Functions of Markov chains

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Abstract

A function of a homogeneous Markov chain preserves the Chapman-Kolmogorov property if and only if the transition matrix of this Markov chain satisfies some conditions. In this paper we give a theorem which describes the structure of a matrix for which the conditions mentioned above are fulfilled.

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1 Introduction

Let $(x_n)_{n\in\mathbf{N}}$ be a homogeneous Markov chain on $(\Omega, \mathcal{K}, P)$ with the state space $X = \{1, 2, \ldots, m\}$, $Y = \{1, \ldots, p\}$, $p < m$, a set and $\varphi : X \to Y$ a surjection. The problem is to find the conditions under which the stochastic process $(\varphi \circ x_n)_{n\in\mathbf{N}}$ has the Chapman-Kolmogorov property, i.e. for any $n_1 < n_2 < n_3$ and any $i, j \in \{1, \ldots, p\}$

$$\begin{aligned} P(\varphi \circ x_{n_3} &= i \mid \varphi \circ x_{n_1} = j) = \\ \sum_{\alpha=1}^{p} P(\varphi \circ x_{n_3} &= i \mid \varphi \circ x_{n_2} = \alpha) P(\varphi \circ x_{n_2} = \alpha \mid \varphi \circ x_{n_1} = j). \end{aligned}$$

In the particular case when $[\mu]Q = [\mu]$, where $\mu = P \circ x_0^{-1}$ is the initial distribution of the Markov chain $(x_n)_{n\in\mathbf{N}}$, $Q = (P(x_{n+1} = j \mid x_n = i))_{1\leq i,j\in\leq m}$, $n \geq 0$, is its transition matrix and $[\mu] = (\mu(\{1\}) \ \ldots \ \mu(\{m\}))$, $\mu(\{i\}) > 0$ for any $i = 1, \ldots, m$, the stochastic process $(\varphi \circ x_n)_{n\in\mathbf{N}}$ has the Chapman-Kolmogorov property if and only if $\varphi(Q^n) = \varphi(Q)^n$, $n \geq 1$ (for the definition of $\varphi(Q)$ see section 2). In a previous paper (see [1]) we proved a theorem that gives conditions under which the relations $\varphi(Q^n) = \varphi(Q)^n$, $n \geq 1$, hold. In the present paper we give a stronger result, this is a theorem that describes the structure of a matrix Q which satisfies the (μ, φ)-condition, i.e. Q is stochastic, $[\mu]Q = [\mu]$ and $\varphi(Q^n) = \varphi(Q)^n$, $n \geq 1$.

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2 The (μ, φ)-condition

As in the previous section, we consider that $(x_n)_{n\in\mathbf{N}}$ is a homogeneous Markov chain on $(\Omega, \mathcal{K}, P)$ with the state space $X = \{1, 2, \dots, m\}$, $\mu = P \circ x_0^{-1}$ its initial distribution and $Q = (P(x_{n+1} = j \mid x_n = i))_{1\leq i,j\in\leq m}$ its transition matrix. We denote $[\mu] = (\mu(\{1\}) \ \dots \ \mu(\{m\}))$ and we suppose that $\mu(\{i\}) > 0$ for any $i = 1, \dots, m$.

Let $Y = \{1, \dots, p\}$, $p < m$, and let $\varphi : X \to Y$ be a surjection such that for $i \in Y$, $\varphi^{-1}(i) = \{t_{i-1}+1,\ t_{i-1}+2,\ \dots, t_i\}$, where $0 = t_0 < t_1 < \dots < t_p = p$.

We denote $[\mu^\varphi] = \left(\mu^\varphi_{i,j}\right)_{1\leq i\leq p,\ 1\leq j\leq m}$, where for each $i \in Y$,

$\mu^\varphi_{i,j} = \dfrac{\mu(\{j\})}{\sum\limits_{s=1}^{t_i - t_{i-1}} \mu(\{t_{i-1}+s\})}$ if $j \in \varphi^{-1}(i)$ and $\mu^\varphi_{i,j} = 0$ if $j \notin \varphi^{-1}(i)$,

$[\mathcal{I}_\varphi] = \left(\varepsilon^\varphi_{i,j}\right)_{1\leq i\leq m,\ 1\leq j\leq p}$, where for each $j \in Y$, $\varepsilon^\varphi_{i,j} = 1$ if $i \in \varphi^{-1}(j)$ and $\varepsilon^\varphi_{i,j} = 0$ if $i \notin \varphi^{-1}(j)$ and

$$\varphi(A) = [\mu^\varphi] \cdot A \cdot [\mathcal{I}_\varphi]$$

if A is a square matrix of order m.

