Contributions to the mathematical theory of physical systems

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Abstract

A short review of the mathematical foundation of physical systems theory is given. It is a rigorous theory, the systems having abstract signal spaces and a general *government*. The contributions of Romanian mathematicians, particularly of the author, to the development of this theory are pointed out.

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1 Introduction

We expose here a theory that represents the mathematical basis of the theory of physical systems, both nonlinear and linear, time-varying and time-invariant, discrete-variable and continuous-variable, unbounded and continuous. Several properties of the systems as: linearity, continuity, time-invariance, causality, passivity and others, are studied with the help of the admittance (impulse-response) function associated to the physical system. In the present theory, this is an operator-valued function, the operator generated by such a function being named Carleman operator (see [3] and [7] for the linear case).

The theory was extended to more general signals and used to the study of the system properties by R. Cristescu [13], D. Wexler (see [19]), I. Cioranescu [13], M. Sabac [24], N. Racoveanu [23], D. Stanomir [26] and the author [2]-[10], the last papers being the support of the present survey. Results in the some direction were obtained by V. Dolezal [15]-[17], J. Sanborn [18], B. Pondelicek [22] and R. Meidan [21].

Notations. We denote by $\Phi(E,G)$ the set of all applications between two given sets E and G. If these sets are vector spaces over a field K, then $L(E,G)$ denotes the vector space of all linear operators from E to G and $E = L_a(E,K)$ is the linear dual of E. If E and G are topological vector spaces, then $L(E,G)$ denotes the vector space of all linear continuous operators from E to G and $E' = L(E,K)$ is the linear topological dual of E.

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2 Time-varying nonlinear physical systems

2.1. The nonlinear systems have as mathematical models the applications $U : E \longrightarrow E$, where E is the set of all inputs, while F is a set that contains all the outputs of the system U. The element $Ue \in F$ is named *the output* of the system U for *the input*, $e \in E$.

2.2. The dual (adjoint or transpose) system is the linear application U^T between the vector spaces $\Phi(F,G)$ and $\Phi(E,G)$, where G is a vector space, defined by the relation:

$$(U^T f)(e) = f(Ue),\ \forall e \in E,\ \forall f \in \Phi(F,G). \tag{1}$$

2.3. Admittance (weight or impulse-response) functions of the systems

2.3.1. The outputs. The outputs are considered to be functions, namely $F \subseteq \Phi(S,G)$, where S is a set. In this case, the outputs can be measured.

2.3.2. The Dirac application. The Dirac application $\delta \in \Phi(F,G)$, concentrated at the element $s \in S$ (model for an impulse at s), is defined by the relation:

$$\delta_s(f) = f(s),\ \forall f \in F. \tag{2}$$

2.3.3. The (unique) admittance function associated to the system U and its dual U^T is the function $\varphi : S \longrightarrow \Phi(E,G)$ given by the formula

$$\varphi(s) = \delta_s \circ U = U^T(\delta_s),\ \forall s \in S \tag{3}$$

(response at the impulses δ_s by the dual system U^T).

2.4. The representations of a nonlinear system $U : E \longrightarrow F$ and its linear dual $U^T : \Phi(F,G) \longrightarrow \Phi(E,G)$, are

$$(Ue)(s) = \delta_s(Ue) = (U^T\delta_s)(e) = \varphi(s)(e) = \varphi(s) \otimes e,\ \forall s \in S,\ \forall s \in S,\ \forall e \in E, \tag{4}$$

$$(U^T f)(e) = f((Ue)(s)) = f(\varphi(s)(e)) = f(\varphi(s) \otimes e) = (\varphi(s) \otimes f)(e), \tag{5}$$

hence

$$U^T(f) = \varphi(s) \otimes f,\ \forall f \in \Phi(F,G). \tag{6}$$

So the systems U and U^T are *governed* by the admittance function (5).

The composition products that appear in the formulae (4) and (5) are defined just by these formulae. For this reason, the admittance functions $\varphi(s)$ are also named (E,F)-*composition functions*, the vector space of these functions being denoted $\Phi C(E,F)$. Hence $\varphi(s) \in \Phi C(E,F)$ if $\varphi(s)(e) \in F,\ \forall e \in E$.

