}{lll} [e_1,e_4]=k_{12}e_6+k_{13}e_7 & [e_1,e_3]=e_4 & [e_1,e_2]=e_3 \\ [e_2,e_3]=k_{44}e_6+k_{24}e_8 & [e_1,e_7]=e_8 & [e_1,e_6]=e_7 \\ [e_3,e_4]=k_{44}e_8 & [e_2,e_5]=e_6 & [e_3,e_5]=e_7 \\ [e_2,e_4]=k_{44}e_7 & [e_4,e_5]=e_8 & \end{array} \tag{8}$$

$$g_6: \begin{array}{ll} [e_1,e_2]=e_3 & [e_3,e_6]=e_8 \\ [e_1,e_4]=k_{21}e_5+(k_{41}-k_{53})e_6 & [e_1,e_5]=e_7 \\ [e_2,e_3]=k_{42}e_5+k_{32}e_6 & [e_1,e_7]=e_8 \\ [e_3,e_4]=k_{53}e_7 & [e_2,e_6]=e_7 \\ [e_2,e_4]=k_{41}e_5+k_{42}e_6+k_{44}e_8 & [e_4,e_5]=e_8 \\ [e_1,e_3]=(k_{41}-2k_{53})e_5+k_{42}e_6+k_{13}e_7 & \end{array} \quad (9)$$

$$g_7: \begin{array}{lll} [e_1,e_2]=-e_3 & [e_1,e_3]=-e_4 & [e_1,e_4]=k_{11}e_5+k_{44}e_7 \\ [e_1,e_5]=e_6 & [e_1,e_6]=e_7 & [e_2,e_3]=-k_{33}e_6+k_{23}e_7 \\ [e_2,e_4]=k_{33}e_7 & [e_2,e_7]=e_8 & [e_3,e_4]=k_{44}e_8 \\ [e_3,e_6]=e_8 & [e_4,e_5]=e_8 & \end{array} \quad (10)$$

$$g_8: \begin{array}{lll} [e_1,e_2]=e_3 & [e_1,e_3]=e_4 & [e_1,e_4]=(k_{33}-k_{44})e_6+k_{13}e_7 \\ [e_1,e_6]=e_7 & [e_1,e_7]=e_8 & [e_2,e_3]=k_{33}e_6+k_{24}e_8 \\ [e_2,e_4]=k_{33}e_7 & [e_2,e_5]=e_6 & [e_2,e_6]=e_8 \\ [e_3,e_4]=k_{44}e_8 & [e_3,e_5]=e_7 & [e_4,e_5]=e_8 \end{array} \quad (11)$$

$$g_9: \begin{array}{lll} [e_1,e_2]=-e_3 & [e_2,e_4]=-0.5k_{43}e_6+k_{33}e_7 & [e_2,e_5]=e_7 \\ [e_1,e_3]=-e_4 & [e_3,e_4]=k_{43}e_7+k_{44}e_8 & [e_3,e_6]=e_8 \\ [e_1,e_5]=e_6 & [e_1,e_4]=1.5k_{43}e_5+k_{44}e_7 & [e_2,e_7]=e_8 \\ [e_1,e_6]=e_7 & [e_2,e_3]=-0.5k_{43}e_5-k_{33}e_6+k_{23}e_7 & [e_4,e_5]=e_8 \end{array} \quad (12)$$

$$g_{10}: \begin{array}{ll} [e_1,e_2]=e_3 & [e_1,e_3]=-2k_{52}e_6+k_{41}e_7+k_{14}e_8 \\ [e_1,e_4]=-k_{52}e_5+k_{22}e_6 & [e_1,e_6]=e_8 \\ [e_2,e_3]=k_{41}e_6+k_{33}e_7 & [e_2,e_4]=k_{41}e_5+k_{43}e_7 \\ [e_2,e_5]=e_6 & [e_2,e_7]=e_8 \\ [e_3,e_4]=k_{52}e_6 & [e_3,e_5]=e_8 \\ [e_4,e_5]=e_7 & [e_4,e_6]=e_8 \end{array} \quad (13)$$

