An algorithm for computing the spectra of discrete time-invariant systems

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Abstract

In this paper is developed an algorithm for computing the spectra of discrete time-invariant systems.In the first part are reviewed the basic theoretical results of the spectra theory of multiplication operator and convolution.The general setup is as follows :for every $\theta : Z \mapsto \mathcal{M}_p$ we associate the corresponding convolution system $K_\theta : \ell^2(Z, C^p) \mapsto \ell^2(Z, C^p)$ which is unitary equivalent with the multiplication system Π_ϕ on $L^2(\mathcal{C}, C^p)$.As a consequence we finally develop a Maple-application which helps us do numerical computation of the spectra.

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I. Introduction

Throughout this paper, Z will be the additive group of integers and $\mathcal{C}$ the unit circle. If $p \in N$, we denote by C^p the usual p-dimensional complex Hilbert space and by $(\mathcal{M}_p, \| \quad \|)$ the Banach algebra of linear bounded operators on C^p. The usual Hilbert spaces of C^p-valued, square-integrable functions on Z and $\mathcal{C}$, are denoted by $(\ell^2(Z, C^p), \| \quad \|_2)$ and $(L^2(\mathcal{C}, C^p), \| \quad \|_2)$, respectively. The total variation of a $\mathcal{M}_p$ -valued sequence, $\theta : Z \mapsto \mathcal{M}_p$ is defined as $\| \theta \|_1 = \sum_{n \in Z} \| \theta(n) \|$. Let $L^1(Z, \mathcal{M}_p) = \{\theta : Z \mapsto \mathcal{M}_p \mid \ \| \theta \|_1 < \infty\}$ be the Banach algebra with the convolution: $(\theta \star \nu)(n) = \sum_{k \in Z} \theta(n-k)\nu(k)$.
The convolution of an element $\theta \in L^1(Z, \mathcal{M}_p)$ with a sequence $x \in \ell^2(Z, C^p)$ is defined by $(\theta \star x)(n) = \sum_{k \in Z} \theta(n-k)x(k)$. By direct computation it can be proved that $\| \theta \star x \|_2 \leq \| \theta \|_1 \| x \|_2$, hence $\theta \star x \in \ell^2(Z, C^p)$. Consequently, one can define the convolution system $K_\theta : \ell^2(Z, C^p) \mapsto \ell^2(Z, C^p)$, $K_\theta x = \theta \star x$. Obviously, K_θ is linear, bounded and $\| K_\theta \| \leq \| \theta \|_1$. A special case of a convolution system is the translation system defined as follows; for every $n \in Z$, let $\delta_n : Z \mapsto \mathcal{M}_p, \delta_n(m) = I$, if $m = n$ and O otherwise. Then $(K_{\delta_n} x)(m) = x(m-n)$ is the translation with n. In particular, K_{δ_1} is the ideal delay. For every $\theta \in L^1(Z, \mathcal{M}_p)$, the following commutativity condition: $K_\theta K_{\delta_1} = K_{\delta_1} K_\theta$ holds, hence the convolution system associated to θ is time-invariant. In this paper we present an algorithm

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(a Maple 8-application) for computing the spectra of these time-invariant, infinite dimensional systems.

II. Basic results

The $\mathcal{Z}$-transform

Let $\mathcal{Z} : \ell^2(Z, C^p) \mapsto L^2(\mathcal{C}, C^p)$ be the $\mathcal{Z}$-transform defined as follows.
For every $x \in \ell^2(Z, C^p)$ and $\xi \in C^p$, the scalar-valued sequence
$x_\xi(n) =< x(n), \xi >$ has a $\mathcal{Z}$-transform, denoted $\mathcal{Z}x_\xi$. The $\mathcal{Z}$-transform of x is given by the relation $< (\mathcal{Z}x)(e^{it}), \xi >= (\mathcal{Z}x_\xi)(n)$, $\forall n \in Z$. On the dense subspace $\ell^1(Z, C^p) \subset \ell^2(Z, C^p)$ the z-transform has the explicit formula

$$(\mathcal{Z}x)(e^{it}) = \sum_{n \in Z} e^{-int} x(n).$$

The map $\mathcal{Z}$ is a Hilbert space isomorphism; its inverse is

$$\mathcal{Z}^{-1} : L^2(\mathcal{C}, C^p) \mapsto \ell^2(Z, C^p),\ (\mathcal{Z}^{-1} f)(n) = \frac{1}{2\pi} \int_0^{2\pi} e^{int} f(e^{it}) dt.$$

For our study we need also the $\mathcal{Z}$-transform of a sequence $\theta \in L^1(Z, \mathcal{M}_p)$, i.e. $(\mathcal{Z}\theta)(z) = \sum_{n \in Z} z^{-n} \theta(n)$, $\forall z \in \mathcal{C}$. Obviously, the series is convergent; moreover, the following equalities hold:
i. $\mathcal{Z}\theta : \mathcal{C} \mapsto \mathcal{M}_p$ is continuous and $\sup_{|z|=1} \| (\mathcal{Z}\theta)(z) \| = \| \mathcal{Z}\theta \|_\infty \leq \| \theta \|_1$.
ii. $\mathcal{Z}(\theta \star \nu) = (\mathcal{Z}\theta)(\mathcal{Z}\nu)$, $\forall \theta, \nu \in L^1(Z, \mathcal{M}_p)$.