2.5. The connections of the time-varying nonlinear systems

2.5.1. In series. The system $U_2 \circ U_1$ obtained by series connection ([6]) of the systems $U_1 : E \longrightarrow F_1 \subseteq \Phi(S,G),\ U_2 : F_2 \longrightarrow F_1 \subseteq \Phi(T,G)$, where T is an set, with the admittance functions $\varphi_1 : S \longrightarrow \Phi(E,G),\ \varphi_2 : T \longrightarrow \Phi(F_1,G)$ has the admittance function $\psi : T \longrightarrow \Phi(E,G)$, given by the formula:

$$\psi(t) = \varphi_1(s) \otimes \varphi_2(t),\ \forall t \in T. \tag{7}$$

2.5.2. In parallel. The system $U_1 + U_2$ obtained by parallel conection ([6]) of the systems $U_1, U_2 : E \longrightarrow F$, with admittance functions $\varphi_1, \varphi_2 : S \longrightarrow \Phi(E,G)$ has the admittance function $\varphi_1 + \varphi_2$.

3 Time invariant nonlinear physical systems

3.1. Sets of functions invariant to the translations. Let S be a commutative group with the unity $s_0 \in S$. The sets of functions $E \subseteq \Phi(S, G)$ will be supposed to be *translation invariant*, namely $e \in E$ and $f \in F$ implies $\tau_s e(t) = e(ts) \in E$, respectively $\tau_s f \in F$, for any $s \in S$.

3.2. The translation of the application $u \in \Phi(E, G)$ with the element $s \in S$ is given by the relation:

$$(\tau_s u)(e(t)) = u(\tau_{s^{-1}} e(t)) = u(e(ts^{-1})),\ \forall t \in S,\ \forall e \in E. \tag{8}$$

3.3. The convolution product between an application $u \in \Phi(E, G)$ named $(E, F)-$ *convolutor* that satisfies the condition $\varphi(s) = \tau_s u \in \Phi C(E, F)$ and a function and a function $e \in E$, is the function $u * e = \tau_s u \otimes e \in F$, hence is given by the relation:

$$(u * e(t))(s) = \tau_s u \otimes e(t) = (\tau_s u)(e(t)) = u(e(ts^{-1})). \tag{9}$$

The convolution product between the applications u and $f \in \Phi(F, G)$ is the application $u * f = \tau_s u \otimes f \in \Phi(E, G)$, given by the relation:

$$(u * f)(e) = (\tau_s u \otimes f)(e) = f((\tau_s u)(e)) = f(u * e),\ \forall e \in E. \tag{10}$$

3.4. The nonlinear operator The nonlinear operator $U : E \longrightarrow F$ is *permutable with the translation* if satisfies the condition:

$$U(\tau_s e) = \tau_s U(e),\ \forall s \in S,\ \forall e \in E. \tag{11}$$

(the system does not change its behavior in time). In this case, its linear dual U^T : $\Phi(F, G) \longrightarrow \Phi(E, G)$ is also permutable with the translations, hence it satisfies the condition:

$$U^T(\tau_s f) = \tau_s U^T(f),\ \forall s \in S,\ \forall f \in \Phi(F, G). \tag{12}$$

3.5. The admittance application of $U : E \longrightarrow F$ is defined by

$$u = U^T(\delta_{s_0}) \in \Phi(E, G). \tag{13}$$

If U is permutable with the translations, then u is an $(E, F)-$ convolutor.

3.6. The representations of a time-invariant nonlinear system and its time-invariant linear dual are

$$Ue = \varphi(s) \otimes f = (\tau_s u) \otimes f = u * e,\ \forall e \in E, \tag{14}$$

$$U^T f = \varphi(s) \otimes f = (\tau_s u) \otimes f = u * f,\ \forall f \in \Phi(F, G). \tag{15}$$

3.7. The connection of the time-invariant nonlinear systems

3.7.1. In series. The time-invariant system $U_2 \circ U_1$ obtained by series connection of the time-invariant systems $U_1 : E \longrightarrow F_1,\ U_2 : F_1 \longrightarrow F_2$ has the admittance $u_1 * u_2 \in \Phi(E, G)$.

3.7.2. In parallel. The time-invariant system $U_1 + U_2$ obtained parallel connection of the time-invariant systems $U_1,\ U_2 : E \longrightarrow F$ with the admittance $u_1, u_2 \in \Phi(E, G)$, has the admittance $u_1 + u_2 \in \Phi(E, G)$.

4 Time-varying linear continuous systems

4.1. Time-varying linear systems E, F, G are vector space on K and $U : E \longrightarrow F$, $U^T : L_a(F,G) \longrightarrow L_a(E,G)$ are linear operators. In case that $F \subseteq \Phi(S,G)$, where S is a set, then $\delta_s \in L_a(F,G)$ and $\varphi(s) = U^T(\delta_s) \in L_a(E,G)$, $\forall s \in S$. Physically, a system is linear if it satisfies the superposition principle.