$$g_{11}: \begin{array}{ll} [e_1,e_2]=e_3 & [e_1,e_3]=-2k_{53}e_6+k_{13}e_7 \\ [e_1,e_4]=-k_{53}e_5+k_{22}e_6+k_{23}e_7 & [e_1,e_5]=e_6 \\ [e_1,e_7]=e_8 & [e_2,e_3]=k_{52}e_6 \\ [e_2,e_4]=k_{52}e_5+k_{42}e_6 & [e_2,e_5]=e_7 \\ [e_3,e_4]=k_{52}e_6+k_{53}e_7 & [e_3,e_5]=e_8 \\ [e_4,e_5]=e_7 & [e_4,e_6]=e_8 \end{array} \quad (14)$$

$$g_{12}: \begin{array}{ll} [e_1,e_2]=e_3 & [e_4,e_7]=e_8 \\ [e_1,e_4]=-k_{52}e_5+k_{22}e_6 & [e_1,e_6]=e_8 \\ [e_2,e_3]=(-k_{33}+k_{41})e_6+k_{33}e_7 & [e_4,e_5]=e_7 \\ [e_2,e_5]=e_6 & [e_2,e_7]=e_8 \\ [e_3,e_4]=k_{52}e_6 & [e_3,e_5]=e_8 \\ [e_2,e_4]=k_{41}e_5+k_{43}e_7 & [e_4,e_6]=e_8 \\ [e_1,e_3]=(k_{33}-k_{41}-2k_{52})e_6+(-k_{33}+k_{41})e_7+ & \\ \quad +k_{14}e_8 & \end{array} \quad (15)$$

$$g_{13}:\quad \begin{array}{lll} [e_1,e_2]=e_3 & [e_3,e_5]=e_8 & [e_2,e_6]=e_7 \\ [e_1,e_4]=-k_{53}e_6 & [e_4,e_6]=e_8 & [e_3,e_4]=k_{53}e_7 \\ [e_2,e_3]=(k_{41}-k_{53})e_6+k_{33}e_7 & [e_2,e_5]=e_6 & [e_4,e_5]=e_7 \\ [e_2,e_4]=k_{41}e_5+k_{43}e_7 & [e_1,e_6]=e_8 & \\ [e_1,e_3]=(k_{41}-k_{53})e_7+k_{14}e_8 & [e_2,e_7]=e_8 & \end{array} \qquad (16)$$

$$g_{14}:\quad \begin{array}{ll} [e_1,e_2]=e_3 & [e_1,e_6]=e_7 \\ [e_1,e_4]=(k_{13}-k_{44})e_5+(-k_{13}+k_{44})e_6+k_{13}e_7 & [e_2,e_6]=e_8 \\ [e_1,e_5]=e_6 & [e_3,e_5]=e_8 \\ [e_2,e_4]=k_{33}e_7 & [e_1,e_3]=-e_4 \\ [e_3,e_4]=k_{44}e_8 & [e_2,e_7]=e_8 \\ [e_4,e_5]=e_8 & [e_3,e_6]=e_8 \\ [e_2,e_3]=-k_{33}e_6+k_{23}e_7 & \end{array} \qquad (17)$$

$$g_{15}:\quad \begin{array}{ll} [e_1,e_2]=e_3 & [e_4,e_6]=e_8 \\ [e_1,e_3]=(-k_{33}-2k_{53})e_6+k_{33}e_7+k_{14}e_8 & [e_3,e_5]=e_8 \\ [e_1,e_5]=e_6 & [e_1,e_7]=e_8 \\ [e_2,e_3]=(-k_{33}+k_{52}-k_{53})e_6+k_{33}e_7 & [e_2,e_5]=e_7 \\ [e_2,e_4]=k_{52}e_5+k_{42}e_6+k_{43}e_7 & [e_2,e_7]=e_8 \\ [e_1,e_4]=-k_{53}e_5+k_{43}e_7 & [e_4,e_5]=e_7 \\ [e_3,e_4]=k_{52}e_6+k_{53}e_7 & [e_4,e_7]=e_8 \end{array} \qquad (18)$$

$$g_{16}:\quad \begin{array}{ll} [e_1,e_2]=-e_3 & [e_2,e_5]=e_7 \\ [e_1,e_4]=1.5k_{43}e_5-1.5k_{43}e_6+(1.5k_{43}+k_{44})e_7 & [e_1,e_6]=e_7 \\ [e_3,e_4]=k_{43}e_7+k_{44}e_8 & [e_4,e_5]=e_8 \\ [e_2,e_4]=0.5k_{43}e_6+k_{33}e_7 & [e_1,e_3]=-e_4 \\ [e_2,e_7]=e_8 & [e_1,e_5]=e_6 \\ [e_3,e_6]=e_8 & [e_2,e_6]=e_8 \\ [e_2,e_3]=-0.5k_{43}e_5-k_{33}e_6+k_{23}e_7 & [e_3,e_5]=e_8 \end{array} \qquad (19)$$

