On the Haar wavelet analysis of jumps

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Abstract

Wavelets give significant information on the evolution of a time series. In particular, due to their localization properties the significant local changes in observed data (both in time and in frequency) can be easily detected by a limited set of their corresponding wavelet coefficients. Some examples will be given, in the following, showing the effectiveness of this method.

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1. Introduction

Haar wavelets (see e.g. [15]) have been proposed [2, 3, 4, 5, 6, 8, 10, 11, 12, 13, 16, 17, 18] for the analysis of localized significant changes in observed data $\boldsymbol{Y}$, with jumps either in time-space or in frequency-space. These changes can be easily detected by analysing the wavelet coefficients (in general), but more in particular by focussing only on a few of them [2, 10].

It is well known [16, 17, 18], that each time-series might be roughly represented by the composition of a sequence of high and low frequency "small" waves with a trend. These waves have bounded frequencies, and are localized according to the fact that what happen at a given time t, in general, has a negligible influence (correlation) with the other data at time $t' \gg t$. Due to these properties, the local properties of the time-series are well described by a reduced number of wavelet coefficients, i.e. one can choose a limited number of the basis (wavelet) functions to locally represent the time-series. As a consequence, the local analysis can be improved by slicing the time-series of length N into a set of small segments of length $p \ \ (= N/\sigma)$ (at a fixed time intervals) and analysing each segment.

We propose a method (see also [2, 10]) based on the following steps:

1. The sequence of N data is sliced into σ short segments of $p \ \ (= N/\sigma)$ data
2. Each segment is transformed by using the discrete wavelet transform into the finite length p vector of the wavelet coefficients. So that locally the time-series is characterized by the wavelet coefficients restricted to the indexed segment.

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3. The jumps (and anomalies) might be easily detected by the relations existing between the wavelet coefficients, i.e. according to the relations among the wavelet coefficients independently on the chosen wavelet basis.

In doing so only the first set of detail coefficients are considered for the analysis and moreover the computational complexity is strongly reduced. With some examples given in the following, we will see that jumps are strictly connected with detail coefficients.

2. Haar wavelet transform

Let $\boldsymbol{Y} \equiv \{Y_i\}$, $(i = 0, \dots, 2^M - 1,\ 2^M = N < \infty,\ M \in \mathbf{N})$, be a real and square summable denoised (see e.g. [2, 3, 4, 8, 7]) time-series $\boldsymbol{Y} \in \mathbf{K}^N \subset \ell^2$ (where $\mathbf{K}$ is a real field); $t_i = i/(2^M - 1)$, is the regular equispaced grid of *dyadic points* on the interval restricted, for convenience and without restriction, to $\Omega = [0, 1]$.

The *Haar scaling function* $\varphi(t)$ is the characteristic function on $[0, 1]$; its family of translated and dilated scaling functions is defined (in $[0, 1]$) as

$$\begin{cases} \varphi^n_k(t) \equiv 2^{n/2}\varphi(2^n t - k)\ , \qquad (0 \le n\ ,\ 0 \le k \le 2^n - 1)\ , \\ \varphi(2^n t - k) = \begin{cases} 1 & , t\ \in \Omega^n_k \\ 0 & , t\ \notin \Omega^n_k\ . \end{cases} \quad , \qquad \Omega^n_k \equiv \left[\dfrac{k}{2^n}, \dfrac{k+1}{2^n}\right)\ , \end{cases} \tag{2.1}$$

The *Haar wavelet* family $\{\psi^n_k(t)\}$ is the orthonormal basis for the $L^2([0, 1])$ functions [15]:

$$\begin{cases} \psi^n_k(t) \equiv 2^{n/2}\psi(2^n t - k)\ , \qquad ||\psi^n_k(t)||_{L^2} = 1\ , \\ \psi^n_k(t) = \begin{cases} -2^{-n/2}\ , & t \in \left[\dfrac{k}{2^n}, \dfrac{k+1/2}{2^n}\right)\ , \\ 2^{-n/2}\ , & t \in \left[\dfrac{k+1/2}{2^n}, \dfrac{k+1}{2^n}\right)\ , \\ 0\ , & \text{elsewhere}\ . \end{cases} \quad , \ (0 \le n\ ,\ 0 \le k \le 2^n - 1)\ , \end{cases} \tag{2.2}$$

Although, without loss of generality, we restrict ourselves to $0 \le n\ ,\ 0 \le k \le 2^n - 1 \Longrightarrow \Omega^n_k \subseteq [0, 1]$, for other integer values of k the family of the Haar scaling functions and wavelets are defined also outside $[0, 1]$ making possible to extend the following considerations to any interval of $\mathbf{R}$.