Proposition. *If μ and φ are as above and $[\mu]Q = [\mu]$, then the stochastic process $(\varphi \circ x_n)_{n\in\mathbf{N}}$ has the Chapman-Kolmogorov property if and only if $\varphi(Q^n) = \varphi(Q)^n$, $n \geq 1$.*

Proof. Let $\Pi_n = (P(x_{\nu+n} = i \mid x_\nu = j))_{1\leq i,j\leq m}$ and $\Phi_n = (P(\varphi \circ x_{\nu+n} = i \mid \varphi \circ x_\nu = j))_{1\leq i,j\leq p}$ $(\nu \in \mathbb{N})$.

$$P(\varphi \circ x_{\nu+n} = i \mid \varphi \circ x_\nu = j) = \frac{P(\varphi \circ x_{\nu+n} = i,\ \varphi \circ x_\nu = j)}{P(\varphi \circ x_\nu = j)} =$$

$$\frac{\sum\limits_{\alpha=1}^{t_i - t_{i-1}} \sum\limits_{\beta=1}^{t_j - t_{j-1}} P(x_{\nu+n} = t_{i-1}+\alpha,\ x_\nu = t_{j-1}+\beta)}{P(x_\nu = t_{j-1}+1) + \dots + P(x_\nu = t_j)} =$$

$$\frac{\sum\limits_{\beta=1}^{t_j - t_{j-1}} \mu(\{t_{j-1}+\beta\}) \left(\sum\limits_{\alpha=1}^{t_i - t_{i-1}} P(x_{\nu+n} = t_{i-1}+\alpha,\ x_\nu = t_{j-1}+\beta) \right)}{\sum\limits_{\beta=1}^{t_j - t_{j-1}} \mu(\{t_{j-1}+\beta\})},$$

$(\mu(\{i\}) = P(x_n = i),\ n \geq 0)$. So we have $\Phi_n = \varphi(\Pi_n)$, $n \geq 1$.

Because $(x_n)_{n\in\mathbf{N}}$ is a Markov chain, it has the Chapman-Kolmokorov property and this property can be written $\Pi_{n_1+n_2} = \Pi_{n_1} \cdot \Pi_{n_2}$, $n_1, n_2 \geq 1$, or, in an equivalent form, $\Pi_n = \Pi_1^n$, $n \geq 1$. Since $\Pi_1 = Q$, we have $\Pi_n = Q^n$.

$(\varphi \circ x_n)_{n\in\mathbf{N}}$ has the Chapman-Kolmogorov property if and only if $\Phi_{n_1+n_2} = \Phi_{n_1} \cdot \Phi_{n_2}$, $n_1, n_2 \geq 1$, or $\Phi_n = \Phi_1^n$, $n \geq 1$, but it can be written $\varphi(\Pi_n) = \varphi(Q)^n$, $n \geq 1$, or $\varphi(Q^n) = \varphi(Q)^n$, $n \geq 1$. Q.E.D.

Remark. If $\varphi(Q^n) = \varphi(Q)^n$, $n \geq 1$, then the spectrum of Q contains the spectrum of $\varphi(Q)$.

Indeed, if $\varphi(Q^n) = \varphi(Q)^n$, $n \geq 1$, and P is a polynomial such that $P(Q) = 0$, then $P(\varphi(Q)) = 0$. If P_Q is the minimal polynomial of Q and $P_{\varphi(Q)}$ the minimal polynomial of $\varphi(Q)$, then $P_{\varphi(Q)}$ is a divisor of P_Q. Consequently the spectrum of Q contains the spectrum of $\varphi(Q)$.

Definition. If Q is a stochastic matrix, $[\mu]Q = [\mu]$ and $\varphi(Q^n) = \varphi(Q)^n$, $n \geq 1$, then we say that Q satisfies the (μ, φ)-condition.

