A universal enveloping algebra for a Lie triple system

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Abstract

A new kind of universal enveloping algebra for a Lie triple system is made evident. This (new) universal enveloping algebra $\mathcal{U}(A)$ for the Lie triple system A is a T-algebra (i.e., it is a commutative nonassociative algebra satisfying to a specific identity) having the property that any Lie triple system homomorphism of A into a T-algebra B extends to an algebra isomorphism of $\mathcal{U}(A)$ into B. The algebra $\mathcal{U}(A)$ is a quotient of the nonassociative tensor algebra $\mathcal{T}\{A\}$ by a suitable two sided ideal and it is a filtered algebra.

It is especially important to solve the problem of representing the ternary composition of a certain Lie triple system by means of an appropriate binary composition defined on its ground space. The aim of this paper is to give some necessary conditions for the existence of a binary composition whose standard associated h-system is a given Lie triple system.

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1 Introduction

Lie triple systems (briefly, Ltss) were first noted by E. Cartan in his study on totally geodesic submanifolds (see [6]); they are subspaces of any Lie algebra which are closed under the ternary composition $[[x, y], z]$. If $\mathcal{G}$ is the Lie algebra of the Lie group G, then Ltss in $\mathcal{G}$ are connected with the totally geodesic subspaces of G in the same way that Lie subalgebras of $\mathcal{G}$ are related to analytic subgroups of G. Ltss can be also defined as the subspaces of any Lie algebra (over a field of characteristic $\neq 2$) mapped into their negatives by an involution of the given Lie algebra. Since the symmetry in a symmetric space give rise to the involution in the Lie algebra of the group defining the symmetric space, the Ltss arisen also in the studies on symmetric spaces [6, 23]. From the algebraic point of view, Ltss were studied by N. Jacobson [14, 15] and W. G. Lister [22]. Actually, they were introduced by N. Jacobson [14] in 1948 as being the abstract algebraic structures describing the subspaces of an associative algebra that are closed relative to the ternary operations $[[x, y], z]$ where $[x, y] = xy - yx$. They had also arisen in a natural way in the study of Jordan algebras and Jordan

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triple systems (Jts). Particular Ltss and Jtss have been early arisen (1938, 1943) in quantum mechanics [9, 19]. It must be also remarked that they arose naturally in the investigation by H. Freudenthal (1954) on the geometries of exceptional simple Lie groups [10]. Ltss were also used in differential geometry by P. I. Kovalev in the study of certain manifolds endowed with affine connections (see [20, 21]). Recently, N. Kamiya and S. Okubo connect Ltss with the study of the Yang-Baxter equation. A crucial moment in the development of Ltss structural theory was the result proved in 1951 by N. Jacobson [14], stating that every Lts $(T, [.,.,.])$ can be embedded into a Lie algebra $(L, [.,.])$ such that its ternary operation is a superposition of Lie brackets of L (i.e., $[x, y, z] = [[x, y], z], \forall\ x, y, z \in T$). This embedding suggests that Ltss are the tangent algebras of particular homogeneous spaces.

Another important class of Ltss is connected with a class of commutative algebras satisfying an identity of degree 4 (see [25]), the so called T-algebras. This connection is yielded by means of the ternary operation $[x, y, z] = x(yz) - y(xz)$ (see Proposition 2.2). On other hand, such a commutative algebra can be associated with every nonassociative algebra (see Proposition 2.4), that allows us to give an embedding result for Ltss similar to the Jacobson's embedding result. More exactly, it is proved the following result: *any Lts can be embedded into a commutative algebra satisfying to a certain identity of degree 4*. Actually, we shall prove that a universal enveloping T-algebra can be associated with every Lts. For a Lie triple system A, the universal enveloping T-algebra $\mathcal{U}(A)$ is a nonassociative algebra having the property that any Lie triple system homomorphism of A into the Lts associated to a nonassociative T-algebra B extends to an algebra isomorphism of $\mathcal{U}(A)$ into B. The algebra $\mathcal{U}(A)$ is a quotient of the nonassociative tensor algebra $\mathcal{T}\{A\}$ by a suitable two sided ideal and it is a filtered algebra.

In this paper we also give some necessary conditions for an Lts be the h-system (see [3]) associated with a binary algebra. The importance of solving this problem comes from the possibility that a Lts can be the tangent structure of a homogeneous space. There exist Ltss for which this problem has a solution but, unfortunately, the problem of representing the ternary composition of a Lie triple system by means of an appropriate binary composition, defined on its ground space, has not always a positive answer (see, for example, the meson triple system).