Multiplication systems

For every continuous map $\phi : \mathcal{C} \mapsto \mathcal{M}_p$, we define the multiplication system associated to ϕ as $\Pi_\phi : L^2(\mathcal{C}, C^p) \mapsto L^2(\mathcal{C}, C^p)$, $(\Pi_\phi f)(z) = \phi(z) f(z)$. The norm of Π_ϕ is $\| \Pi_\phi \| = \| \phi \|_\infty = \sup_{|z|=1} \| \phi(z) \|$.
The main result on the spectrum of the multiplication system is the following

Theorem 1.

i. The system Π_ϕ is invertible as an element of the Banach algebra of all linear operators on $L^2(\mathcal{C}, C^p)$ if and only if $\phi(z)$ is an invertible element of the Banach algebra $\mathcal{M}_p$.
ii. If $\sigma(T)$ is the spectrum of an arbitrary operator T, then $\sigma(\Pi_\phi) = \bigcup\limits_{|z|=1} \sigma(\phi(z))$.
The theoretical foundation of our algorithm for computing the spectra of time-invariant systems is the following result.

Theorem 2.

Let $\theta \in L^1(Z, \mathcal{M}_p)$ and $x \in \ell^2(Z, C^p)$.
i. $\left(\mathcal{Z}(\theta \star x)\right)(z) = \left((\mathcal{Z}\theta)(z)\right)\left((\mathcal{Z}x)(z)\right)$, $\forall z \in \mathcal{C}$.
ii. The systems K_θ and $\Pi_{\mathcal{Z}\theta}$ are unitarily equivalent: $\mathcal{Z} K_\theta \mathcal{Z}^{-1} = \Pi_{\mathcal{Z}\theta}$.

Proof. i. We prove the equality for x in the dense subspace $\ell^1(Z, C^p)$; for every $\xi \in C^p$ and $z \in \mathcal{C}$, we have:

$$< \left(\mathcal{Z}(\theta \star x)\right)(z), \xi >=< \sum_{n \in Z} z^{-n} (\theta \star x)(n), \xi >=$$

$$=< \sum_{n\in Z} z^{-n} \sum_{k\in Z} \theta(k)x(n-k), \xi >=< \sum_{k\in Z} \theta(k) \left(\sum_{n\in Z} z^{-n} x(n-k) \right), \xi >=$$

$$=< \sum_{k\in Z} \theta(k) \left(\sum_{m\in Z} z^{-(k+m)} x(m) \right), \xi >=< \left(\sum_{k\in Z} z^{-k}\theta(k) \right) \left(\sum_{m\in Z} z^{-m} x(m) \right), \xi >=$$

$$=< ((\mathcal{Z}\theta)(z))\,((\mathcal{Z}x)(z)), \xi > .$$

ii. Obviously, for every $f \in L^2(\mathcal{C}, C^p)$ we have:

$$\left(\mathcal{Z}K_\theta\mathcal{Z}^{-1}\right) f = \mathcal{Z}K_\theta(\mathcal{Z}^{-1}f) = \mathcal{Z}(\theta \star \mathcal{Z}^{-1}f) = (\mathcal{Z}\theta)\, f = \Pi_{\mathcal{Z}\theta} f.$$

The consequence of the above theorem is the spectrum of the convolution system.

Theorem 3.

The unitary equivalence of K_θ and $\Pi_{\mathcal{Z}\theta}$ and Theorem 1 imply the equality:

$$\sigma(K_\theta) = \bigcup_{|z|=1} \sigma((\mathcal{Z}\theta)(z)).$$

III. The algorithm

According to the results of the previous section (especially the last theorem), the steps of computing the spectrum of a time invariant system K_θ for a fixed $\theta \in L^1(Z, \mathcal{M}_p)$ are:

i. Read the sequence of matrices $(\theta(n))_{n\in Z}$.
ii. Compute the $\mathcal{Z}$-transform of θ, i.e. $\mathcal{Z}\theta$.
iii. For every $t \in [0, 2\pi]$, compute the spectrum (eigenvalues) of the matrix $(\mathcal{Z}\theta)(e^{it}) \in \mathcal{M}_p$.
iv. The spectrum of the time-invariant system is $\sigma(K_\theta) = \bigcup_{t\in[0,2\pi]} \sigma\left((\mathcal{Z}\theta)(e^{it})\right)$.