We want to know which is the structure of a matrix Q that satisfies the (μ, φ)-condition, provided that μ and φ are as in the beginning of this section. The theorem which we will prove in the following section will bring us very near to this goal.

3 The structure of a stochastic matrix which satisfies the (μ, φ)-condition

If A is a square matrix and λ is a eigenvalue of A, then we denote by $m_A(\lambda)$ the multiplicity order of λ as a root of the characteristic polynomial of A.

If $u,\ v \in \mathcal{M}_{m\times 1}(\mathbb{R})$ we use the notation ${}_{\varphi}u = [\mu^{\varphi}] \cdot u$ and ${}^t v^{\varphi} = {}^t v \cdot [\mathcal{I}_{\varphi}]$.

Theorem. *Let μ be a distribution on X with $\mu(\{i\}) > 0$, $i \in X$, and $\varphi : X \to Y$ a surjection as above. Then the matrix Q satisfies the (μ, φ)-condition, all the eigenvalues of Q are real and $m_{\varphi(Q)}(\lambda) = m_Q(\lambda)$ for any eigenvalue λ of $\varphi(Q)$ if and only if Q satisfies the following conditions:*

1) $Q = \displaystyle\sum_{i=1}^{k} \left(\lambda_i \sum_{j=n_{i-1}+1}^{n_i} u_j {}^t v_j + \sum_{j=n_{i-1}+1}^{n_i} \varepsilon_j u_j {}^t v_{j+1} \right)$, $\lambda_1 = 1,\ \lambda_2, \dots, \lambda_k \in \mathbb{R}$, $u_j,\ v_j \in \mathcal{M}_{m\times 1}(\mathbb{R})$ *and* $\varepsilon_1 = 0$, $\varepsilon_j \in \{0,1\}$ *for any* $j = 2, ..., m$, $0 = n_0 < n_1 < \dots < n_k = m$.

2) ${}^t v_i u_j = \delta_{ij}$, $i, j \in \{1, \dots, m\}$.

3) ${}^t u_1 = (1\ \dots\ 1)$, ${}^t v_1 = [\mu]$.

4) *There is* $l \in \{1, \dots, k\}$ *such that* $p = n_l$.

5) *If* $i \geq l+1$, *then* $\displaystyle\sum_{j=n_{i-1}+1}^{n_i} {}_{\varphi}u_j {}^t v_j^{\varphi} = 0$ *and*

$$\sum_{j=n_{i-1}+1}^{n_i-q} \varepsilon_j \varepsilon_{j+1} \dots \varepsilon_{j+q-1}\, {}_{\varphi}u_j {}^t v_{j+q}^{\varphi} = 0 \text{ for any } q = 1, \dots, n_i - n_{i-1} - 1.$$

6) *The matrix Q has only positive entries.*

Proof. First we suppose that the matrix Q satisfies the (μ, φ)-conditions, all the eigenvalues of Q are real and $m_{\varphi(Q)}(\lambda) = m_Q(\lambda)$ for any eigenvalue λ of $\varphi(Q)$.

Let $\{\lambda_1, ..., \lambda_k\}$ be the spectrum of Q and $p_Q(\lambda) = (\lambda - \lambda_1)^{s_1} ... (\lambda - \lambda_k)^{s_k}$ the minimal polynomial of Q. If $\dfrac{1}{p_Q(\lambda)} = \dfrac{a_1(\lambda)}{(\lambda - \lambda_1)^{s_1}} + \dots + \dfrac{a_k(\lambda)}{(\lambda - \lambda_k)^{s_k}}$, where $a_1(\lambda), ..., a_k(\lambda)$ are polynomials, then

$$1 = S_1(\lambda) + \dots + S_k(\lambda). \tag{3.1}$$

where $S_i(\lambda) = a_i(\lambda)(\lambda - \lambda_1)^{s_1} ... (\lambda - \lambda_{i-1})^{s_{i-1}} (\lambda - \lambda_{i+1})^{s_{i+1}} ... (\lambda - \lambda_k)^{s_k}$.