2 Preliminaries

A *Lie triple system* (briefly, Lts) is a vector space V over the field K, with a ternary composition $[.,.,.] : V \times V \times V \to V, (a, b, c) \to [a, b, c]$, which is trilinear and satisfies the following axioms:

$$
\begin{array}{ll}
(Lts.1) & [x, x, y] = 0, \\
(Lts.2) & [x, y, z] + [y, z, x] + [z, x, y] = 0, \\
(Lts.3) & [x, y, [u, v, w]] = [[x, y, u], v, w] + [u, [x, y, v], w] + [u, v, [x, y, w]],
\end{array}
$$

$\forall\ x, y, z, u, v, w \in V$. For every $x, y \in V$ we can define the vector space endomorphism $D_{(x,y)} : V \to V$ by $D_{(x,y)}(z) = [x, y, z]$; the equation (Lts. 3) says that $D_{(x,y)}$ is a Lts-derivation which is called a *inner derivation* of V. Moreover, the axiom (Lts. 3) assures that the usual bracket of endomorphisms endows $\mathcal{D} = Span_K\{D_{(x,y)} \mid x, y \in V\}$ with a Lie algebra structure; $\mathcal{D}$ is the so-called *inner derivation algebra* and it is also denoted by $InnDer(V)$. Consequently, it becomes clear that the possibility of using the Lie algebra representation theory in the study of Ltss arose.

Let $\mathcal{LTS}_K$ denote the category of Lie triple systems over K with the Lts homomorphisms as morphisms.

If L is a Lie algebra with product $[a, b]$, then the ternary composition $[a, b, c] = [[a, b], c]$ satisfies the above identities. N. Jacobson shown in 1951 [15] that any Lts may be considered as a subspace of a Lie algebra L in such a way that $[a, b, c] = [[a, b], c]$. The construction is: let L be the vector space direct sum $T \oplus (T \otimes T)^-$, where $(T \otimes T)^-$ is the factor space of $T \otimes T$ modulo the subspace of all $\sum_i a_i \otimes b_i$ such that $\sum_i [a_i, b_i, x] = 0$, for all $x \in T$. The product $[x, y]$ in L is defined by

$$[a, b] = (a \otimes b)^-, \quad \forall a, b \in T,$$

$$\big[(\textstyle\sum_i [a_i, b_i]), c\big] = \textstyle\sum_i [a_i, b_i, c], \ \forall a_i, b_i, c \in T, \tag{2.1}$$

$$\big[c, (\textstyle\sum_i [a_i, b_i])\big] = -\textstyle\sum_i [a_i, b_i, c], \ \forall a_i, b_i, c \in T,$$

$$\big[(\textstyle\sum_i [a_i, b_i]), (\textstyle\sum_j [c_j, d_j])\big] = \textstyle\sum_{i,j} [[a_i, b_i, c_j], d_j] - \textstyle\sum_{i,j} [[a_i, b_i, d_j], c_j],$$

$\forall\ a_i, b_i, c_i, d_i \in T$. L is then a Lie algebra, called the *standard enveloping Lie algebra* of T, and it is denoted by $L_s(T)$ or merely L_s. It is clear that the natural map of T into $L_s(T)$ is 1-1. One can also construct a universal enveloping Lie algebra $L_u(T)$, with the property that any homomorphism of T into a Lie algebra extends to a homomorphism of $L_u(T)$ and a universal associative algebra (with identity) $\mathcal{U}(T)$ with the same property relative to the homomorphisms of T into associative algebras. $\mathcal{U}(T)$ is then the universal associative algebra of $L_u(T)$, the natural maps of T into $\mathcal{U}(T)$ and $L_u(T)$ are 1-1 and $L_u(T) = T \oplus [T, T]$.

A Lts is called Abelian if $[x, y, z] = 0$ for any triple (x, y, z). Of course, every K-module T is an Abelian Lts by defining the Lts-bracket as $[x, y, z] = 0$. In this case, $L_s(T)$ identifies with the Abelian Lie algebra having underlying vector space T. Also, $L_u(T) = T \oplus \Lambda^2 T$, with $[x, y] = x \wedge y$ for $x, y \in T$ and with all other products vanishing.

In what follows we prove the existence of a new kind of universal enveloping algebra for Ltss, that is neither a Lie algebra nor an associative algebra; it is a commutative algebra satisfying to a certain identity, i.e., it is the so called *T-algebra* (see Definition 2.1). The origin for the construction of such a universal enveloping is the following classifying result given by J.M. Osborn in 1965 [25].

Theorem 2.1. *Let $A(\cdot)$ be a commutative (nonassociative) algebra with unity element over a field of characteristic not 2 or 3, and let A satisfies an identity of degree ≤ 4 not implied by the commutative law. Then A satisfies at least one of the following three identities*

$$(x^2 \cdot x) \cdot x = x^2 \cdot x^2,$$

$$2(yx \cdot x) \cdot x + y \cdot (x^2 \cdot x) = 3(y \cdot x^2) \cdot x, \tag{2.2}$$

$$2(y^2 \cdot x) \cdot x - 2(yx \cdot y) \cdot x + 2(yx \cdot x) \cdot x - y^2 \cdot x^2 + yx \cdot yx = 0.$$

Definition 2.1. Any commutative algebra (with or without identity element) satisfying the identity (2.2_2) is called a *T-algebra*.

Let us denote by $T - \mathcal{ALG}_K$ the category of T-algebras over K with the algebra homomorphisms as morphisms.

