

# On metrical linear connections with torsion in Riemannian Geometry

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## Abstract

On a Riemannian manifold endowed with a metrical linear connection with nonvanishing torsion, there are computed the first and second variation of energy and there is proven that the main results related to geodesics and Jacobi fields known for the Levi-Civita connection (including Morse's index theorem), hold true also in this case.

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**Key words:** Riemannian manifold, torsion, geodesic, Jacobi field, energy, exponential map.

## 1 Introduction

The study of the geometry of Riemannian manifolds, endowed with linear connections with nonvanishing torsions is requested by some situations when the Levi-Civita connection is not advantageous, such as: 1) in the geometry of higher order ([5]), or 2) in the geometry of the total space of a vector bundle ([6]).

In these two particular cases, the equations of geodesics and the equations of deviations of geodesics are established by V. Balan, R. Miron, P. Stavrinou, and Gr. Tsagas, [1], [2], [7], by defining geodesics as extremal curves of the distance Lagrangian

$$L(c) = \int_0^1 \sqrt{\langle \dot{c}, \dot{c} \rangle}.$$

In our paper, geodesics are defined as extremal curves of the energy Lagrangian  $E(c) = \int_0^1 \langle \dot{c}, \dot{c} \rangle$ ; in the following, we compute the first and the second variation of energy, deduce the equations of geodesics, define Jacobi fields and generalize some classical results related to geodesics and Jacobi fields, for any Riemannian manifold, endowed with a metrical linear connection (not necessarily the Levi-Civita one).

Throughout the paper, by "classical case", we mean the case of the Levi-Civita connection. Also, by "smooth", we mean " $C^\infty$ -differentiable". The techniques used below can be used even in the cases when the arc-length is not invariant to reparametrizations (for example, in the geometries of higher order, [9]).

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## 2 Geodesics and the exponential map

Let  $M$  be a real differentiable manifold of dimension  $n$  and class  $C^\infty$ , endowed with a Riemannian metric  $g$  and  $\langle X, Y \rangle = g(X, Y)$ ,  $\forall X, Y \in \mathcal{X}(M)$ . The coordinates of a point  $p \in M$  in a local chart  $(U, \phi)$  will be denoted by  $\phi(p) = (x^i)$ . Let  $\partial_i = \frac{\partial}{\partial x^i}$  be the natural basis of  $\mathcal{X}(M)$  and  $D$ , a metrical linear connection on  $M$ , of coefficients  $\Gamma_{jk}^i$ :  $D\partial_k \partial_j = \Gamma_{jk}^i \partial_i$ .

We denote by  $T_{jk}^i$  the coefficients of the torsion tensor of  $D$  ( $T(\partial_k, \partial_j) = T_{jk}^i \partial_i$ ) and by  $R_{jkl}^i$ , the coefficients of the curvature tensor  $R$ :  $R(\partial_l, \partial_k) \partial_j = R_{jkl}^i \partial_i$ .

Let  $c : [0, 1] \rightarrow M$  be a piecewise smooth curve and  $0 = t_0 < t_1 < \dots < t_k = 1$  a division of  $[0, 1]$  so that  $c$  be of class  $C^\infty$  on each  $[t_{i-1}, t_i]$ ,  $i = 1, \dots, k$ . A variation of  $c$  [4], is a mapping  $\alpha : (-\varepsilon, \varepsilon) \times [0, 1] \rightarrow M$ , (where  $\varepsilon > 0$ ), with the properties: 1)  $\alpha(0, t) = c(t)$ ,  $\forall t \in [0, 1]$  and 2)  $\alpha$  is continuous on each  $(-\varepsilon, \varepsilon) \times [t_{i-1}, t_i]$ ,  $\forall i = 1, \dots, k$ . We denote by  $\bar{\alpha}$  the mapping defined on  $(-\varepsilon, \varepsilon)$  by  $\bar{\alpha}(u)(t) = \alpha(u, t)$ .

If  $\alpha(u, 0) = p$ ,  $\alpha(u, 1) = q$ ,  $\forall u \in (-\varepsilon, \varepsilon)$ , the variation  $\alpha$  has fixed endpoints.