$$g_{17}:\quad \begin{array}{ll} [e_1,e_2]=-e_3 & [e_1,e_6]=e_7 \\ [e_1,e_5]=e_6 & [e_2,e_7]=e_8 \\ [e_2,e_3]=-k_{33}e_6-k_{34}e_7 & [e_3,e_6]=e_8 \\ [e_2,e_4]=k_{33}e_7+k_{34}e_8 & [e_4,e_5]=e_8 \\ [e_3,e_4]=k_{44}e_8 & [e_1,e_7]=e_8 \\ [e_1,e_3]=-e_4 & [e_2,e_6]=e_8 \\ [e_1,e_4]=(k_{13}-k_{33}-k_{44})e_5+ & [e_3,e_5]=e_8 \\ \qquad +(-k_{13}+k_{33}+k_{44})e_6+k_{13}e_7 & \end{array} \qquad (20)$$

$$g_{18}: \begin{array}{ll} [e_1,e_2]=-e_3 & [e_1,e_6]=e_7 \\ [e_1,e_4]=1.5k_{43}e_5+k_{43}e_6+(k_{33}-k_{43}+k_{44})e_7 & [e_1,e_5]=e_6 \\ [e_2,e_4]=0.5k_{43}e_6+k_{33}e_7+k_{34}e_8 & [e_2,e_6]=e_8 \\ [e_2,e_3]=-0.5k_{43}e_5-k_{33}e_6-k_{34}e_7 & [e_3,e_6]=e_8 \\ [e_2,e_5]=e_7 & [e_1,e_7]=e_8 \\ [e_3,e_4]=k_{43}e_7+k_{44}e_8 & [e_2,e_7]=e_8 \\ [e_1,e_3]=-e_4 & [e_4,e_5]=e_8 \end{array} \tag{21}$$

$$g_{19}: \begin{array}{ll} [e_1,e_4]=-k_{53}e_5+(wk_{42}+wk_{43}-wk_{54})e_7 & [e_1,e_6]=e_8 \\ [e_2,e_3]=(-wk_{33}+k_{52}-wk_{53})e_6+k_{33}e_7+k_{34}e_8 & [e_1,e_2]=e_3 \\ [e_1,e_7]=e_8 & [e_2,e_5]=e_7 \\ [e_2,e_4]=k_{52}e_5+k_{42}e_6+k_{43}e_7 & [e_2,e_7]=we_8 \\ [e_3,e_4]=k_{52}e_6+k_{53}e_7+k_{54}e_8 & [e_3,e_5]=e_8 \\ [e_4,e_6]=e_8 & [e_4,e_5]=e_7 \\ [e_1,e_5]=e_6 & [e_4,e_7]=we_8 \\ [e_1,e_3]= & \\ =[(w^3-w^2)k_{33}-(w^2+1)k_{52}+(w^3-2)k_{53}]e_6+ & \\ +[(w-w^2)k_{33}+wk_{52}-w^2k_{53}]e_7. & \end{array} \tag{22}$$

These algebras depend on the number of parameters and in this paper we consider that all parameters are non zero. From these 19 Nilpotent Lie algebras we found that the Lie algebras $g_1, g_2, g_3, g_7, g_9, g_{10}, g_{11}, g_{12}, g_{14}, g_{15}, g_{16}, g_{17}, g_{18}, g_{19}$ are characteristically Nilpotent. Next we will give the representation matrices of the derivations of these characteristically Nilpotent Lie algebras. The representation matrices A_i, $i = 1$, 2, 3, 7, 9, 10, 11, 12, 14, 15, 16, 17, 18, 19 correspond to the Lie algebras g_i, $i = 1$, 2, 3, 7, 9, 10, 11, 12, 14, 15, 16, 17, 18, 19.