Proposition 2.2. *Let $A(\cdot)$ be a T-algebra over K. Then A becomes a Lts $\mathcal{LT}(A)$ where the underlying K-module is A and the Lts-composition is defined by*

$$[x, y, z] = x \cdot yz - y \cdot xz, \forall x, y, z \in A. \tag{2.3}$$

Proof. In [25] it is proved that (2.2_2) is equivalent with

$$(x \cdot yz)w + (x \cdot zw)y + (x \cdot wy)z = (y \cdot xz)w + (y \cdot zw)x + (y \cdot wx)z, \tag{2.4}$$

$\forall x, y, z, w \in A$. More suitable for our purpose is the following form of (2.4):

$$x \cdot (y \cdot zw) - y \cdot (x \cdot zw) - z \cdot (x \cdot yw) + z \cdot (y \cdot xw) - \tag{2.5}$$
$$-(x \cdot yz)w + (y \cdot xz)w = 0,$$

$\forall x, y, z, w \in A$. (Lts.1) and (Lts.2) yield as consequences of (2.3) and commutativity of " $\cdot$ ". By a straightforward computation one gets

$$[x, y, [u, v, w]] - [[x, y, u], v, w] - [u, [x, y, v], w] - [u, v, [x, y, w]] =$$
$$= f(x, y, u, v \cdot w) - v \cdot f(x, y, u, w) - f(x, y, u, v) \cdot w - f(x, y, v, uw) +$$
$$+ u \cdot f(x, y, v, w) + f(x, y, v, u) \cdot w,$$

where

$$f(x, y, z, w) = x \cdot (y \cdot zw) - y \cdot (x \cdot zw) - z \cdot (x \cdot yw) + \tag{2.6}$$
$$+ z \cdot (y \cdot xw) - (x \cdot yz)w + (y \cdot xz)w.$$

Consequently, (Lts.3) holds on A.

Therefore, Proposition 2.2 assures us that there exists a covariant functor **LT** : $T - \mathcal{ALG}_K \to \mathcal{LTS}_K$.

Later, will be useful the following result.

Proposition 2.3. *Any inner derivation of $\mathcal{LT}(A)$ is a derivation for $A(\cdot)$.*

Proof. Taking into account that $D_{(x,y)} = [L_x, L_y]$ it can be proved that (2.5) is just the equation $D_{(x,,y)}(zw) = D_{(x,y)}(z) \cdot w + z \cdot D_{(x,y)}(w)$.

Proposition 2.2 gives the suggestion to associate a Lts with any binary (nonassociative and non-commutative) algebra.

Let $B(\cdot)$ be a nonassociative algebra over a field K of characteristic not 2 or 3 and I be its two-sided ideal generated by $\{x \cdot y - y \cdot x, f(x, y, z, w) | \ \forall x, y, z, w \in B\}$. We set $\widetilde{B} = B/I$ and define the binary composition

$$\widetilde{B} \times \widetilde{B} \ni (\overline{x} = x + I, \overline{y} = y + I) \to \overline{x} \cdot \overline{y} = x \cdot y + I \in \widetilde{B}. \tag{2.7}$$

Proposition 2.4. *$\widetilde{B}(\cdot)$ is a commutative algebra that satisfies the identity* (2.2_2).

Proof. The assertion is a consequence of the following identities:

$$\overline{x}\cdot\overline{y} = x\cdot y + I = y\cdot x + (x\cdot y - y\cdot x) + I = \overline{y}\cdot\overline{x}, \forall\ \overline{x},\overline{y}\in\widetilde{B},$$

$$f(\overline{x},\overline{y},\overline{z},\overline{w}) = f(x,y,z,w) + I = I, \forall\ \overline{x},\overline{y},\overline{z},\overline{w}\in\widetilde{B}.$$

Remark 2.5. If $B(\cdot)$ has the identity element 1 then $\widetilde{B}(\cdot)$ has the identity element $1+I$.

Remark 2.6. Proposition 2.4 is a proof for the existence of algebras satisfying identity (2.2_2).

Corollary 2.7. *$\widetilde{B}$ becomes a Lts relative to the ternary operation defined by*

$$[\overline{x},\overline{y},\overline{z}] = \overline{x}\cdot\overline{yz} - \overline{y}\cdot\overline{xz}, \forall\overline{x},\overline{y},\overline{z}\in\widetilde{B}.$$

Now, the question arises, whether the functor **LT** has or not an adjoint; i.e., does there exists a functor $\mathbf{U}$:$\mathcal{LTS}_K \to T-\mathcal{ALG}_K$ with property

$$Hom_{Lts}(T,\mathbf{LT}(A)) \backsimeq Hom_{Alg}(\mathbf{U}(T),A),$$

or, more exactly, is there a functor **U** which is left adjoint to **LT**? This problem give rise to the construction of the universal enveloping T-algebra.

To this end we shall introduce, following closely Shestakov [29], the nonassociative tensor algebra over a vector space.