The deviation vector field attached to  $\alpha$  is

$$(2.1) \quad W(t) = \frac{\partial \alpha}{\partial u}(0, t)$$

(if  $\alpha$  has fixed endpoints, then  $W(0) = W(1) = 0$ ) and the tangent vector field, [7],  $V = \dot{c}$ . In local writing,

$$\dot{c} = V = V^i \partial_i, \quad W = W^i \partial_i.$$

Let also

$$(2.2) \quad A := \frac{DV}{dt} = A^i \partial_i,$$

be the covariant acceleration, where, for  $X \in \mathcal{X}(M)$ , we denoted  $\frac{DX}{dt} := D_{\dot{c}}X$ , and

$$(2.3) \quad \Delta_t X = X(t_+) - X(t_-), \quad t \in [0, 1], \quad X \in \mathcal{X}(M)$$

the jump of  $X \in \mathcal{X}(M)$  in  $t$ .

The energy of the curve  $c$  is, [4],  $E(c) = \int_0^1 \langle V, V \rangle dt = \int_0^1 g_{ij} V^i V^j dt$ .

Taking into account that  $D$  is metrical, one obtains:

**Theorem 1. (The first variation of the energy):** *If  $c : [0, 1] \rightarrow M$  is a piecewise smooth path and  $\alpha : (-\varepsilon, \varepsilon) \times [0, 1] \rightarrow M$  is a variation with fixed endpoints of  $c$ , then*

$$(2.4) \quad \frac{1}{2} \frac{dE(\bar{\alpha}(u))}{du} \Big|_{u=0} = - \sum_{i=0}^{k-1} \langle W, \Delta_{t_i} V \rangle + \int_0^1 \langle T(W, V), V \rangle - \langle W, A \rangle dt.$$

If the curve  $c$  is  $C^\infty$ -smooth on the whole  $[0, 1]$ , then, in the previous equality, the term  $\sum_{i=0}^{k-1} \langle W, \Delta_{t_i} V \rangle$  vanishes.

By writing the term  $\langle T(W, V), V \rangle$  in the form  $\langle F, W \rangle$ , one obtains that:

1.  $F = F^i \partial_i$ , given by

$$(2.5) \quad F^i = g^{ik} g_{hl} T_{jk}^h V^j V^l$$

is a vector field along  $c$ , which does not depend on the variation  $\alpha$  of  $c$ .

2. There holds the equality  $\langle T(W, V), V \rangle = \langle W, F \rangle$ .

By *geodesic* of  $M$  we shall mean an extremal curve  $c : [0, 1] \rightarrow M$  of the energy  $E$ , of class  $C^\infty$  on the whole  $[0, 1]$ . Taking into account the previous results, we get

**Theorem 2.** *The  $C^\infty$ -smooth curve  $c : [0, 1] \rightarrow M$ ,  $t \mapsto (x^i(t))$  is a geodesic of  $M$  if and only if  $\frac{D}{dt} \frac{dc}{dt} = F$ , or, in local coordinates,*

$$(2.6) \quad \frac{DV^i}{dt} = F^i$$

(2.6) is an ODE system of order two, with the unknown real functions  $x^i$ ,  $i = 1, \dots, n$ . In conditions of regularity, the associated Cauchy problem has a unique solution; this is, for any  $p \in M$ , and  $V \in T_p M$ , there uniquely exists a geodesic  $c : (-\varepsilon, \varepsilon) \rightarrow M$ , with the initial conditions

$$(2.7) \quad c(0) = p_0, V(0) = V_0.$$

Furthermore, the solution depends smoothly on the initial conditions 2.7.