IV. A Maple 8-application

[> with(linalg):
[> p := (a non-negative integer that represents the row dimension and the column dimension of the matrix $\theta(n)$):
[> $\mathcal{Z}\theta(z)$:= matrix(p, p): (square matrix)
[> for i from 1 to p do
[> for j from 1 to p do
[> $\theta[i,j](n)$:= (the entries of the matrix $\theta(n)$)
[> with(inttrans):
[> g :=ztrans($\theta[i,j](n), n, z$): (we compute $\mathcal{Z}$-transform for every entry of the matrix $\theta(n)$)
[> $\mathcal{Z}\theta[i,j](z)$:= g: (every entry of the matrix $\mathcal{Z}\theta(z)$ is the $\mathcal{Z}$-transform that we already computed)
[> od:
[> od:
[> print($\mathcal{Z}\theta(z)$); (the programme will post the matrix ($\mathcal{Z}\theta(z)$))
[> with(linalg):
[> eigenvals($\mathcal{Z}\theta(z)$); (the programme will post the eigenvalues of the matrix $\mathcal{Z}\theta(z)$))
[> $f1$:= subs(z=exp(I*t),(we write here the first eigenvalue which is a function whose

argument is z));
. . .
[> fp := subs(z=exp(I*t),(we write here the p-th eigenvalue which is a function whose argument is z)); ($f1, \dots, fp$ are functions with the argument t)
[> plot ($\{f1, \dots, fp\}$,t = 0..2*Pi);(the programme will draw the graphics of functions $f1, \dots, fp$ on the same system of coordinates , so this will be the spectrum of the invariant system K_θ)

V.Examples

1. Let

$$\theta(n) = \begin{pmatrix} 0 & \frac{1}{n^2+1} \\ \frac{1}{n^2+1} & 0 \end{pmatrix}$$

With the algorithm in the previous section we obtain:

$$\mathcal{Z}\theta_{11}(z) = \mathcal{Z}\theta_{22}(z) = 0,$$

$$\mathcal{Z}\theta_{12}(z) = \mathcal{Z}\theta_{21}(z) = hypergeom([1, I, -I], [1 - I, 1 + I], 1/z)$$

and
$\sigma(\mathcal{Z}\theta(z)) = \{ -hypergeom([1, I, -I], [1 - I, 1 + I], 1/z), hypergeom([1, I, -I], [1- -I, 1 + I], 1/z)\}$

So $\sigma(K_\theta) = \bigcup_{|z|=1} \sigma((\mathcal{Z}\theta)(z)) = \bigcup_{|z|=1} \{ -hypergeom([1, I, -I], [1 - I, 1+ +I], 1/z), hypergeom([1, I, -I], [1 - I, 1 + I], 1/z)\}$

2. Let

$$\theta(n) = \begin{pmatrix} 0 & \frac{1}{n^2+1} & \frac{1}{n^2+1} \\ \frac{1}{n^2+1} & 0 & \frac{1}{n^2+1} \end{pmatrix}$$

Then

$$\mathcal{Z}\theta_{11}(z) = \mathcal{Z}\theta_{22}(z) = \mathcal{Z}\theta_{33}(z) = 0$$

$\mathcal{Z}\theta_{12}(z) = \mathcal{Z}\theta_{13}(z) = \mathcal{Z}\theta_{21}(z) = \mathcal{Z}\theta_{23}(z) = \mathcal{Z}\theta_{31}(z) = \mathcal{Z}\theta_{32}(z) =$
$= -hypergeom([1, I, -I], [1 - I, 1 + I], 1/z)$
and
$\sigma(\mathcal{Z}\theta(z)) = \{2 \cdot hypergeom([1, I, -I], [1 - I, 1 + I], 1/z), -hypergeom([1, I, -I], [1- -I, 1 + I], 1/z), -hypergeom([1, I, -I], [1 - I, 1 + I], 1/z)\}$

So $\sigma(K_\theta) = \bigcup_{|z|=1} \sigma((\mathcal{Z}\theta)(z)) = \bigcup_{|z|=1} \{2 \cdot hypergeom([1, I, -I], [1 - I, 1+ +I], 1/z), -hypergeom([1, I, -I], [1 - I, 1 + I], 1/z), -hypergeom([1, I, -I], [1- -I, 1 + I], 1/z)\}$