The *discrete Haar wavelet transform* is the operator $\mathcal{W}^N : \mathbf{K}^N \subset \ell^2 \to \mathbf{K}^N \subset \ell^2$ which associates to a given finite energy vector $\boldsymbol{Y}$ the finite energy vector of the *wavelet coefficients* $\{\alpha\ ,\ \beta^n_k\}$:

$$\mathcal{W}^N\boldsymbol{Y} = \{\alpha, \beta^0_0, \dots, \beta^{M-1}_{2^{M-1}-1}\}\ , \qquad \boldsymbol{Y} = \{Y_0,\ Y_1,\ \dots,\ Y_{N-1}\}\ , \quad (2^M = N)\ . \tag{2.3}$$

Thanks to the so-called pyramidal algorithm, the $N \times N$ matrix $\mathcal{W}^N$ can be computed by the recursive formula, that was summarized in the recursive product of matrices (see e.g. [2, 3, 4, 6])

$$\mathcal{W}^N \boldsymbol{Y} \equiv \left[\prod_{k=1}^{M} \Big((P_{2^k} \oplus I_{2^M - 2^k})(H_{2^k} \oplus I_{2^M - 2^k}) \Big) \right] \boldsymbol{Y} \ , \tag{2.4}$$

being $\oplus$ the direct sum, I the identity matrix, P and H the matrix:

$$H_2 = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} , \quad H_4 = H_2 \oplus H_2 = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 & 0 \\ 0 & 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & 0 & -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}, \dots ,$$

of the lattice coefficients.

For example, with $N = 4$, $M = 2$, assuming the empty set $I_0 \equiv \emptyset$ as the neutral term for the direct sum $\oplus$, it follows from (2.4)

$$\mathcal{W}^4 = \prod_{k=1,2} \left[(P_{2^k} \oplus I_{4-2^k})(H_{2^k} \oplus I_{4-2^k}) \right] = \left[(P_2 \oplus I_2)(H_2 \oplus I_2) \right]_{k=1} \left[(P_4 \oplus I_0)(H_4 \oplus I_0) \right]_{k=2} \ ,$$

that is,

$$\mathcal{W}^4 = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} . \tag{2.5}$$

Analogously, with $N = 8$, $M = 3$, it is (from (2.4))

$$\begin{aligned} \mathcal{W}^8 &= \prod_{k=1,2,3} \left[(P_{2^k} \oplus I_{8-2^k})(H_{2^k} \oplus I_{8-2^k}) \right] \\ &= \left[(P_2 \oplus I_6)(H_2 \oplus I_6) \right]_{k=1} \left[(P_4 \oplus I_4)(H_4 \oplus I_4) \right]_{k=2} \left[(P_8 \oplus I_0)(H_8 \oplus I_0) \right]_{k=3} \ , \end{aligned}$$

that is

$$\mathcal{W}^8 = \begin{pmatrix} \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} \\ -\frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} . \tag{2.6}$$

3. Adapted Haar discrete wavelet transform

The wavelet transform (2.3) implies the computation of $N = 2^M$ wavelet coefficients, at the resolution M, with N basis functions $\psi_k^n(t)$ involved, and a computation complexity $\mathcal{O}(N^2)$. However, as shown in [2, 3, 4], if we consider only $p = 2^m \leq N$ basis functions the complexity reduces to $\mathcal{O}(pN)$. This corresponds to slicing the data with a fixed window, as it is usually done, for instance, in the local sine and cosine transforms or in the wavelet packet decomposition [1, 14]. As a further genralization we can consider a slicing with changing length. With the adapted Haar transform it is possible both to reduce the number of basis functions and the computational complexity.

Algorithm 1. *Let the set $\boldsymbol{Y} = \{Y_i\}$ of N data, segmented into σ segments (in general) of different length, each segment $\boldsymbol{Y}^s$, $s = 0, \dots, \sigma - 1$ is made of $p_s = 2^{m_s}$, $(\sum_s p_s = N)$, data:*

$$\boldsymbol{Y} = \{Y_i\}_{i=0,\dots,N-1} = \bigoplus_{s=0}^{\sigma-1} \{\boldsymbol{Y}^s\} \ , \qquad \boldsymbol{Y}^s \equiv \{Y_{sp_s},\ Y_{sp_s+1}, \dots ,\ Y_{sp_s+p_s-1}\} \ ,$$

being, in general, $p_s \neq p_r$. The adapted discrete Haar wavelet transform of $\boldsymbol{Y}$ is $\mathcal{W}^{p_s,\sigma}\boldsymbol{Y}$, being explicitly