3 The nonassociative tensor algebra

Recall, following Shestakov [29], the definition and some properties of the nonassociative tensor algebra associated with any K-module.

Consider the *nonassociative tensor algebra* $\mathcal{T}\{V\}$ of a K-module V as being

$$\mathcal{T}\{V\} = V\oplus(V\otimes V)\oplus ...\oplus V^{\otimes_n}\oplus ...$$

where, for $n\gneq 1$

$$V^{\otimes_n} = \sum_{i=1}^{n-1} V^{\otimes_i}\otimes V^{\otimes_{n-i}},$$

and the product of $v\in V^{\otimes_i}$ and $w\in V^{\otimes_j}$ is defined as $v\cdot w = v\otimes w$. $\mathcal{T}\{V\}$ is the free nonassociative K-algebra over V.

The algebra $\mathcal{T}\{V\}$ has the following universal property: for any K-linear mapping of V into an arbitrary K-algebra A there exists a unique extension of it to an algebra homomorphism from $\mathcal{T}\{V\}$ to A. More precisely, to any K-algebra A and any K-linear map $f: V\to A$ there exists a unique algebra homomorphism $f_0:\mathcal{T}\{V\}\to A$ extending f. In other words, the functor $\mathcal{T}$ is left adjoint to the underlying functor to K-vector space which forgets the algebra structure. This assertion is easily proved by observing that $f_0([v_1\otimes ...\otimes v_n])$ may be defined by $[f(v_1)f(v_2)...f(v_n)]$ (here $[v_1\otimes ...\otimes v_n]$ denotes the tensor product of the ordered set $(v_1,...,v_n)$ realized in pairs delimited by parenthesis, and $[f(v_1)f(v_2)...f(v_n)]$ is the element obtained in A by composing the elements $f(v_i)$ and preserving the parenthesis from $[v_1 \otimes ... \otimes v_n])$. $\mathcal{T}\{V\}$ has a natural (positive) $\mathbb{Z}$-grading: $\mathcal{T}\{V\} = \oplus_{i\in\mathbb{Z}} V^{\otimes_i}$ where $V^{\otimes_i} = 0$ if $i \leqslant 0$. This grading induces in $\mathcal{T}\{V\}$ the ascending filtration

$$T_1 \subseteq T_2 \subseteq ... \subseteq T_n \subseteq ...,$$

where $T_k = \oplus_{i=1}^{k} V^{\otimes_i}$. If J is a two-sided ideal of $\mathcal{T}\{V\}$, $U = U(V) = \mathcal{T}\{V\}/J$ and $\pi : \mathcal{T}\{V\} \to U$ be the natural projection, then the grading of $\mathcal{T}\{V\}$ induces the ascending filtration in $U(V)$

$$U_1 \subseteq U_2 \subseteq ... \subseteq U_n \subseteq ...,$$

with $U_k = \pi(T_k)$. Let denote by i the restriction of π on V (here V is considered a K-submodule of $\mathcal{T}\{V\}$), i.e. $i(v) = \pi(v) = v + J$, $\forall\ v \in V$. Remark that U is generated by $U_1 = i(V)$. Consider now the $\mathbb{Z}$-graded algebra $gr(U) = \oplus_{i\in\mathbb{Z}}(grU)_i$ associated with the filtered algebra U. Its components are defined by the condition: $(grU)_n = 0$, for $n \leqslant 0$, $(grU)_1 = U_1$ and $(grU)_i = U_i/U_{i-1}$ for $i \gneqq 1$. If $\overline{a} = a + U_{i-1} \in (grU)_i, \overline{b} = b + U_{j-1} \in (grU)_j$, then

$$\overline{a} \cdot \overline{b} = ab + U_{i+j-1}$$

.
Notice that $i(V) = U_1 = (grU)_1$, hence the problem of the injectivity of i is reduced to the structure of the graded algebra $grU(V)$. Usually, a graded algebra associated with a filtered one is more easy to deal with, so we may turn our attention to the algebra $grU(V)$.

4 Universal enveloping algebra

We shall associate to each Lts A over K an (nonassociative) algebra which is generated as "freely" as possible by A subject to the ternary relations in A.

Definition 4.1. Let A be a Lts over K. The pair $(\mathcal{U}, i)$ where $\mathcal{U}$ is a T-algebra over K and i is a Lts-homomorphism from A to $\mathcal{LT}(\mathcal{U})$ (i.e., it is a linear map such that

$$(4.1) \qquad i([x,y,z]) = i(x)\cdot[i(y)\cdot i(z)] - i(y)\cdot[i(x)\cdot i(z)],\ \forall\ x,y,z \in A)$$

is called a *universal enveloping T-algebra* of A if the following property holds: for any T-algebra B and any Lts-homomorphism j from A to $\mathcal{LT}(B)$, there exists a unique homomorphism of T-algebras $\Phi : \mathcal{U} \to B$ such that $\Phi \circ i = j$.