From relation (3.1) we get

$$I_n = S_1(Q) + \dots + S_k(Q). \tag{3.2}$$

Since for $i \neq j$, $p_Q(\lambda)$ is a divisor of $S_i(\lambda)S_j(\lambda)$ and $p_Q(Q) = 0$, we have $S_i(Q)S_j(Q) = 0$.

Let J be the Jordan form of Q and $U, V \in \mathcal{M}_n(\mathbb{R})$ such that $U \cdot V = I_n$ and $Q = U \cdot J \cdot V$. Then

$$I_n = U \cdot [S_1(J) + \ldots + S_k(J)] \cdot V. \tag{3.3}$$

Because $U \cdot V = I_n$, we have also $V \cdot U = I_n$ and then from (3.3) we obtain

$$I_n = S_1(J) + \ldots + S_k(J). \tag{3.4}$$

Let $J = B(\lambda_1) \oplus \ldots \oplus B(\lambda_k)$, where for every $t = 1, \ldots, k$, $B(\lambda_t) = C_1(\lambda_t) \oplus \ldots \oplus C_{c_t}(\lambda_t)$ and $C_\alpha(\lambda_t)$ is a Jordan cell of order o_α and $o_1 + \ldots + o_{c_t} = m_Q(\lambda_t)$. We put $m_t = m_Q(\lambda_t)$. Since $s_t = \max\{o_1, \ldots, o_{c_t}\}$, it follows that $(B(\lambda_t) - \lambda_t I_{m_t})^{s_t} = 0$. Because $(J - \lambda_t I_n)^{s_t} = (B(\lambda_1) - \lambda_t I_{m_1})^{s_t} \oplus \ldots \oplus (B(\lambda_k) - \lambda_t I_{m_k})^{s_t}$, the block which corresponds to λ_t in the matrix $(J - \lambda_t I_n)^{s_t}$ is null. The matrix $S_i(J)$ is made up from blocks which have the same orders as the blocks of the matrix J and all those blocks (of $S_i(J)$) which correspond to the eigenvalues $\lambda_1, \ldots, \lambda_{i-1}, \lambda_{i+1}, \ldots, \lambda_k$ are null. Since all the matrices $S_1(J), \ldots, S_{i-1}(J), S_{i+1}(J), \ldots, S_k(J)$ have the block which corresponds to λ_i null, taking into consideration the relation (3.4) it follows that the block which corresponds to λ_i in the matrix $S_i(J)$ is I_{m_i}, i.e. $S_i(J) = O_{m_1} \oplus \ldots \oplus O_{m_{i-1}} \oplus I_{m_i} \oplus O_{m_{i+1}} \oplus \ldots \oplus O_{m_k}$ (O_m is the null matrix of order m).

If we put ${}^t e_j = (0 \ldots 0\,1\,0 \ldots 0)$, where 1 fills the j position, and $n_0 = 0$, $n_j = m_1 + \ldots + m_j$, $j = 1, \ldots, k$, ($n_k = m$), then we have $S_i(J) = \sum_{j=n_{i-1}+1}^{n_i} e_j\, {}^t e_j$.

From relation (3.2) we get $Q = QS_1(Q) + \ldots + QS_k(Q)$ and then $Q = \lambda_1 S_1(Q) + (Q - \lambda_1 I_m)S_1(Q) + \ldots + \lambda_k S_k(Q) + (Q - \lambda_k I_m)S_k(Q)$.

If $U = [u_1 \ldots u_n]$ and ${}^t V = [v_1 \ldots v_n]$ ($u_1, \ldots, u_n$ are the columns of U and ${}^t v_1, \ldots, {}^t v_n$ are the rows of V), then

$$S_i(Q) = U \cdot S_i(J) \cdot V = \sum_{j=n_{i-1}+1}^{n_i} u_j\, {}^t v_j$$

and

$$(Q - \lambda_i I_m)S_i(Q) = \sum_{j=n_{i-1}+1}^{n_i} (Q - \lambda_i I_m) u_j\, {}^t v_j = \sum_{j=n_{i-1}+1}^{n_i - 1} \varepsilon_j u_j\, {}^t v_{j+1},$$

where $\varepsilon_j \in \{0, 1\}$ for $j = n_{i-1} + 1, \ldots, n_i - 1$. So we obtain

$$Q = \sum_{i=1}^{k} \left(\lambda_i \sum_{j=n_{i-1}+1}^{n_i} u_j\, {}^t v_j + \sum_{j=n_{i-1}+1}^{n_i} \varepsilon_j u_j\, {}^t v_{j+1} \right).$$