$$A_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -a_{76}k_{42}-a_{75}k_{52} & -a_{75}k_{51}-a_{76}k_{52} & a_{53} & a_{54} & 0 & 0 & 0 & 0 \\ -a_{76}k_{22}-a_{75}k_{42} & -a_{76}k_{42}-a_{75}k_{52} & a_{63} & a_{53} & 0 & 0 & 0 & 0 \\ a_{71} & a_{72} & a_{73} & a_{74} & a_{75} & a_{76} & 0 & 0 \\ a_{81} & a_{82} & a_{83} & a_{84} & a_{85} & a_{86} & 0 & 0 \end{bmatrix},$$

where

$a_{53} = a_{86}k_{42} + a_{85}k_{52}$,

$a_{63} = a_{86}k_{22} + a_{85}k_{42}$,

$a_{54} = a_{85}k_{51} + a_{86}k_{52}$.

$$A_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -a_{65} & a_{86} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_{87}k_{52} & -a_{85}k_{31} - a_{87}k_{41} & a_{65}k_{31} & a_{54} & 0 & 0 & 0 & 0 \\ a_{61} & a_{83} & a_{86}k_{12} - a_{87}k_{52} & a_{64} & a_{65} & 0 & 0 & 0 \\ a_{71} & -a_{85}k_{41} - a_{87}k_{43} & a_{65}k_{41} - 2a_{86}k_{52} & a_{74} & 0 & 0 & 0 & 0 \\ a_{81} & a_{82} & a_{83} & a_{84} & a_{85} & a_{86} & a_{87} & 0 \end{bmatrix},$$

where

$a_{54} = a_{65}k_{41} - a_{86}k_{52}$,
$a_{64} = a_{71} + a_{87}k_{23} + a_{86}k_{22} - a_{85}k_{52}$.

$$A_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -a_{65} & a_{86} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_{87}k_{52} & -a_{87}k_{41} & 0 & a_{63} & 0 & 0 & 0 & 0 \\ a_{61} & a_{83} & a_{86}k_{12} - a_{87}k_{52} & a_{64} & a_{65} & 0 & 0 & 0 \\ a_{71} & -a_{85}k_{41} - a_{87}k_{43} & a_{65}k_{41} - 2a_{86}k_{52} & a_{74} & a_{87} & 0 & 0 & 0 \\ a_{81} & a_{82} & a_{83} & a_{84} & a_{85} & a_{86} & a_{87} & 0 \end{bmatrix},$$

where

$a_{63} = a_{65}k_{41} - a_{86}k_{52}$,
$a_{64} = a_{71} + a_{87}k_{23} + a_{86}k_{22} - a_{85}k_{52}$.

$$A_7 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_{87} & a_{43} & 0 & 0 & 0 & 0 & 0 & 0 \\ a_{41} & 0 & a_{43} & 0 & 0 & 0 & 0 & 0 \\ -a_{85}k_{11} & a_{52} & 0 & -a_{43}k_{11} & 0 & 0 & 0 & 0 \\ -a_{41}k_{44} + a_{84} & a_{41}k_{33} & a_{63} & 0 & -a_{43} & 0 & 0 & 0 \\ a_{83} & a_{72} & a_{87}k_{23} & a_{74} & 0 & -a_{43} & 0 & 0 \\ a_{81} & a_{82} & a_{83} & a_{84} & a_{85} & a_{41} & a_{87} & 0 \end{bmatrix},$$

where

$a_{52} = a_{43}k_{44} + a_{74} - a_{87}k_{33}$,
$a_{63} = -a_{43}k_{44} - a_{74}$.

$$A_9 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_{31} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1.5a_{85}k_{43} & a_{74} - a_{31}k_{33} & -0.5a_{31}k_{43} & 0 & 0 & 0 & 0 & 0 \\ a_{84} & -0.5a_{85}k_{43} & -a_{74} & 0.5a_{31}k_{43} & 0 & 0 & 0 & 0 \\ a_{83} & a_{72} & a_{31}k_{23} - a_{85}k_{43} & a_{74} & a_{31} & 0 & 0 & 0 \\ a_{81} & a_{82} & a_{83} & a_{84} & a_{85} & 0 & a_{31} & 0 \end{bmatrix},$$

$$A_{10} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -a_{87} & a_{32} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_{51} & a_{32}k_{41} - a_{86}k_{41} & 0 & a_{54} & 0 & 0 & 0 & 0 \\ a_{61} & a_{62} & a_{63} & a_{64} & a_{87} & 0 & 0 & 0 \\ a_{71} & a_{72} & a_{73} & a_{74} & -a_{32} + a_{86} & 0 & 0 & 0 \\ a_{81} & a_{82} & a_{83} & a_{84} & a_{85} & a_{86} & a_{87} & 0 \end{bmatrix},$$