Uniqueness. The uniqueness of the universal enveloping T-algebra $(\mathcal{U}, i)$ of A is easily proved in a standard way. Indeed, if $(\mathcal{V}, j)$ is another universal enveloping T-algebra for A, we get the homomorphisms $\Phi : \mathcal{U} \to \mathcal{V}$, $\Psi : \mathcal{V} \to \mathcal{U}$ such that $\Phi \circ i = j$ and $\Psi \circ j = i$. It results: $(\Psi \circ \Phi) \circ i = i$ and $(\Phi \circ \Psi) \circ j = j$. By definition 4.1, there is a unique map $F : \mathcal{U} \to \mathcal{U}$ such that $F \circ i = i$. But $1_{\mathcal{U}}$ and $\Psi \circ \Phi$ both do the trick, so $\Psi \circ \Phi = 1_{\mathcal{U}}$. Similarly, it is proved $\Phi \circ \Psi = 1_{\mathcal{V}}$.

Existence. The existence of a suitable pair (U, i) is also not difficult to establish. Let $\mathcal{T}\{A\}$ be the nonassociative tensor algebra on A and let J be the two sided ideal in $\mathcal{T}\{A\}$ generated by $\{x \otimes y - y \otimes x, x \otimes (y \otimes z) - y \otimes (x \otimes z) - [x, y, z], f^{\otimes}(x, y, z, w) |\ x, y, z, w \in A\}$, where A is considered naturally imbedded in $\mathcal{T}\{A\}$and $f^{\otimes}$ is formally obtained from f in (2.6) by changing "$\cdot$" with "$\otimes$". We define the *universal enveloping algebra* $\mathcal{U}(A)$ of A to be the quotient of the nonassociative tensor algebra $\mathcal{T}\{A\}$ by the ideal J, i.e., $\mathcal{U}(A) = \mathcal{T}\{A\}/J$. Let $\pi : \mathcal{T}(A) \to \mathcal{U}(A)$ be the canonical homomorphism. Notice that $J \subset \otimes_{i=1}^{\infty} T^{\otimes_i}(A)$, so π maps $T^1(A)$ isomorphically into $U_1(A) = A$.

Proposition 4.1. *$(\mathcal{U}, i)$, where $i : A \to \mathcal{U}(A)$ is the restriction to A of the natural projection $\pi : \mathcal{T}(A) \to \mathcal{U}(A)$, is a universal enveloping T-algebra of A.*

Proof. Since $\mathcal{U}(A)$ is a commutative algebra over K satisfying the second identity (2.2), according Corollary 2.7, $\mathcal{U}(A)$ carries a natural Lts-structure such that i is a Lts-monomorphism. Let $j : A \to B$ be as in Definition 4.1. The universal property of $\mathcal{T}\{A\}$ yields an algebra homomorphism $\Phi' : \mathcal{T}\{A\} \to B$ which extends j. The special property (4.1) of j forces all $x \otimes y - y \otimes x, x \otimes (y \otimes z) - y \otimes (x \otimes z) - [x, y, z], f^{\otimes}(x, y, z, w)$ to lie in $\ker \Phi'$, so Φ' induces an algebra homomorphism $\Phi : \mathcal{U}(A) \to B$ such that $\Phi \circ i = j$. Actually, Φ is a Lts-homomorphism. The uniqueness of Φ is evident since $Im\ i$ generate $\mathcal{U}(A)$.

Therefore, the functor **U** is a left adjoint to the functor **LT**.

Theorem 4.2. *1. The T-algebra $\mathcal{U} = \mathcal{U}(A)$ is generated by $i(A)$.*
2. Let A_1 and A_2 be two Ltss, $(\mathcal{U}_1, i_1)$, $(\mathcal{U}_2, i_2)$-their corresponding universal enveloping T-algebras, and $\alpha : A_1 \to A_2$ be a T-algebra homomorphism. Then, there exists a unique T-homomorphism $\alpha' : \mathcal{U}_1 \to \mathcal{U}_2$ such that $\alpha' \circ i_1 = i_2 \circ \alpha$.
3. Let B be a two-sided ideal of A and let $\mathcal{R}$ be the two-sided ideal of $\mathcal{U}$ generated by $i(B)$. If $a \in A$, then $j : a + B \to i(a) + \mathcal{R}$ is a Lts-homomorphism of A/B and $\mathcal{LT}(\mathcal{V})$, where $\mathcal{V} = \mathcal{U}/\mathcal{R}$, and $(\mathcal{V}, j)$ is the universal enveloping T-algebra for A/B.
4. The T-algebra $\mathcal{U}$ has a uniquely defined antiautomorphism π such that $\pi \circ i = -i$; moreover, $\pi^2 = 1$.
5. If D is a derivation for the Lts A then there exists a uniquely defined derivation D' for the T-algebra $\mathcal{U}$ such that $D' \circ i = i \circ D$.