Since Q is a stochastic matrix, one of its eigenvalues $\lambda_1, \ldots, \lambda_k$ is equal to 1. Let us suppose that $\lambda_1 = 1$. Because all the eigenvalues of Q are real, it follows that $\lambda_2, \ldots, \lambda_k \in \mathbb{R}$. We have $u_j,\ v_j \in \mathcal{M}_{m \times 1}(\mathbb{R})$, $\varepsilon_j \in \{0, 1\}$ for any j, and $0 = n_0 <$

$n_1 < \ldots < n_k = m$. So Q has the same form as in 1), but we have to show that $\varepsilon_1 = 0$. This fact will be proved below.

From $V \cdot U = I_n$ we obtain ${}^t v_i u_j = \delta_{ij}$, $i, j \in \{1, \ldots, m\}$ and therefore the condition 2) is fulfilled.

u_1 is a right eigenvector for the eigenvalue 1 and ${}^t v_1$ is a left eigenvector for the same eigenvalue. Because a right eigenvector for the eigenvalue 1 is ${}^t(1 \ldots 1)$, ${}^t v_1 u_1 = 1$, $[\mu]Q = [\mu]$ and $[\mu]\cdot\, {}^t(1 \ldots 1) = 1$, we can take ${}^t u_1 = (1 \ldots 1)$ and ${}^t v_1 = [\mu]$. Now the condition 3) is satisfied.

We have ${}^t v_1 Q = {}^t v_1 + \varepsilon_1\, {}^t v_2$, ${}^t v_1 = [\mu]$ and $[\mu]Q = [\mu]$. It results that $\varepsilon_1\, {}^t v_2 = 0$ and because ${}^t v_2 \neq 0$, we have $\varepsilon_1 = 0$. So the condition 1) is completely satisfied.

The remark made in the previous section tell us that the spectrum of Q contains the spectrum of $\varphi(Q)$. Because $\varphi(Q)$ is a stochastic matrix, $\lambda_1 = 1$ belongs to the spectrum of $\varphi(Q)$. Let $\{\lambda_1, \ldots, \lambda_l\}$ be the spectrum of $\varphi(Q)$. We have $m_{\varphi(Q)}(\lambda_1) + \ldots + m_{\varphi(Q)}(\lambda_l) = p$ (p is the order of $\varphi(Q)$) and because $m_{\varphi(Q)}(\lambda_i) = m_Q(\lambda_i)$, $i = 1, ..., l$, we get $m_Q(\lambda_1) + \ldots + m_Q(\lambda_l) = p$, this is $n_l = m_1 + \ldots + m_l = p$, ($m_i = m_Q(\lambda_i)$) and so the condition 4) is fulfilled.

We have

$$\varphi(Q) = \sum_{i=1}^{k} (\lambda_i S_i(\varphi(Q)) + (\varphi(Q) - \lambda_i I_p) S_i(\varphi(Q)))$$

(because $\varphi(P(Q)) = P(\varphi(Q))$ for any polynomial P).

Since for $i \geq l+1$ the minimal polynomial of $\varphi(Q)$ is a divisor of S_i, $S_i(\varphi(Q)) = 0$ and $(\varphi(Q) - \lambda_i I_p) S_i(\varphi(Q)) = 0$.

On the other hand,

$S_i(\varphi(Q)) = \varphi(S_i(Q)) = \varphi\left(\sum_{j=n_{i-1}+1}^{n_i} u_j\, {}^t v_j\right) = \sum_{j=n_{i-1}+1}^{n_i} {}_{\varphi}u_j\, {}^t v_j^{\varphi}$ and

$((\varphi(Q) - \lambda_i I_p) S_i(\varphi(Q)))^q = \varphi\left(((Q - \lambda_i I_m) S_i(Q))^q\right) =$

$\sum_{j=n_{i-1}+1}^{n_i - q} \varepsilon_j \varepsilon_{j+1} \ldots \varepsilon_{j+q-1}\, {}_{\varphi}u_j\, {}^t v_{j+q}^{\varphi}$, and so we obtain the condition 5). The condition 6) is obvious satisfied because Q is a stochastic matrix.