**Remark 3.** *If the curve  $c : [0, 1] \rightarrow M$ ,  $t \mapsto (x^i(t))$  is a geodesic, then, for any  $\lambda > 0$ , the curve  $\bar{c} : [0, \frac{1}{\lambda}] \rightarrow M$ ,  $\bar{x}^i(\frac{t}{\lambda}) := x^i(t)$ , is a geodesic, too.*

Thus, we can state

**Theorem 4.** *Let  $p \in M$  and  $V \in T_p M$ . There exists an  $\varepsilon > 0$  so that, if  $\|V\| < \varepsilon$ , there uniquely exists the geodesic  $c : (-2, 2) \rightarrow M$ ,  $t \mapsto (x^i(t))$ , with the initial conditions*

$$c(0) = p, \frac{dc}{dt}(0) = V.$$

The point  $c(1)$  is called the *exponential* of  $V \in T_p M$  in  $p$  and will be denoted by

$$(2.8) \quad c(1) = \exp_p(V).$$

The exponential map in  $p \in M$  is generally defined only for small values of  $\|V\|$ ; obviously, if it exists, the value  $\exp_x(V)$  is unique. Furthermore, if  $c$  is a geodesic of  $M$  with  $p = c(0)$ ,  $V = \dot{c}(0)$ , then

$$(2.9) \quad c(t) = \exp_p(tV).$$

In the case of the Levi-Civita connection, along any geodesic, we have  $\|V\| =$  constant, as a consequence of the fact that  $\frac{DV}{dt} = 0$ . Though, in our case,  $\frac{DV}{dt}$  does not generally vanish, we still can prove

**Proposition 5.** *Let  $p \in M$  and  $V \in T_p M$ . The geodesic  $t \mapsto c(t) = \exp_p(tV_0)$  has the property  $\left\| \frac{dc}{dt} \right\| = \|V_0\| = \text{constant}$ .*

**Proof.** Let  $V = \frac{d \exp_p(tV_0)}{dt}$ , the velocity vector field of  $c$  and

$$\alpha(u, t) = \exp_p((u + t)V_0), \quad u \in (-\varepsilon, \varepsilon),$$

where  $\varepsilon > 0$ . Then,  $\alpha$  is a variation of  $c$ , having as deviation vector field

$$W = \frac{\partial \alpha}{\partial u} \Big|_{u=0} = \frac{d \exp_p(sV_0)}{ds} \Big|_{s=t} \cdot \frac{\partial s}{\partial u} \Big|_{u=0} = \frac{d \exp_p(tV_0)}{dt} = V.$$

Since  $c$  is a geodesic, then  $\frac{DV}{dt} = F$ , where  $\langle F, W \rangle = \langle T(W, V), V \rangle$ ; from  $W = V$ , there follows  $T(W, V) = 0$ , this is,  $\langle F, V \rangle = \left\langle \frac{DV}{dt}, V \right\rangle = 0$ . Now, the conclusion follows immediately.  $\square$

### 3 Minimal geodesics

For a curve  $c : [0, 1] \rightarrow M$ , let  $L(c) = \int_0^1 \sqrt{\langle \dot{c}, \dot{c} \rangle}$  denote its length. The proofs of the following results are similar to those in the classical case, [4]:

1.  $(L(c))^2 \leq E(c)$ , with equality if and only if  $\|\dot{c}\| = \text{constant}$ .
2. The geodesic  $t \mapsto c(t) = \exp_p(tV_0)$  is minimal for the energy  $E(c)$  if and only if  $c$  is minimal for the length  $L(c)$ . Such a geodesic will be called in the following, a geodesic of minimal length (or, simply, *minimal*).
3. For any  $p \in M$ , there exists a neighbourhood  $\mathcal{W}$  and an  $\varepsilon > 0$ , so that :  
 a) any two points of  $\mathcal{W}$  can be joined by a unique geodesic of length  $< \varepsilon$ ; b)  
 if  $t \mapsto \exp_{q_1}(tV)$  is the geodesic joining  $q_1, q_2 \in \mathcal{W}$ , then the correspondence  $(q_1, q_2) \mapsto (q_1, V)$  is differentiable; c) for any  $q \in \mathcal{W}$ , the application  $\exp_p$  maps the open ball of radius  $\varepsilon$  from  $T_q M$  diffeomorphically into an open set  $U_q \supset \mathcal{W}$ .