Proof. 1. Let $\mathcal{V}$ be the subalgebra of $\mathcal{U}$ generated by $i(A)$. The map i can be viewed as a mapping between Lts A and $\mathcal{LT}(\mathcal{V})$. Then, there exists a uniquely defined T-homomorphism i' of T-algebras $\mathcal{U}$ and $\mathcal{V}$ such that $i = i' \circ i$. Since $i = 1_{\mathcal{U}} \circ i$ and i' can be considered as mapping of $\mathcal{U}$ in $\mathcal{U}$, by taking into account of the uniqueness condition for such a T-homomorphism, it follows $i' = 1_{\mathcal{U}}$. Consequently, $\mathcal{U} = 1_{\mathcal{U}}(\mathcal{U}) = i'(\mathcal{U}) \subseteq \mathcal{V}$, that means $\mathcal{V} \equiv \mathcal{U}$.

2. If α is a Lts-homomorphism of A_1 into A_2, then $i_2 \circ \alpha$ is a Lts-homomorphism of A_1 and $\mathcal{LT}(\mathcal{U}_2)$. Then, there exists a uniquely defined T-homomorphism α' from $\mathcal{U}_1$ into $\mathcal{U}_2$ such that $\alpha' \circ i_1 = i_2 \circ \alpha$.

3. Let us denote that the mapping $a \to i(a) + \mathcal{R}$ from A to $\mathcal{V} = \mathcal{U}/\mathcal{R}$ is a Lts-homomorphism of A into $\mathcal{LT}(\mathcal{V})$. Since $i(B) \subseteq \mathcal{R}$, B is carried in 0 by this homomorphism. Consequently, we obtain the induced Lts-homomorphism $a + B \to i(a) + \mathcal{R}$ from Lts A/B to $\mathcal{LT}(V)$. This is just the mapping j. Let now θ be a Lts-homomorphism of the Ltss A/B and $\mathcal{LT}(U)$, where U is a T-algebra. Then, the mapping $\eta : a \to \theta(a + B)$ is a Lts-homomorphism of A into $\mathcal{LT}(U)$. Consequently,

there exists a T-homomorphism $\eta' : \mathcal{U} \to U$ such that $\eta' \circ i = \eta$. If $b \in B$, then $\eta b = 0$, so that $i(b) \in \ker \eta'$. Then, $\mathcal{R} \subseteq \ker \eta'$ and, consequently, we obtain the induced T-algebra homomorphism $\theta' : u + \mathcal{R} \to \eta'(u)$ from $\mathcal{V} = \mathcal{U}/\mathcal{R}$ into U. Further, it results $\theta = \theta' \circ j$. We must prove now that θ' is uniquely defined. For this end it is enough to prove that $j(A/B)$ generates $\mathcal{V}$. According to 1, $\mathcal{U}$ is generated by $i(A)$, so it follows that $\mathcal{V}$ is generated by elements of the form $j(A+B)$, i.e., by the set $j(A/B)$.

4. In order to prove the existence of this antiautomorphism π we consider, for any $n \geqslant 1$, the map defined by

$$(x_1, ..., x_n) \to (-1)^n [x_n \otimes x_{n-1} \otimes ... \otimes x_1],$$

it is n-multilinear from $A \times ... \times A$ into $T^n\{A\}$ and hence extends to a linear map of $T^n\{A\}$ into itself by

$$[x_1 \otimes ... \otimes x_n] \to (-1)^n [x_n \otimes x_{n-1} \otimes ... \otimes x_1]$$

(here the presence of brackets $[,]$ tells us that the places of the parenthesis $(,)$ in $[x_1 \otimes ... \otimes x_n]$ are preserved in $[x_n \otimes ... \otimes x_1]$). Taking the direct sum of these maps as n varies (and for any positioning of the brackets) we obtain a linear map of $T\{A\}$ into itself (sending 1 into 1, if it is the case). It is clear that this map is an antiautomorphism and extends $x \to -x$ in $T^1\{A\}$. Composing it with passage to the quotient by J, we obtain an antiautomorphism of $T\{A\}$ into $\mathcal{U}(A)$. Its kernel is an ideal. To show that the map descends to $\mathcal{U}(A)$ it is enough to show that each generator of J maps to 0. But each generator maps in $T\{A\}$ to itself and then maps to 0 in $\mathcal{U}(A)$. Hence the "transpose" map π descents to $\mathcal{U}(A)$. It is clearly of order two and thus it is a one-one onto. The uniqueness of π is the result of the fact that $i(A)$ generates $\mathcal{U}(A)$.