where

$$a_{51} = -a_{32}k_{52} + a_{86}k_{52}, \qquad a_{71} = a_{62} - a_{83} + a_{32}k_{14},$$
$$a_{54} = a_{87}k_{41} - a_{32}k_{52}, \qquad a_{73} = a_{87}k_{33} + a_{32}k_{41},$$
$$a_{61} = a_{64} - a_{86}k_{22} + a_{85}k_{52}, \qquad a_{74} = a_{62} + a_{85}k_{41} + a_{87}k_{43}.$$
$$a_{63} = a_{87}k_{41} - a_{32}k_{52} - a_{86}k_{52},$$

$$A_{11} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_{31} & a_{32} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_{86}k_{53} & -a_{86}k_{52} & 0 & a_{54} & 0 & 0 & 0 & 0 \\ a_{61} & -a_{86}k_{42} - a_{85}k_{52} & a_{63} & a_{64} & a_{32} & 0 & 0 & 0 \\ a_{71} & a_{83} & a_{73} & a_{74} & -a_{31} + a_{86} & 0 & 0 & 0 \\ a_{81} & a_{82} & a_{83} & a_{84} & a_{85} & a_{86} & a_{32} & 0 \end{bmatrix},$$

where

$$a_{54} = -a_{31}k_{52} - a_{32}k_{53},$$
$$a_{61} = -a_{32}k_{23} + a_{74} - a_{86}k_{22} + a_{85}k_{53},$$
$$a_{63} = -a_{31}k_{52} - a_{86}k_{52} - 2a_{32}k_{53},$$
$$a_{73} = a_{32}k_{13} - a_{86}k_{53}.$$

$$A_{12} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -a_{32} + a_{86} - a_{87}, & a_{32} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -a_{32}k_{52} + a_{86}k_{52}, & a_{52} & 0 & a_{54} & 0 & 0 & 0 & 0 \\ a_{61} & a_{62} & a_{63} & a_{64} & a_{32} - a_{86} + a_{87} & 0 & 0 & 0 \\ a_{71} & a_{72} & a_{73} & a_{74} & -a_{32} + a_{86} & 0 & 0 & 0 \\ a_{81} & a_{82} & a_{83} & a_{84} & a_{85} & a_{86} & a_{87} & 0 \end{bmatrix},$$

where

$$a_{52} = a_{32}k_{41} - a_{86}k_{41},$$
$$a_{54} = -a_{86}k_{41} + a_{87}k_{41} + a_{32}(k_{41} - k_{52}),$$
$$a_{61} = a_{64} - a_{71} - a_{86}k_{22} + a_{85}k_{52},$$

$a_{63} = a_{87}(-k_{33} + k_{41}) + a_{86}(k_{33} - k_{41} - k_{52}) - a_{32}k_{52}$,
$a_{72} = -a_{62} + a_{74} - a_{85}k_{41} - a_{87}k_{43}$,
$a_{73} = -a_{86}k_{33} + a_{87}k_{33} + a_{32}k_{41}$,
$a_{83} = a_{62} - a_{71} + a_{32}k_{14}$.

$$A_{14} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_{31} & a_{43} & 0 & 0 & 0 & 0 & 0 & 0 \\ -a_{31} + a_{86} & 0 & a_{43} & 0 & 0 & 0 & 0 & 0 \\ a_{51} & a_{52} & 0 & a_{43}(k_{44} - k_{13}) & 0 & 0 & 0 & 0 \\ a_{61} & a_{63} + a_{86}k_{33} & a_{63} & a_{43}(k_{13} - k_{44}) & -a_{43} & 0 & 0 & 0 \\ -a_{61} + a_{83} & a_{72} & a_{73} & -a_{63} - a_{43}k_{13} & 0 & -a_{43} & 0 & 0 \\ a_{81} & a_{82} & a_{83} & a_{84} & a_{85} & a_{86} & a_{31} & 0 \end{bmatrix},$$

where
$a_{51} = (a_{86} - a_{13} - a_{85})(k_{13} - k_{44})$,
$a_{52} = -a_{63} - a_{31}k_{33}$,
$a_{73} = -a_{63} + a_{31}(k_{23} - k_{33})$,
$a_{84} = a_{61} - a_{31}k_{13} + a_{86}k_{13} - a_{85}(k_{13} - k_{44})$.