Let  $\mathcal{W}$  and  $\varepsilon$  as above. If we suppose, as in [4], that the geodesic in a) is entirely contained in  $\mathcal{W}$ , there also holds

**Theorem 6.** *If  $\gamma : [0, 1] \rightarrow M$  is the geodesic of length  $< \varepsilon$  joining  $p, q \in \mathcal{W}$  and  $\omega : [0, 1] \rightarrow M$ , an arbitrary curve joining  $p$  and  $q$ , then*

$$\int_0^1 \left\| \frac{d\gamma}{dt} \right\| dt \leq \int_0^1 \left\| \frac{d\omega}{dt} \right\| dt.$$

The proof follows the same three steps as that in the classical case ([4]); though, we must essentially take into account the fact that the linear connection  $D$  is **not** torsionless:

**1.** We show that, on the neighbourhood  $U_q$  of  $q$ , any geodesic through  $q$  is orthogonal onto the hypersurface  $S = \{\exp_q(V) \mid V \in T_q(M), \|V\| = \text{constant}\}$ . This is equivalent to the fact that, on the parametrized surface  $f(r, \theta) = \exp_q(rV(\theta))$ ,  $r \in (0, \varepsilon)$ ,  $\theta \in [0, 1]$  (where  $V(\theta) \in T_qM$ ,  $\|V(\theta)\| \equiv 1$ ,) the curves  $r = r_0$  are orthogonal to the curves  $\theta = \theta_0$ . Thus, we must prove that

$$V = \frac{\partial f}{\partial r} \text{ and } W = \frac{\partial f}{\partial \theta}$$

are orthogonal.

If we consider  $f(r, \theta)$  as a variation (without fixed endpoints) of the radial geodesic  $r \mapsto \exp_q(rV(\theta_0))$  (i.e.,  $\theta = \theta_0$ ), then  $W|_{\theta_0}$  is the corresponding deviation vector field. Let  $\theta = \theta_0$  be arbitrary. Since

$$f_{\theta_0} : [0, r_0] \rightarrow M, f_{\theta_0}(r) = \exp_q(rV(\theta_0)),$$

(where  $r_0 \in (0, \varepsilon)$ ) is a geodesic, we have  $\frac{DV}{\partial r}|_{\theta=\theta_0} = F$ , with  $\langle F, W \rangle|_{\theta=\theta_0} = \langle T(W, V), V \rangle|_{\theta=\theta_0}$ . Then, in  $\theta = \theta_0$ , we have

$$\begin{aligned} \frac{\partial}{\partial r} \langle V, W \rangle &= \left\langle \frac{DV}{\partial r}, W \right\rangle + \left\langle V, \frac{DW}{\partial r} \right\rangle = \langle F, W \rangle + \left\langle T(V, W) + \frac{DV}{\partial \theta}, V \right\rangle = \\ &= \langle F, W \rangle - \langle T(W, V), V \rangle + \left\langle \frac{DV}{\partial \theta}, V \right\rangle = \left\langle \frac{DV}{\partial \theta}, V \right\rangle. \end{aligned}$$

From  $\left\langle \frac{DV}{\partial \theta}, V \right\rangle = \frac{1}{2} \frac{\partial}{\partial \theta} \|V\|^2 = 0$ , we obtain  $\frac{\partial}{\partial r} \langle V, W \rangle|_{\theta=\theta_0} = 0$ , which means that  $\langle V, W \rangle$  does not depend on  $r$ . Evaluating in  $r = 0$  we have  $\langle V, W \rangle|_{r=0} = 0$ ; since  $\theta_0$  is arbitrary, it follows  $\langle V, W \rangle = 0$ .

**2.** Let  $\omega : [0, 1] \rightarrow U_q$  be any piecewise smooth path. Then, any point  $\omega(t)$  can be uniquely written as  $\omega(t) = \exp_q(r(t)V(t))$ , with  $V \in T_qM$ ,  $\|V\| = 1$ .

Then,  $\int_0^1 \left\| \frac{d\omega}{dt} \right\| dt \geq |r(1) - r(0)|$ , where equality holds if and only if  $r(t)$  is monotone and  $V(t) = \text{constant}$  (i.e., if  $\omega$  is a radial geodesic).