5. Let D be a derivation of the Lts A. We construct the algebra $\mathcal{U}_2$ of 2×2-matrices with entries in the universal enveloping T-algebra $\mathcal{U} = \mathcal{U}(A)$ defined by the usual matrix composition connected with the binary product in $\mathcal{U}$. Let us consider the mapping

$$\theta : a \to \begin{bmatrix} i(a) & i(D(a)) \\ 0 & i(a) \end{bmatrix}$$

of A into $\mathcal{U}_2$. It is a linear mapping satisfying the property

$$[\theta(x), \theta(y), \theta(z)] = \theta([x, y, z]).$$

Then the vector subspace $\widetilde{\mathcal{U}}_2$ of $\mathcal{U}_2$ consisting from the matrices of the form $\begin{bmatrix} x & y \\ 0 & x \end{bmatrix}$ with $x, y \in \mathcal{U}$ is a T-algebra. Consequently, θ is a Lts-homomorphism from A to $\mathcal{LT}(\widetilde{\mathcal{U}}_2)$. In order to prove this assertion we remark that the following two identities holds in any T-algebra:

$$(yx \cdot x)z = 2(y \cdot xz)x - (xz \cdot x)y - (yz \cdot x)x + (y \cdot x^2)z,$$

$$2(yx \cdot z)x - 2(y \cdot zx)x - (y \cdot x^2)z + (z \cdot x^2)y = 0, \forall x, y, z \in \mathcal{U}.$$

This identities are obtained from (2.5) changing $z \to x$, $w \to z$ and $x \to z$, $w \to x$, respectively. It follows that there is a T-algebra homomorphism θ' of $\mathcal{U}$ into $\widetilde{\mathcal{U}}_2$ such that $\theta = \theta' \circ i$. Since

$$\theta'(i(a)) = \begin{bmatrix} i(a) & i(D(a)) \\ 0 & i(a) \end{bmatrix}$$

and the elements $i(a)$ generates $\mathcal{U}$ we have for any $x \in \mathcal{U}$

$$\theta'(x) = \begin{bmatrix} x & y \\ 0 & x \end{bmatrix}$$

where y is uniquely determined by x. We put $y = D'x$ and a straightforward computation shows that D' is a derivation of $\mathcal{U}$. Then it results $D' \circ i = i \circ D$. The uniqueness of D' follows from the fact that $i(A)$ generates $\mathcal{U}$ and the derivation is determined by its action on a set of generators.

5 The representability of Lts composition by binary operations

Obviously, it is especially important to solve, for any Lts $(T, [.,.,.])$, the following problem.

Problem : Exists or not a binary K-algebra $T(\cdot)$ whose associated h.s. be just $(T, [.,.,.])$?

In what follows, we try to give some necessary conditions for the existence of a binary composition whose standard associated h.s. is a given Lie triple system.

Let $(T, [.,.,.])$ be finite dimensional Lts, $n = dim_K T$ and $B = \{e_1, e_2, ..., e_n\}$ be a basis for it. The equations

$$[e_i, e_j, e_k] = t^s_{ijk} e_s, \; i, j, k \in \{1, 2, ..., n\}$$

define the structure constants of Lts T. Then, the linear operators $D_{ij} := D_{e_i e_j}$ with $i \lneq j$ gives a family of generators for Lie algebra $\mathcal{D}$. If $T(\cdot)$ is a commutative algebra, then its structure constants are defined by

$$e_i e_j = a^s_{ij} e_s, \; i, j \in \{1, 2, ..., n\}.$$

The solving of our problem consists in solving the following quadratic algebraic system

$$a^q_{ip} a^p_{jk} - a^q_{jp} a^p_{ik} = t^q_{ijk}, \tag{5.1}$$

which is a very difficult problem. We can pass to the easier problem of solving a linear homogeneous system. Indeed, a necessary condition for $(T, [.,.,.])$ be the standard (see [3]) associated h-system with $T(\cdot)$ is

$$[D_{ij}, L_{e_k}] - L_{[e_i, e_j, e_k]} = 0, \; \forall i, j, k \in \{1, 2, ..., n\}, i \lneq j. \tag{5.2}$$

Using the before defined structure constants we get

$$t^q_{ijp} a^p_{ks} - t^p_{ijs} a^q_{kp} = t^p_{ijk} a^q_{ps}, i, j, k, p, q, s \in \{1, 2, ..., n\}, i \lneq j. \tag{5.3}$$

(5.3) is an algebraic system of $n^3\frac{n^2-n}{2}$ linear equations with $n\frac{n^2+n}{2}$ unknown numbers a_{ij}^k. Of course, the study of the compatibility of this system is, in general, a difficult problem but in concrete cases it can be solved. In case (5.3) is a compatible system, its solutions $\widetilde{a}_{ij}^k$ must satisfy necessarily the conditions

$$\widetilde{a}_{ip}^q\widetilde{a}_{jk}^p - \widetilde{a}_{jp}^q\widetilde{a}_{ik}^p = t_{ijk}^q. \tag{5.4}$$

6 Uniqueness of a solution

In case when the Problem has a solution, the problem of the uniqueness of this solution arises.