$$A_{15} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_{31} & a_{32} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_{86}k_{53} & -a_{86}k_{52} & 0 & -a_{31}k_{52} & -a_{32}k_{53} & 0 & 0 & 0 \\ a_{61} & a_{62} & a_{63} & a_{64} & a_{32} & 0 & 0 & 0 \\ a_{71} & a_{72} & a_{73} & a_{74} & -a_{31} + a_{86} & 0 & 0 & 0 \\ a_{81} & a_{82} & a_{83} & a_{84} & a_{85} & a_{86} & \alpha_{87} & 0 \end{bmatrix},$$

where
$a_{61} = -a_{71} + a_{74} + a_{31}k_{43} - a_{32}k_{43} - a_{86}k_{43} + a_{85}k_{53}$,
$a_{62} = -a_{72} + a_{74} + a_{86}(-k_{42} - k_{43}) + a_{31}k_{43} - a_{32}k_{43} - a_{85}k_{52}$,
$a_{63} = -a_{86}k_{52} - a_{32}(k_{33} + 2k_{53}) + a_{31}(k_{33} - k_{52} + k_{53})$,
$a_{73} = -a_{31}k_{33} + a_{32}k_{33} - a_{86}k_{53}$,
$a_{83} = -a_{71} + a_{72} + a_{32}k_{14}$,
$\alpha_{87} = -a_{31} + a_{32} + a_{86}$.

$$A_{16} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_{31} & a_{43} & 0 & 0 & 0 & 0 & 0 & 0 \\ a_{41} & 0 & a_{43} & 0 & 0 & 0 & 0 & 0 \\ a_{51} & a_{52} & -0.5a_{41}k_{43} & -1.5a_{43}k_{43} & 0 & 0 & 0 & 0 \\ a_{61} & a_{62} & a_{63} & a_{64} & -a_{43} & 0 & 0 & 0 \\ -a_{61} + a_{83} & a_{72} & a_{73} & a_{74} & a_{31} & -a_{43} & 0 & 0 \\ a_{81} & a_{82} & a_{83} & a_{84} & a_{85} & a_{31} + a_{41} & a_{31} & 0 \end{bmatrix},$$

where

$a_{51} = 1.5(a_{41} - a_{85})k_{43}$,
$a_{52} = -a_{63} - a_{31}k_{33} + 0.5a_{41}k_{43}$,
$a_{62} = a_{63} + a_{31}k_{33} + 0.5a_{41}(2k_{33} + k_{43}) - 0.5a_{85}k_{43}$,
$a_{64} = 0.5(3a_{43} + a_{31})k_{43}$,
$a_{73} = -a_{63} + a_{41}k_{43} + 0.5a_{43}(3k_{43} + 2k_{44})$,
$a_{74} = -a_{63} + a_{87}k_{23} - a_{87}k_{33}$,
$a_{84} = a_{61} - 1.5a_{85}k_{43} + 0.5a_{41}(3k_{43} + 2k_{44})$.

$$A_{17} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_{31} & a_{43} & 0 & 0 & 0 & 0 & 0 & 0 \\ -a_{31} + a_{86} & 0 & a_{43} & 0 & 0 & 0 & 0 & 0 \\ a_{51} & a_{52} & 0 & a_{54} & 0 & 0 & 0 & 0 \\ a_{61} & a_{62} & -a_{74} - a_{43}k_{13} & a_{64} & -a_{43} & 0 & 0 & 0 \\ a_{71} & a_{72} & a_{73} & a_{74} & 0 & -a_{43} & 0 & 0 \\ a_{81} & a_{82} & a_{83} & a_{84} & a_{85} & a_{86} & a_{31} - a_{43} & 0 \end{bmatrix},$$

where
$a_{51} = a_{74} + a_{43}k_{13} + a_{86}(k_{13} - k_{33} - k_{44}) + a_{31}(k_{44} - k_{13}) + a_{85}(-k_{13} + k_{33} + k_{44})$,
$a_{52} = a_{74} - 2a_{87} + a_{43}k_{13} - a_{31}k_{33}$,
$a_{54} = a_{43}(-k_{13} + k_{33} + k_{44})$,
$a_{62} = -a_{74} - a_{43}k_{13}$,
$a_{64} = a_{43}(k_{13} - k_{33} - k_{44})$,
$a_{71} = -a_{61} + a_{72} + a_{83} + a_{31}k_{34} - a_{86}k_{34}$,
$a_{73} = a_{74} + a_{43}k_{13} - a_{31}(k_{33} + k_{34})$,
$a_{84} = a_{61} + a_{86}(k_{13} - k_{33}) + a_{31}(k_{33} + k_{34} - k_{13}) + a_{85}(k_{33} + k_{44} - k_{13})$.