$$\begin{cases} \mathcal{W}^{p_s,\sigma} \equiv \bigoplus_{s=0}^{\sigma-1} \mathcal{W}^p_s & , \quad \boldsymbol{Y} = \bigoplus_{s=0}^{\sigma-1} \boldsymbol{Y}^s \ , \\ \mathcal{W}^{p_s,\sigma}\boldsymbol{Y} & = \left(\bigoplus_{s=0}^{\sigma-1} \mathcal{W}^{p_s} \right) \boldsymbol{Y} = \left(\bigoplus_{s=0}^{\sigma-1} \mathcal{W}^{p_s} \boldsymbol{Y}^s \right) \ , \\ \mathcal{W}^{2^m_s}\boldsymbol{Y}^s & = \left\{ \alpha_0^{0(s)},\ \beta_0^{0(s)},\ \beta_0^{1(s)},\ \beta_1^{1(s)}, \dots ,\ \beta_{2^{m_s-1}-1}^{m_s-1(s)} \right\} \ , \end{cases}$$

where $(2^{m_s} = p_s,\ \sum_{s=0}^{\sigma-1} p_s = N)$. In particular, when $p_s = p = N/\sigma,\ s = 0, \dots, \sigma - 1, (\sigma > 1)$, the adapted wavelet transform reduces to the so-called reduced (windowed) transform [2, 3, 4, 5, 6, 8, 7] where the slices of data have the same length. When $p_s = p = N,\ \sigma = 1$, the above coincides with the ordinary wavelet transform. In other words, the adapted wavelet transform is a wavelet transform with a flexible window.

In general, for the vector of 2^M elements, $\boldsymbol{Y} = \{Y_i\}_{i=0,\dots,2^M-1}$, the Haar wavelet transform is the vector $\mathcal{W}^{2^M}\boldsymbol{Y}$, while there are different adapted wavelet transforms that can be done with one of the following matrices, even if the resulting piecewise interpolation will be the same. There follows that, in order to transform the vector $\boldsymbol{Y}$, instead to use the $N \times N$ matrix $\mathcal{W}^N$ up to the resolution $M = \log_2 N$, we can use the $N \times N$ matrix $\mathcal{W}^{p_s,\sigma}$, which is the direct sum of $p_s \times p_s$ matrices: $\mathcal{W}^{p_s,\sigma} = \bigoplus_{s=0}^{\sigma-1} \mathcal{W}^{p_s}$.

For example the reduced wavelet transform $\mathcal{W}^{4,2}$, compared with $\mathcal{W}^8$ of equation (2.6), is:

$$\mathcal{W}^{4,2} = \mathcal{W}^4 \oplus \mathcal{W}^4 = \begin{pmatrix} \boldsymbol{W}^4 & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{W}^4 \end{pmatrix}$$

and explicitly

$$
\text{(3.1)} \quad \mathcal{W}^{4,2} = \mathcal{W}^4 \oplus \mathcal{W}^4 = \begin{pmatrix}
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\
-\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\
-\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\
0 & 0 & 0 & 0 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\
0 & 0 & 0 & 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}
\end{pmatrix} .
$$

More in general for σ segments of length 4, we have the 4σ order matrix

$$
\mathcal{W}^{4,\sigma} = \mathcal{W}^4 \oplus \mathcal{W}^4 = \begin{pmatrix}
\mathcal{W}^4 & & \mathbf{0} \\
& \ddots & \\
\mathbf{0} & & \ddots \\
& & \mathcal{W}^4
\end{pmatrix}
$$

4. Jumps detection in time and frequency

In order to show the application of the above wavelet technique let us first show how the adapted wavelet transform focuses on the local anomalies.

Let us consider first a sharp jump in time-space and for this let us take the 4-length segment $\boldsymbol{Y}$ where is located the jump X i.e.

$$
\text{(4.1)} \qquad |X| \gg |Y_0| \cong |Y_1| \cong |Y_2| \ ,
$$

We can have three different situation:

1. X is the third or fourth value of the sequence. Let it be

$$
\boldsymbol{Y} = \{Y_0,\ Y_1,\ Y_2,\ X\} \ .
$$

By explicitly computing the wavelet coefficients we have:

$$\begin{aligned}
\alpha &= \frac{1}{4}\left(Y_0 + Y_1 + Y_2 + X\right) \ , \\
\beta_0^0 &= \frac{1}{2}\left(-Y_0 - Y_1 + Y_2 + X\right) \ , \\
\beta_0^1 &= \frac{1}{\sqrt{2}}\left(Y_1 - Y_0\right), \\
\beta_1^1 &= \frac{1}{\sqrt{2}}\left(X - Y_2\right),
\end{aligned}$$

so that, according to (4.1),

$$|\alpha| \gg 0 \ , \ |\beta_0^0| \gg 0 \ , \ \beta_0^1 \cong 0 \ , \ |\beta_1^1| \gg 0$$