4.2. Carleman operators. We suppose that E, G are (separated) topological vector spaces. The operator $U : E \longrightarrow F \subseteq \Phi(S,G)$ is named *Carleman* ([7]) if there is a function $\varphi : S \longrightarrow L(E,G)$, named *generating function* or *kernel* of U, so that

$$U(e)(s) = \varphi(s)(e) \otimes e,\ \forall s \in S,\ \forall e \in E. \tag{16}$$

This is equivalent with the fact that the operator U is continous when the space F is endowed with the pointwise convergence topology. Every Carleman operator is linear. The initial notion was given by T. Carleman in 1932 in the case $E = F = L^2(a,b)$ (see 5.2.2. below). Several extensions were given for example on [20] and [27]. In the general case, presented here, this notion was introduced and studied by the author in [23], without the actual denomination, also in [7] and in other papers. The notion of Carlemn operator is taken as a *mathematical model for a linear physical system*, the function φ being its admittance function.

4.3. Continuous systems. We suppose that F is a topological vector space having the topology finer than the pointwise convergence topology. With some necessary precautions it will be also considered the case when F is composed from classes of functions equal almost everywhere on a measure space S.

4.3.1. Every linear continuous operator $U : E \longrightarrow F$ is Carleman. This means that every linear continuous system has the admittance function given by

$$\varphi(s) = \delta_s \circ U \in L(E,G),\ \forall s \in S. \tag{17}$$

A system is continuous if close outputs correspond to close inputs (in the sense of the considered topologies).

4.3.2. Every Carleman operator is closed, hence is continuous if the pair of spaces (E,F) fulfil the closed graph theorem ([9]).

4.3.3. The dual system $U^T : L(F,G) \longrightarrow L(E,G)$has the representation (5) while the admittance φ is given by (3).

5 Examples of time-varying linear continuous systems.

5.1. The scalar case $G = K$. In this case $U : E \longrightarrow F$, $U^T = U' : F' \longrightarrow E'$, $\delta_s \in F'$, $\varphi(s) = U'(\delta_s) = e'_s \in E'$, $(U(e))(s) = e'_s(e) = e'_s \otimes e)$, $(U'(f'))(e) = (e'_s \otimes f')(e) = f'(e'_s(e))$, $\forall s \in S$, $\forall e \in E$, $\forall f' \in F'$. For simplicity, all the following examples will be considered only in the real scalar case (hence $G = K = \mathbb{R}$).

5.2. *E* Hilbert space. Taking into account the isomorphism between E and its dual E' given by the Riesz theorem, we have $U : E \longrightarrow F$, $U' : F' \longrightarrow E$, $\varphi(s) = U'(\delta_s) = e_s \in E$, $(U(e))(s) = (e_s) \otimes e = (e_s, e)$, $(U'(f'))(e) = ((e_s) \otimes f')(e) = f'((e_s, s))$, $\forall s \in S$, $\forall e \in E$, $\forall f' \in F'$, where $(\cdot,\cdot)$ denotes the scalar product in E.

5.2.1. Matricial systems. Discrete variable systems. Let $S = \{1, \dots, m\}$, $E = \mathbb{R}^n$, $F = \mathbb{R}^m$. Then $\varphi = A = [a_{i,j}] \in M_{m,n}[\mathbb{R}]$, where $\varphi(s) = L_s = [a_{s,1} \dots a_{s,n}] \in M_{1,n}[\mathbb{R}] \cong \mathbb{R}^n \cong (\mathbb{R}^n)'$, $s = 1, \dots, m$. The linear (continuous) system $U : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ has the representation $U(X) = \varphi \otimes X = AX$, $\forall X \in M_{n,1}[\mathbb{R}] \cong \mathbb{R}^n$. The system obtained by the series connection of the systems governed by the matrices $A \in M_{m,n}$, $B \in M_{n'k}$ is governed by the matrix $AB \in M_{m,k}$. The system obtained by the parallel connection of the systems governed by the matrices $A, B \in M_{m,n}$ is governed by the matrix $A + B \in M_{m,n}$.