$$A_{18} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_{87} + a_{43} & a_{43} & 0 & 0 & 0 & 0 & 0 & 0 \\ a_{41} & 0 & a_{43} & 0 & 0 & 0 & 0 & 0 \\ a_{51} & a_{52} & a_{53} & -1.5a_{43}k_{43} & 0 & 0 & 0 & 0 \\ a_{61} & a_{62} & a_{63} & 0.5(a_{87} - a_{43})k_{43} & -a_{43} & 0 & 0 & 0 \\ a_{71} & a_{72} & a_{73} & a_{74} & a_{43} + a_{87} & -a_{43} & 0 & 0 \\ a_{81} & a_{82} & a_{83} & a_{84} & a_{85} & a_{86} & a_{87} & 0 \end{bmatrix},$$

where
$a_{51} = 0.5a_{74} - a_{87}k_{33} - a_{41}k_{43} - 1.5a_{85}k_{34} + a_{43}(k_{44} - k_{43})$,
$a_{52} = a_{74} - a_{87}k_{33} - 0.5a_{41}k_{43} - a_{43}k_{43} + a_{43}k_{44}$,
$a_{53} = -0.5(a_{43} + a_{87})k_{43}$,
$a_{62} = a_{41}k_{33} - 0.5a_{85}k_{43}$,
$a_{63} = -\alpha_{74} + a_{41}k_{43} + \alpha_{43}(-k_{33} + k_{43} - k_{44})$,
$a_{71} = -a_{61} + a_{72} + a_{83} - a_{41}k_{34}$,
$a_{73} = a_{74} - a_{87}(k_{33} + k_{34}) - a_{41}k_{43} - a_{85}k_{43} - a_{43}(k_{34} + k_{43} - k_{44})$,
$a_{84} = a_{61} - a_{74} + a_{87}(k_{33} + k_{34}) + a_{85}k_{43} + a_{43}(k_{34} + k_{43} - k_{44}) + a_{41}(k_{43} + k_{44})$,

$a_{86} = a_{41} + a_{43} + a_{87}$.

$$A_{19} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_{86}k_{53} & -a_{86}k_{52} & 0 & 0 & 0 & 0 & 0 & 0 \\ a_{61} & a_{62} & -a_{86}k_{52} & a_{64} & 0 & 0 & 0 & 0 \\ a_{71} & a_{72} & -a_{86}k_{53} & a_{74} & a_{86} & 0 & 0 & 0 \\ a_{81} & a_{82} & a_{83} & a_{84} & a_{85} & a_{86} & a_{86}w & 0 \end{bmatrix},$$

where

$a_{61} = a_{64} + a_{74} + a_{85}k_{53} - a_{71}w - a_{86}w(k_{42}w + k_{43}w - k_{54}w)$,
$a_{62} = -a_{85}k_{52} - a_{72}w + a_{74}w + a_{86}(-k_{42} - k_{43}w)$,
$a_{83} = -a_{85}k_{52} + a_{72}(1 - w) - a_{71}w + a_{74}w + a_{86}(-k_{42} - k_{43}w)$.

Now, we can state the following theorem:

Theorem 1 *Let g be a Nilpotent Lie algebra of dimension eight over the field K of characteristic zero, whose maximal abelian ideal is of dimension four. There are nineteen such Lie algebras $g_1, ..., g_{19}$, which are given by 3.1-3.19. From these the following fourteen g_1, g_2, g_3, g_7, g_9, g_{10}, g_{11}, g_{12}, g_{14}, g_{15}, g_{16}, g_{17}, g_{18}, g_{19}, are characteristically Nilpotent Lie algebras.*

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Authors' addresses:

Gr.Tsagas, Th.Tapanidis and Ch.Christophoridou
Mathematics Division
School of Technology
Aristotle University of Thessaloniki
Thessaloniki 54124, Greece
Email tsagas@eng.auth.gr, chrisi@eng.auth.gr

S.Kortesis and A.Synefaki
Informatics Division
School of Technology
Aristotle University of Thessaloniki
Thessaloniki 54124, Greece
email stk@eng.auth.gr, aspa@eng.auth.gr