This condition holds, analogously if X is the third value of the sequence, i.e.

$$\boldsymbol{Y} = \{Y_0, \ Y_1, \ X, \ Y_2\} \ .$$

2. X is the first or second value of the sequence, like e.g.

$$\boldsymbol{Y} = \{Y_0, \ X, \ Y_1, \ Y_2\} \ .$$

In this case there results:

$$|\alpha| \gg 0 \ , \ |\beta_0^0| \gg 0 \ , \ |\beta_0^1| \gg 0 \ , \ \beta_1^1 \cong 0$$

Therefore we can conclude that if one of the detail coefficient β_k^1 , $k = 0, 1$ is very large and the other is very small than we can assume that there is a significant jump.More in general for a different scale analysis, by using the wavelet transform $\mathcal{W}^N$, $(N = 2^M)$ we can say that the condition for the existence of a jump is that all the finest detail coefficients β_k^{M-1} are vanishing but one which is very large.

As a second case, let us consider a sharp jump in frequency-space and for this let us take the 4-length segment $\boldsymbol{Y}^s$ with

$$\boldsymbol{Y}^s = \{0, \ \sin\frac{1}{3}\omega, \ \sin\frac{2}{3}\omega, \ \sin\omega\} \ ,$$

and the subsequent segment

$$\boldsymbol{Y}^{s+1} = \{0, \ \sin\frac{1}{3}\Omega, \ \sin\frac{2}{3}\Omega, \ \sin\Omega\} \ .$$

with different frequency. The corresponding wavelet coefficients are:

$$\begin{aligned}
\alpha^{(s)} &= \frac{1}{4}\left(\sin\frac{1}{3}\omega + \sin\frac{2}{3}\omega + \sin\omega\right) \ , \\
\beta^{(s)}{}_0^0 &= \frac{1}{2}\left(-\sin\frac{1}{3}\omega + \sin\frac{2}{3}\omega + \sin\omega\right) \ , \\
\beta^{(s)}{}_0^1 &= \frac{1}{\sqrt{2}}\sin\frac{1}{3}\omega, \\
\beta^{(s)}{}_1^1 &= \frac{1}{\sqrt{2}}\left(-\sin\frac{2}{3}\omega + \sin\omega\right),
\end{aligned}$$

and analogously for the slice $s+1$. There follows that, the change in the frequency can be immediately detected by the ratio

$$\frac{\beta^{(s+1)}{}^1_0}{\beta^{(s)}{}^1_0} = \frac{\sin\frac{1}{3}\Omega}{\sin\frac{1}{3}\omega}$$

which is different from 1 in case of a frequency variation. Thus if the frequency is growing we have

$$\alpha^{(s+1)} > \alpha^{(s)}\ ,\ \beta^{(s+1)}{}^0_0 > \beta^{(s)}{}^0_0\ ,\ \beta^{(s+1)}{}^1_0 > \beta^{(s)}{}^1_0\ ,\ \beta^{(s+1)}{}^1_1 > \beta^{(s)}{}^1_1\ .$$

the contrary if the frequency is lowering (see Fig. 2).

5. Examples of jump detection by wavelets

Taking, e.g. the sequence of numbers $\{1\,,2\,,3\,,9,\ 10\,,11\,,12\,,13\}$, the reduced wavelet transform gives as wavelet coefficients $\{1.5,\ 0.7071\}$, $\{6,\ 4.2426\}$, $\{10.5,\ 0.7071\}$, $\{12.5,\ 0.7071\}$. The first coefficient of each group reflects the average value (see e.g [2, 3, 4, 9]) while the second coefficient is proportional to the difference of each pair of values. Thus we can say that the second segment with the highest coefficient $\beta^0_0 = 4.2426$, shows the presence of a jump. Applying the adapted wavelet tranform $\mathcal{W}^{(2,4,2),3}$ we have the following coefficients $\{1.5,\ 0.7071\}$, $\{8.25,\ ,4.5,\ 4.2426,\ 0.7071\}$, $\{12.5,\ 0.7071\}$. In this case both wavelet coefficients of the second slice, namely $\beta^0_0 = 4.5$, $\beta^1_0 = 4.2426$, represent the jump in correspondence on the third-fourth element of the initial sequence of numbers.