5.2.2. Integral systems governed. Continuous variable systems. Let $S = (a, b)$ with $-\infty \le a < b \le \infty$, $E = F = L^2(a, b)$. Then the linear continuous operator $U : L^2(a, b) \longrightarrow L^2(a, b)$ has the representation

$$(U(e(t)))(s) = \int_a^b K(s,t)e(t)\mathrm{d}t,\ \forall e(t) \in L^2(a,b),$$

the admittance function being $\varphi(s) = K(s,t) \in (L^2(a,b))' \cong L^2(a,b)$, for almost all $s \in (a,b)$. This is the initial notion defined by T. Carleman. The system obtained by series connection of the systems that have the admittance functions $\varphi_1(s) = K_1(s,t)$, $\varphi_2(s) = K_2(s,t)$, has the admittance function $\varphi_1(s) \otimes \varphi_2(r) = \int_a^b K_1(s,r)K_2(r,t)\mathrm{d}r$, named *Fredholm product*, while in the case of parallel connection the admittance function is $K_1(s,t) + K_2(s,t)$.

5.3. Distributional systems. Let $S = \mathbb{R}^n$, $G = K$, $E = C_0^\infty(\mathbb{R}^m) = D(\mathbb{R}^m)$, $F = D(\mathbb{R}^n)$. Every linear continuous operator $U : D(\mathbb{R}^m) \longrightarrow D(\mathbb{R}^n)$ and its dual $U^T = U' : D'(\mathbb{R}^n) \longrightarrow D'(\mathbb{R}^m)$ have the representations $(Ue)(s) = e'_s(e) = e'_s \otimes e$, $(U'f')(e) = f'(e'_s(e)) = (e'_s \otimes f')(e)$, $\forall s \in \mathbb{R}^n$, $\forall e \in D(\mathbb{R}^m)$, $\forall f' \in D'(\mathbb{R}^n)$, where the admittance function $\varphi(s) = U'(\delta_s) = e'_s \in D'(\mathbb{R}^m)$, $\forall s \in \mathbb{R}^n$ is a indefinite differentiable distributional function, weak with compact support. Here $\delta_s \in D'(\mathbb{R}^n)$ is the Dirac distribution concentrated at $s \in \mathbb{R}^n$ (impulse at s). The same holds for other spaces from distribution theory or Sobolev spaces.

5.3.1. Systems multiplicatively governed. If $n = m$ and $\alpha \in C^\infty(\mathbb{R}^n)$, the operator $Ue = \alpha e$, $\forall e \in D(\mathbb{R}^n)$ is Carleman with the admittance function $\varphi(s) = \alpha(s)\delta_s \in D'(\mathbb{R}^n)$, $\forall s \in \mathbb{R}^n$ and $U'(f') = \alpha f'$, $\forall f' \in D'(\mathbb{R}^n)$.

5.3.2. Convolutional systems. If $n = m$ and $u \in E'(\mathbb{R}^n)$, then the convolution operator $Ue = u * e$, $\forall e \in D(\mathbb{R}^n)$ is Carleman with the admittance function $\varphi(s) = \tau_s u \in D'(\mathbb{R}^n)$, $\forall s \in \mathbb{R}^n$ and the dual $U'f' = u * f'$, $\forall f' \in D'(\mathbb{R}^n)$.

5.3.3. The derivative operator as distributional systems. If $n = m = 1$, the derivative operator $Ue(x) = \dfrac{\mathrm{d}e(x)}{\mathrm{d}x}$, $\forall e(x) \in D(\mathbb{R})$ is a Carleman operator $U : D(\mathbb{R}) \longrightarrow D(\mathbb{R})$, with the admittance function $\varphi(s) = \dfrac{\mathrm{d}\delta_s(x)}{\mathrm{d}x} \in D'(r)$, $\forall s \in \mathbb{R}$ and the dual $U' : D'(\mathbb{R}) \longrightarrow D'(\mathbb{R})$, given by $U'(f') = \dfrac{\mathrm{d}f'}{\mathrm{d}x}$, $\forall f' \in D'(\mathbb{R})$.

5.3.4. Differential governed systems. If $n = m$, the differential operator $U'(f'(x)) = \sum\limits_{|k| \le q} a_k(x) \dfrac{\partial^{|k|} f'(x)}{\partial x_1^{k_1} \cdots \partial x_n^{k_n}} \in D'(\mathbb{R}^n)$, $\forall f'(x) \in D'(\mathbb{R}^n)$, $x = (x_1, \dots, x_n) \in \mathbb{R}^n$,

$k = (k_1, \dots, k_n) \in N^n,\ |k| = \sum_{j=1}^{n} k_j$, is the dual of a Carleman operator with the admittance function $\varphi(s) = \sum_{|k|\le q} a_k(s) \dfrac{\partial^{|k|}\delta_s(x)}{\partial x_1^{k_1} \cdots \partial x_n^{k_n}} \in D'(\mathbb{R}^n),\ \forall s \in \mathbb{R}^n$.