We suppose now that the conditions 1), ... 6) are satisfied.

Since ${}^t v_i u_1 = 0$ for any $i = 2, \ldots, m$ and ${}^t u_1 = (1\ \ldots\ 1)$, the sum of all the terms in the same row of a matrix $u_i \cdot\, {}^t v_i$ or $u_i \cdot\, {}^t v_{i+1}$ with $i \geq 2$ is equal to 0. Because the sum of all the terms in the same row of the matrix $u_1 \cdot\, {}^t v_1$ is equal to 1 ($u_1 = (1\ \ldots\ 1)$, ${}^t v_1 = [\mu]$) and the matrix Q has only positive terms, Q is a stochastic matrix.

From ${}^t v_1 u_1 = 1$ and ${}^t v_1 u_i = 0$ for $i \geq 2$, we obtain ${}^t v_1 Q = {}^t v_1 + \varepsilon_1\, {}^t v_2$. Because ${}^t v_1 = [\mu]$ and $\varepsilon_1 = 0$ we have $[\mu]Q = [\mu]$.

Let us denote $B_i = \sum_{j=n_{i-1}+1}^{n_i} u_j\, {}^t v_j$, $N_i = \sum_{j=n_{i-1}+1}^{n_i - 1} \varepsilon_j u_j\, {}^t v_{j+1}$ and $A_i = \lambda_i B_i + N_i$, $i = 1, \ldots, k$. From 2) we obtain for any $i, j \in \{1, \ldots, k\} : B_i \cdot B_j = \delta_{ij} B_i$, $N_i \cdot N_j = 0$ for $i \neq j$, $N_i^{q_i} = 0$ where $q_i = n_i - n_{i-1}$, $N_i \cdot B_j = B_j \cdot N_i = 0$ for $i \neq j$ and $N_i \cdot B_i = B_i \cdot N_i = N_i$. Then

$$Q^n = \sum_{i=1}^{k} A_i^n = \sum_{i=1}^{k} (\lambda_i^n B_i + \sum_{q=1}^{\min(n,q_i-1)} \mathrm{C}_n^q \lambda_i^{n-q} N_i^q).$$

We see that $N_i^q = \sum_{j=n_{i-1}+1}^{n_i-q} \varepsilon_j \varepsilon_{j+1} \ldots \varepsilon_{j+q-1}\, u_j\, {}^t v_{j+q}$ and then the condition 5) implies $\varphi\,(B_i) = 0$ and $\varphi\,(N_i^q) = 0$ for $i \geq l+1$. We have

$$\varphi\,(Q^n) = \sum_{i=1}^{l} (\lambda_i^n \varphi\,(B_i) + \sum_{q=1}^{\min(n,q_i-1)} \mathrm{C}_n^q \lambda_i^{n-q} \varphi\,(N_i^q)).$$

We will show that $\varphi\,(Q)^n$ has the same form. Since $UV = I_m$, $\sum_{i=1}^{m} u_i\ {}^t v_i = I_m$ and consequently $\varphi\left(\sum_{i=1}^{m} u_i\ {}^t v_i\right) = \varphi(I_m)$. Taking into consideration the conditions 4) and 5), $\sum_{i=1}^{p}\ {}_\varphi u_i\ {}^t v_i^\varphi = I_p$ (I_p is the unit matrix of order p). This relation implies ${}^t v_i^\varphi\ {}_\varphi u_j = \delta_{ij}$, $i, j \in \{1, \ldots, p\}$, and we have $\varphi\,(Q) = \sum_{i=1}^{l} (\lambda_i \varphi\,(B_i) + \varphi\,(N_i))$, $\varphi\,(B_i) = \sum_{j=n_{i-1}+1}^{n_i} {}_\varphi u_j\, {}^t v_j^\varphi$,
$\varphi\,(N_i) = \sum_{j=n_{i-1}+1}^{n_i-1} \varepsilon_j\, {}_\varphi u_j\, {}^t v_{j+1}^\varphi$. Therefore we can follow the same way as we followed when we found the form of Q^n and we get

$$\varphi\,(Q)^n = \sum_{i=1}^{l} (\lambda_i^n \varphi\,(B_i) + \sum_{q=1}^{\min(n,q_i-1)} \mathrm{C}_n^q \lambda_i^{n-q} \varphi\,(N_i)^q).$$

But $\varphi\,(N_i)^q = \varphi\,(N_i^q)$ and then $\varphi\,(Q^n) = \varphi\,(Q)^n$.