Let us suppose that $T(\cdot)$ and $T(*)$ have the same associated Lts. Then, the so-called deformation algebra $T(\bullet)$ of this pair of algebras is defined by

$$x \bullet y = x \cdot y - x * y,\ \forall x, y \in T.$$

Of course, $\mathcal{D}$ must be a Lie algebra of derivations for $T(\bullet)$, too. Further,

$$[x, y, z]_\bullet = x \bullet (y \cdot z) - y \bullet (x \cdot z) + x \cdot (y \bullet z) - y \cdot (x \bullet z),\ \forall x, y, z \in T.$$

The 3-linear mapping $[.,.,.]_\bullet$ satisfies the identities

$$[x, y, z]_\bullet + [x, y, z]_\bullet = 0,\ \forall x, y \in T,$$

$$[x, y, z]_\bullet + [y, z, x]_\bullet + [z, x, y]_\bullet = o,\ \forall x, y \in T,$$

but $(T, [.,.,.]_\bullet)$ is not necessarily a Lts.

7 Examples

Example 1. Let T be a finite dimensional real vector space and $\omega : T \to \mathbb{R}$ be a linear form. The ternary composition defined on T by

$$[x, y, z] = \frac{1}{4}\omega(z)[\omega(y)x - \omega(x)y],\ \forall x, y, z \in T,$$

endows T with a Lts-structure. On other hand, the binary algebra $T(\cdot)$ with the binary multiplication defined by

$$2x \cdot y = \omega(x)y + \omega(y)x,\ \forall x, y \in T$$

has $(T, [.,.,.])$ as its associated h-system. Indeed, a straightforward computation proves that

$$[x, y, z] = x \cdot (y \cdot z) - y \cdot (x \cdot z),\ \forall x, y, z \in T,$$

$$[x, y, z \cdot v] - z \cdot [x, y, v] = [x, y, z].v,\ \forall x, y, z \in T.$$

Consequently, the posed Problem for the Lts $(T, [.,.,.])$ has a solution.

The following lemma is useful to prove some negative examples.

Lemma 7.1. *Let V be a finite dimensional real vector space and $D \in EndV$ be an endomorphism which has the set of all eigenvalues $\Lambda = \{\lambda_i \mid i = 1, 2, ..., n\} \subset \mathbb{R}^*$ satisfying the conditions $\mid \Lambda \mid= n$ and $\lambda_i + \lambda_j \notin \Lambda$, $\forall i, j = 1, 2, ..., n$. Then, the only commutative binary algebra on V having D as an own derivation is the null-algebra.*

Proof. Let us consider a basis $B = \{e_1, e_2, ..., e_n\}$ consisting from the eigenvectors of D. Then, $D(e_i \cdot e_j) = (\lambda_i + \lambda_j)e_i \cdot e_j$ holds. Since $\lambda_i + \lambda_j \notin \Lambda, \forall i, j = 1, 2, ..., n$ it results $e_i \cdot e_j = 0, \forall i, j = 1, 2, ..., n$.

Example 2. A particular example of Lts has arisen in quantum mechanics; it is the so-called *meson triple system*. It was been introduced by R.J.Duffin in 1938 [6] by means of the following multiplication table

$$[e_i, e_j, e_k] = \delta_{ki}e_j - \delta_{kj}e_i$$

on a finite dimensional real vector space T with the basis $B = \{e_1, ..., e_n\}$. We shall prove that the meson triple system cannot be obtained from a binary algebra by means of the standard construction [3]. Indeed, $D_{e_1e_2}(e_1) = e_2$, $D_{e_1e_2}(e_2) = -e_1$, $D_{e_1e_2}(e_k) = 0, k \gneqq 2$; moreover, for $i \lneqq j$, $D_{e_ie_j}(e_i) = e_j$, $D_{e_ie_j}(e_j) = -e_i$, $D_{e_ie_j}(e_k) = 0, k \neq i, j$. If $e_i \cdot e_j = \sum_{k=1}^{n} a_{ij}^k e_k$ with $a_{ij}^k = a_{ji}^k$ define a commutative binary multiplication on T then, imposing that every $D_{e_ie_j}$ is a derivation for $T(\cdot)$, we shall obtain the following equalities:

$$D_{e_1e_2}(e_1^2) = 2D_{e_1e_2}(e_1) \cdot e_1 \quad \Leftrightarrow \quad a_{11}^1 e_2 - a_{11}^2 e_1 = 2e_1 \cdot e_2$$

$$D_{e_1e_2}(e_1 \cdot e_2) = D_{e_1e_2}(e_1) \cdot e_2 + e_1 \cdot D_{e_1e_2}(e_2) \quad \Leftrightarrow \quad a_{12}^1 e_2 - a_{12}^2 e_1 = e_2^2 - e_1^2$$

$$D_{e_1e_2}(e_2^2) = 2D_{e_1e_2}(e_2) \cdot e_2 \quad \Leftrightarrow \quad a_{22}^1 e_2 - a_{22}^2 e_1 = -2e_1 \cdot e_2$$

By equating coefficients of these equalities it results $e_1 \cdot e_2 = 0$ and $e_1^2 = e_2^2 = \sum_{k=3}^{n} a_{11}^k e_k$. Similarly, we shall obtain $e_i \cdot e_j = 0$ for $i \neq j$ and $e_1^2 = e_2^2 = ... = e_n^2 = 0$. Consequently, there is no commutative binary algebra on T whose associated Lts be $(T, [., ., .])$.

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