5.3.5. Difference governed systems. If $n = m = 1$, the difference operator $U'(f'(x)) = \sum_{k=1}^{q} a_k(x) f'(x - x_k) \in D'(\mathbb{R}),\ \forall f' \in D'(\mathbb{R})$, where $x_k \in \mathbb{R},\ k = 1, \dots, q$, is the dual of a Carleman operator with the admittance function $\varphi(s) = \sum_{k=1}^{q} a_k(x)\delta(x - x_k) \in D'(\mathbb{R}),\ \forall s \in \mathbb{R}$.

5.4. Fourier transform of admittance functions ([3]). We consider the Fourier transform

$$F(e(t)) = \int_{\mathbb{R}^n} e^{i(t,s)} e(t)\mathrm{d}t \in Z(\mathbb{R}^n) = F(D(\mathbb{R}^n)),\ \forall e(t) \in D(\mathbb{R}^n).$$

The function $\psi = F(\varphi) : \mathbb{R}^n \longrightarrow Z'(\mathbb{R}^n)$, defined by the formula

$$(\psi(t))(Fe) = F((\varphi(s))(e)),\ \forall e \in D(\mathbb{R}^n),\ \forall s, t \in \mathbb{R}^n,$$

is the *Fourier transform* of the admittance function $\varphi : \mathbb{R}^n \longrightarrow D'(\mathbb{R}^n)$. If $\chi : \mathbb{R}^n \longrightarrow D'(\mathbb{R}^n)$ is a distributional arbitrary function, we have $F(\varphi \otimes \chi) = F(\varphi) \otimes F(\chi)$. Particularly, if $u \in E'(\mathbb{R}^n)$ and $v \in D'(\mathbb{R}^n)$, having $\varphi(s) = \tau_s u,\ \chi(s) = \tau_s v$, it is obtained $F(u * v) = F(u)F(v)$.

5.5. Laplace transform of admittance functions ([4]). We consider the Laplace transform

$$L(e(t)) = \int_{-\infty}^{\infty} e^{-ts} e(t)\mathrm{d}t \in L(D(\mathbb{R})),\ \forall e(t) \in D(\mathbb{R}),\ s \in C,$$

complex variable. The function $\psi = L\varphi : C \longrightarrow (L(D(\mathbb{R})))'$, defined by the formula

$$(\psi(s))(Le) = L((\varphi(t)))(e),\ \forall e \in D(\mathbb{R}),\ \forall t \in \mathbb{R},\ \forall s \in C,$$

is the *Laplace transform* of the admittance function $\varphi : \mathbb{R} \longrightarrow D'(\mathbb{R})$. It satisfies analogous properties as the Fourier transform.

6 Time-invariant linear continuous systems

. In the hypotheses made both in Sections 3 and 4, we consider the statements

(i) $U : E \longrightarrow F$ is a convolution operator;

(ii) $U^T : L(F, G) \longrightarrow L(E, G)$ is a convolution operator;

(iii) U commutes with the translations;

(iv) U^T commutes with the translations.

Then ([8]) there holdtime the following implications (i)$\Longleftrightarrow$(ii)$\Longrightarrow$(iii)$\Longleftrightarrow$(iv). If U is a Carleman operator, then the above implications are equivalent.

7 Causality of the time-varying systems relative to a scale of sets

We encourage of reader to see [16] and [21] for a particular linear distributional setting. We further consider fulfilled the hypotheses from the Section 4.

7.1. Scale of sets. It is named *scale of sets* the family of sets $S_a \subseteq S,\ \forall a \in S$, with the properties:

i) $a \in S_a,\ b \in S_a \Longrightarrow S_b \subseteq S_a$

ii) $\overline{I}S_a = S_a$;

iii) there exists $s_0 \in S$ so that $\overline{C}S_a - \lambda s_0 \subseteq CS_a,\ \forall \lambda > 0$,

where $\overline{I}, \overline{C}$ denotes the adherence of the interior, respective of the complementary of the set.

7.2. Examples of scales of sets

7.2.1. Let S be a topological space and $\Gamma \subseteq S$ a *cone* ($s \in \Gamma,\ \lambda \geq 0 \Longrightarrow \lambda s \in \Gamma$), closed, convex and having the interior non-void. Then the sets $s + \Gamma,\ \forall s \in S$ make a scale.