**3)** If  $q, q' \in U_q$  and  $\omega : [0, 1] \rightarrow U_q \subset M$  any piecewise smooth path with  $\omega(0) = q$ ,  $\omega(1) = q'$  and  $\gamma$  - the unique geodesic joining  $q$  and  $q'$ , then we can write

$$q' = \exp_q(rV), \text{ with } r \in (0, \varepsilon), \|V\| = 1.$$

Then, for any  $\delta > 0$ ,  $\omega$  contains a segment which joins the spheres of radius  $\delta$ , respectively,  $r$ , centered in  $q$ , and lying between these spheres. The length of this segment is  $l(\delta) \geq r - \delta$ . Taking  $\delta \rightarrow 0$ , the proof is completed.

**Corollary 7.** *If  $c : [0, 1] \rightarrow M$ , with  $c(0) = p$ ,  $c(1) = q$ , has the length less than any other curve joining  $p$  and  $q$ , then  $c$  is a geodesic.*

## 4 Jacobi fields and conjugate points

Let  $c : [0, 1] \rightarrow M$  be a geodesic and  $\alpha : U \times [0, 1] \rightarrow M$ , be a 2-parameter variation of  $c$  by piecewise smooth curves, with fixed endpoints,  $U$  being a neighbourhood of  $(0, 0) \in \mathbb{R}^2$ . Let

$$W_1(t) := \frac{\partial \alpha}{\partial u_1}(0, 0, t), \quad W_2(t) := \frac{\partial \alpha}{\partial u_2}(0, 0, t)$$

and  $\bar{\alpha}$ , the mapping defined on  $U$  by  $\bar{\alpha}(u_1, u_2)(t) = \alpha(u_1, u_2, t)$ .

Let  $E_{**}$  be the hessian of  $E$ , [4], [3], namely

$$(4.10) \quad E_{**}(W_1, W_2) := \frac{\partial^2 E(\bar{\alpha}(u_1, u_2))}{\partial u_1 \partial u_2} \Big|_{(0,0)}.$$

Let  $\mathcal{F} = \mathcal{F}^i \partial_i$  the vector field given by

$$\left\langle T \left( \frac{\partial \alpha}{\partial u_2}, \frac{\partial \alpha}{\partial t} \right), \frac{\partial \alpha}{\partial t} \right\rangle = \left\langle \mathcal{F}, \frac{\partial \alpha}{\partial u_2} \right\rangle,$$

or, in local coordinates:  $\mathcal{F}^i = g^{ik} g_{hl} T_{jk}^h \frac{\partial \alpha^j}{\partial t} \frac{\partial \alpha^l}{\partial t}$ .

There obviously holds  $\mathcal{F}(0, 0, t) = F(t)$ . There holds

**Theorem 8.** *If  $c : [0, 1] \rightarrow M$  is a geodesic and  $\alpha : U \times [0, 1] \rightarrow M$  is a 2-parameter variation of  $c$  by piecewise smooth curves,  $U$  being a neighbourhood of  $(0, 0) \in \mathbb{R}^2$ , then the hessian  $E_{**}$  is given by:*

$$(4.11) \quad \frac{1}{2} E_{**}(W_1, W_2) = - \sum_{i=1}^{k-1} \left\langle W_2, \Delta_{t_i} \left( T(W_1, V) + \frac{DW_1}{dt} \right) \right\rangle + \int_0^1 \left\langle W_2, \frac{D\mathcal{F}}{\partial u_1} \Big|_{u_1=0} + R(V, W_1)V - \frac{D}{dt} T(W_1, V) - \frac{D^2 W_1}{dt^2} \right\rangle dt,$$

where  $0 = t_0 < t_1 < \dots < t_k = 1$  is a division of  $[0, 1]$  so that  $\alpha$  be  $\mathcal{C}^\infty$  on each  $U \times (t_{i-1}, t_i)$ ,  $i = 1, \dots, k$ .

If we consider only variations  $\bar{\alpha} = \bar{\alpha}(u_1, u_2)$  by  $\mathcal{C}^\infty$ -smooth curves on the whole  $[0, 1]$  of  $c$ , then in 4.11, the term  $\sum_{i=1}^{k-1} \left\langle W_2, \Delta_{t_i} \left( T(W_1, V) + \frac{DW_1}{dt} \right) \right\rangle$  vanishes.