As a second example, let us use the following function (see Figure 1):

$$y(t) = \begin{cases} t & ,\quad t \in (0, t^\star] \\ t^\star + t & ,\quad t \in (t^\star, 1] \\ 0 & ,\quad \text{elsewhere}\ , \end{cases}$$

in order to construct a sequence $\boldsymbol{Y}$ having a jump in correspondence of $t^\star$. The data $\boldsymbol{Y}$ are then defined by a sufficient fine dyadic discretization (512 points) of $y(t)$. By computing the wavelet coefficients it is easily (and exactly) shown in each coefficient where the jump occurs (see Figure 1, bottom left).

Frequency sorting can be considered as the problem of detecting the point where there is a sharp change in frequency in the signal, or, equivalently a jump in the frequency.

Let us consider the following function

$$y(t) = \begin{cases} S_1(t) & ,\quad t \in (0, t^\star] \\ S_2(t) & ,\quad t \in (t^\star, 1] \\ 0 & ,\quad \text{elsewhere}\ , \end{cases}$$

being

$$S_1(t) = A_1 \cos\omega t \quad , \quad S_2(t) = A_2 \cos\omega t\ .$$

It is well known that the Fourier transform does not allow to single out the point $t^\star$ where there is a jump in the frequency. This point, instead can be easily detected

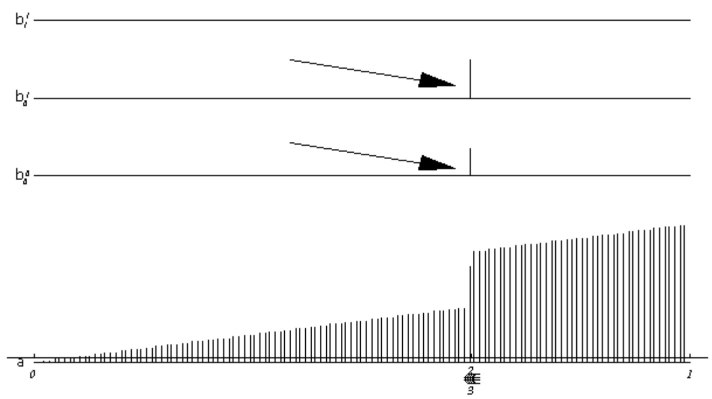

Figure 1: *Wavelet jump detection in time* $t^{\star} = 2/3$.

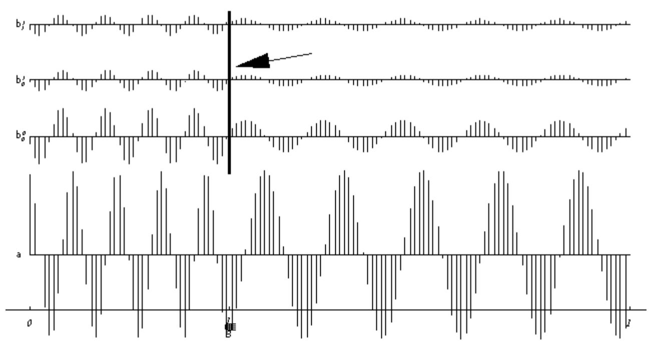

Figure 2: *Wavelet jump detection in frequency* $t^{\star} = 1/3$

by the above wavelet technique. Assuming $A_1 = A_2 = \sqrt{2}$, $\omega_1 = 27\pi$, $\omega_2 = 15.2\pi$ and discretize the function $y(t)$, in a sufficiently fine dyadic grid (512 points) of the interval $[0, 1]$, it is recovered (see Figure 2) the point $1/3$ where the frequency changes.

Finally in Figure 3 it is shown how sharp jumps can be detected in a time series of experimental data. Since they are, in general, uncorrelated there will be more frequent chenges both in time-space and in frequency-space.

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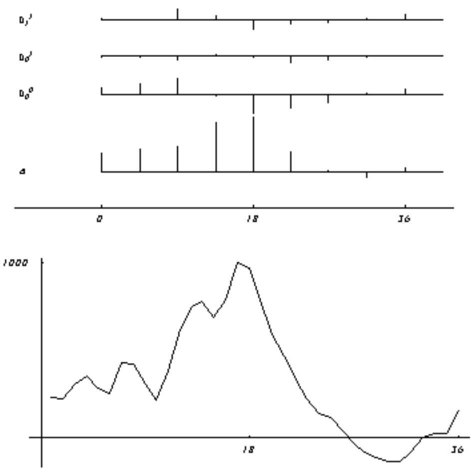

Figure 3: *Wavelet jump detection in a time series of experimental data.*

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