7.2.2. If $S = \mathbb{R},\ \Gamma = [0, \infty)$ the sets $s + \Gamma = [0, \infty),\ \forall s \in \mathbb{R}$ make a scale.

7.2.3. If $S = \mathbb{R}^4,\ \Gamma = \{(x, y, z, t) \in \mathbb{R}^4,\ x^2 + y^2 + z^2 \geq c^2 t^2\}$, the light cone, then the sets $s + \Gamma,\ \forall s \in \mathbb{R}^4$, make a scale.

7.3. Causal nonlinear time-varying systems. If $E, F \subseteq \Phi(S, G)$, with G vector space, the operator $U : E \longrightarrow F$ is named *causal* relative to the scale S_a if for every $a \in S$, $e \in E$ and $e(t) = 0$, $\forall t \in CS_a$ then $(Ue)(t) = 0,\ \forall t \in CS_a$.

If U is linear, this implies that for every $a \in S$ we have

$$e_1, e_2 \in E,\ e_1(t) = e_2(t),\ \forall t \in CS_a \Rightarrow (Ue_1)(t) = (Ue_2)(t),\ \forall t \in CS_a$$

(hence in the past of every moment a to equal causes correspond via the system equal effects).

7.4. Null applications on open sets. It is named the *support* of a function $e \in E$ the adherence in S of the point $s \in S$ at which $e(s) \neq 0$. It is said that an application $e' \in \Phi(E, G)$ is *null* on an open set $A \subseteq S$ if for every function $e \in E$ having its support in the set A, we have $e'(e) = 0$.

7.5. Causal dual systems. It is said that the dual system $U^T : \Phi(F, G) \longrightarrow \Phi(E, G)$ is *causal* relative to the scale S_a if for every $a \in S$ we have $f' \in \Phi(F, G),\ f' = 0$ on $CS_a \Rightarrow U^T(f')$ on CS_a.

7.6. Criteria of causality. In the hypotheses by the Section 4, if $U : E \longrightarrow F$ is a Carleman operator, then the following sentences are equivalent (i) U is causal relative to the scale S_a; (ii) U^T is causal relative to the scale S_a; (iii) $\varphi(a) = 0$ on S_a for every $a \in S$.

7.7. Causal time-invariant systems. In the hypotheses from Section 6, if $U : E \longrightarrow F$ is the operator $Ue = u * e,\ \forall e \in E$, then the following sentences are equivalent (i) U is causal relative to the scale S_a; (ii) U^T is causal relative to the same scale; (iii) $u = 0$ on C.

8 Passivity of the time-varying systems relative to a scale of sets.

We encourage the reader to check see [17] and [21] for a particular linear distributional setting.

We consider fulfilled the hypotheses from Section 4 and we assume that the spaces $E, F \subseteq \Phi(S, \mathbb{R})$ are composed by integrable functions defined almost everywhere on a measure space S.

8.1. Passivity. The operator is named *passive* relative to the scale of measurable sets $S_a,\ \forall a \in S$ if $\int\limits_{CS_a} (Ue)(s)e(s)\mathrm{d}s \geq 0,\ \forall e \in E,\ \forall a \in S$. Hence at every moment a, every input e provides to the system a energy.

8.2. Causal and passivity. Every time-varying linear passive system is causal, relative to the same scale (which in the classic case is stated by the Youla, Castriota-Carlin theorem).

8.3. Weak passive systems are the generalization of the systems that satisfy the condition $\int\limits_S (Ue)(s)(e)\mathrm{d}s \geq 0,\ \forall e \in E$.

8.4. Examples

8.4.1. For $S = \mathbb{R}^n$, the systems $Ue = (-1)^k \dfrac{\partial^{2k} e(s)}{\partial s_i^{2k}}$ and, particularly, $-\Delta = -\sum\limits_{i=1}^{n} \dfrac{\partial^2}{\partial s_i^2}$ are weak passive.

8.4.2. Passivity and weak passivity. Every passive system relative to a scale that satisfies some special conditions, is weak passive.

8.4.3. If the system U that satisfies some special condition is weak passive and causal, then it is passive, relative to the same scale.

8.4.4. Weak passivity of the time-invariant system.

If $S = \mathbb{R}^n$, then the time-invariant system $Ue = u * e,\ \forall e \in E$ is weak passive if $u\left(\int\limits_R e(s)e(t+s)\mathrm{d}s\right) \geq 0,\ \forall e \in E$.

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