$E_{**}(W_1, W_2)$  is a symmetric bilinear form. By considering its associated quadratic form  $E_{**}(W, W)$ , one obtains

**Proposition 9.** *Let  $c : [0, 1] \rightarrow M$  be a geodesic. Then  $E_{**}$  is positively semidefinite along  $c$  if and only if  $c$  is minimal..*

**Definition 10.** *A  $\mathcal{C}^\infty$ -smooth vector field  $J = J^i \partial_i$  along the geodesic  $c$  is called a **Jacobi field** if it satisfies the equation*

$$(4.12) \quad \frac{D^2 J}{dt^2} + \frac{D}{dt} T(J, V) = R(V, J)V + D_J \mathcal{F}.$$

In local coordinates, 4.12 writes

$$(4.13) \quad \frac{D^2 J^i}{dt^2} + \frac{DT^i}{dt} = R^i + \sum_{\beta=0}^2 J^\beta D_{\partial_j} \mathcal{F}^i, i = 1, \dots, n,$$

where  $T^i := V^j J^k T_{jk}^i$ ,  $R^i := -V^h V^j J^k R_{hjk}^i$ .

A Jacobi field is uniquely determined by the initial conditions  $J^i(0)$ ,  $\frac{DJ^i(0)}{dt} \in T_{c(0)}M$ ,  $i = 1, \dots, n$ .

Let  $c$  be a geodesic with no self-intersections. Two distinct points of the geodesic  $c$ ,  $p = c(a)$ ,  $q = c(b)$ ,  $a, b \in [0, 1]$  are *conjugate* points along the geodesic  $c$ , if there exists a nonvanishing Jacobi field  $J$  along  $c$  with

$$(4.14) \quad J(a) = 0, J(b) = 0.$$

The dimension of the space of Jacobi fields with the above property is called the *multiplicity* of  $p$  and  $q$  as conjugate points.

If  $c : [0, 1] \rightarrow M$  is a geodesic and  $T\Omega_c$  is the set of the (smooth) vector fields  $X$  along  $c$  with  $X(0) = X(1) = 0$ , then

$$\ker E_{**} = \{W_1 \in T\Omega_c \mid E_{**}(W_1, W_2) = 0, \forall W_2 \in T\Omega_c\},$$

is the kernel of  $E_{**}$ ; let  $\nu = \dim \ker E_{**}$  denote its nullity (if  $\nu > 0$ , then  $E_{**}$  is degenerate).

In the following, we shall expose some results which hold also in the case of a linear connection with nonvanishing torsions, the proof being similar to that in the classical case:

- Let  $c$  be a geodesic. Then: 1) For a  $\mathcal{C}^\infty$ -smooth vector field  $W_1$  along  $c$ , there holds the equivalence:  $W_1 \in \ker E_{**} \Leftrightarrow W_1$  is a Jacobi field along  $c$ . 2)  $E_{**}$  is degenerate if and only if the endpoints  $p$  and  $q$  of  $c$  are conjugate along  $c$ . 3) The nullity of  $E_{**}$  is equal to the multiplicity of  $p$  and  $q$  as conjugate points.
- If  $\alpha$  is a 1-parameter variation of  $c$  through geodesics, then the deviation vector field  $W$  is a Jacobi field along  $c$ . Conversely, any Jacobi field along the geodesic  $c$  can be written as the deviation vector field of a variation (not necessarily with fixed endpoints) of  $c$  through geodesics of  $M$ .
- A Jacobi field  $J$  along the geodesic  $c : [0, 1] \rightarrow M$  is uniquely determined by its values at the endpoints of the geodesic.
- Let  $c : [0, 1] \rightarrow M$  be a geodesic and  $p = c(0)$ . Then, for any  $W \in T_p M$ , the Jacobi field with  $X(0) = 0$ ,  $\frac{DX}{dt}(0) = W$  is given by

$$(4.15) \quad X(t) = (D \exp_p)(t\dot{c}(0))(tW).$$

- Finally, we mention that **Morse's index theorem** can be proved exactly as in the classical case.

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