The condition 1) gives in fact a Jordan form of the matrix Q and we see that $\lambda_1, ..., \lambda_k$ are the eigenvalues of this matrix and all are real. Using the same argument we get $m_Q(\lambda_i) = n_i - n_{i-1}$.

Since $\varphi(Q) = \sum_{i=1}^{l} \left(\lambda_i \sum_{j=n_{i-1}+1}^{n_i} {}_\varphi u_j\, {}^t v_j^\varphi + \sum_{j=n_{i-1}+1}^{n_i-1} \varepsilon_j\, {}_\varphi u_j\, {}^t v_{j+1}^\varphi \right)$ and ${}^t v_i^\varphi\ {}_\varphi u_j = \delta_{ij}$, $i, j \in \{1, \ldots, p\}$, here we have a Jordan form of the matrix $\varphi(Q)$ and we can see that $m_{\varphi(Q)}(\lambda_i) = n_i - n_{i-1} = m_Q(\lambda_i)$, $i = 1, ..., l$. Q.E.D.

Example. Let $X = \{1, 2, 3, 4\}$, μ such that $[\mu] = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{pmatrix}$, $Y = \{1, 2\}$

and $\varphi : X \to Y$, $\varphi(1) = \varphi(2) = 1$, $\varphi(3) = \varphi(4) = 2$. Then

$$[\mu^{\varphi}] = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix} \quad \text{and} \quad [\mathcal{I}_{\varphi}] = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix}.$$

Let
${}^t u_1 = (1 \quad 1 \quad 1 \quad 1)\,,\ {}^t u_2 = (1 \quad 0 \quad 0 \quad -1)$,

${}^t u_3 = \left(\frac{1}{2} \quad -\frac{1}{2} \quad \frac{1}{2} \quad -\frac{1}{2}\right),\ {}^t u_4 = (3 \quad -1 \quad -1 \quad -1)$,

${}^t v_1 = \left(\frac{1}{4} \quad \frac{1}{4} \quad \frac{1}{4} \quad \frac{1}{4}\right),\ {}^t v_2 = (0 \quad 1 \quad 0 \quad -1)\,,$

${}^t v_3 = (0 \quad -1 \quad 1 \quad 0),\ {}^t v_4 = \left(\frac{1}{4} \quad -\frac{1}{4} \quad -\frac{1}{4} \quad \frac{1}{4}\right)$.

Consider the matrix

$$Q = u_1{}^t v_1 + \lambda_2 u_2{}^t v_2 + \lambda_3(u_3{}^t v_3 + u_4{}^t v_4) + u_3{}^t v_4.$$

We have $n_1 = 1$, $n_2 = 2$, $n_3 = 4$, $l = 3$, $p = n_2$. If we put $\lambda_2 = \frac{3}{8}$ and $\lambda_3 = \frac{1}{4}$, then we get

$$Q = u_1{}^t v_1 + \frac{3}{2} u_2{}^t v_2 + \frac{1}{4}(u_3{}^t v_3 + u_4{}^t v_4) + u_3{}^t v_4 = \begin{pmatrix} \frac{9}{16} & \frac{3}{16} & \frac{1}{16} & \frac{3}{16} \\ \frac{1}{16} & \frac{9}{16} & \frac{5}{16} & \frac{1}{16} \\ \frac{5}{16} & \frac{1}{16} & \frac{5}{16} & \frac{5}{16} \\ \frac{1}{16} & \frac{3}{16} & \frac{5}{16} & \frac{7}{16} \end{pmatrix}$$

and all the conditions 1),...,6) are fulfilled. Consequently we have $[\mu]Q = [\mu]$ and $\varphi(Q^n) = \varphi(Q)^n$, $n \geq 1$.

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