3. Let

$$\theta(n) = \begin{pmatrix} \frac{1}{n^2+1} & \frac{1}{n^4+1} \\ \frac{1}{n^4+1} & \frac{1}{n^2+1} \end{pmatrix}$$

Then

$$\mathcal{Z}\theta_{11}(z) = \mathcal{Z}\theta_{22}(z) = hypergeom([1, I, -I], [1 - I, 1 + I], 1/z),$$

$\mathcal{Z}\theta_{12}(z) = \mathcal{Z}\theta_{21}(z) = hypergeom([1, 1/2 \cdot I \cdot 2^{\frac{1}{2}} - 1/2 \cdot 2^{\frac{1}{2}}, -1/2 \cdot I \cdot 2^{\frac{1}{2}} + 1/2 \cdot 2^{\frac{1}{2}}, 1/2 \cdot 2^{\frac{1}{2}} + 1/2 \cdot I \cdot 2^{\frac{1}{2}}, -1/2 \cdot 2^{\frac{1}{2}} - 1/2 \cdot I \cdot 2^{\frac{1}{2}}], [1/2 \cdot I \cdot 2^{\frac{1}{2}} - 1/2 \cdot 2^{\frac{1}{2}} + 1, 1/2 \cdot 2^{\frac{1}{2}} + 1/2 \cdot I \cdot 2^{\frac{1}{2}} + 1, -1/2 \cdot I \cdot 2^{\frac{1}{2}} + 1/2 \cdot 2^{\frac{1}{2}} + 1, -1/2 \cdot 2^{\frac{1}{2}} - 1/2 \cdot I \cdot 2^{\frac{1}{2}} + 1], 1/z)$

and

$\sigma(\mathcal{Z}\theta(z)) = \{\mathit{hypergeom}([1,-I,I],[1-I,1+I],1/z)+\mathit{hypergeom}([1,1/2\cdot I\cdot 2^{\frac{1}{2}}-1/2\cdot 2^{\frac{1}{2}},-1/2\cdot I\cdot 2^{\frac{1}{2}}+1/2\cdot 2^{\frac{1}{2}},1/2\cdot 2^{\frac{1}{2}}+1/2\cdot I\cdot 2^{\frac{1}{2}},-1/2\cdot 2^{\frac{1}{2}}-1/2\cdot I\cdot 2^{\frac{1}{2}}],[1/2\cdot I\cdot 2^{\frac{1}{2}}-1/2\cdot 2^{\frac{1}{2}}+1,1/2\cdot 2^{\frac{1}{2}}+1/2\cdot I\cdot 2^{\frac{1}{2}}+1,-1/2\cdot I\cdot 2^{\frac{1}{2}}+1/2\cdot 2^{\frac{1}{2}}+1,-1/2\cdot 2^{\frac{1}{2}}-1/2\cdot I\cdot 2^{\frac{1}{2}}+1],1/z),\mathit{hypergeom}([1,-I,I],[1-I,1+I],1/z)-\mathit{hypergeom}([1,1/2\cdot I\cdot 2^{\frac{1}{2}}-1/2\cdot 2^{\frac{1}{2}},-1/2\cdot I\cdot 2^{\frac{1}{2}}+1/2\cdot 2^{\frac{1}{2}},1/2\cdot 2^{\frac{1}{2}}+1/2\cdot I\cdot 2^{\frac{1}{2}},-1/2\cdot 2^{\frac{1}{2}}-1/2\cdot I\cdot 2^{\frac{1}{2}}],[1/2\cdot I\cdot 2^{\frac{1}{2}}-1/2\cdot 2^{\frac{1}{2}}+1,1/2\cdot 2^{\frac{1}{2}}+1/2\cdot I\cdot 2^{\frac{1}{2}}+1,-1/2\cdot I\cdot 2^{\frac{1}{2}}+1/2\cdot 2^{\frac{1}{2}}+1,-1/2\cdot 2^{\frac{1}{2}}-1/2\cdot I\cdot 2^{\frac{1}{2}}+1],1/z)\}$

So $\sigma\left(K_\theta\right) = \bigcup_{|z|=1}\sigma\left((\mathcal{Z}\theta)(z)\right) = \bigcup_{|z|=1}\{\mathit{hypergeom}([1,-I,I],[1-I,1+I],1/z)+\mathit{hypergeom}([1,1/2\cdot I\cdot 2^{\frac{1}{2}}-1/2\cdot 2^{\frac{1}{2}},-1/2\cdot I\cdot 2^{\frac{1}{2}}+1/2\cdot 2^{\frac{1}{2}},1/2\cdot 2^{\frac{1}{2}}+1/2\cdot I\cdot 2^{\frac{1}{2}},-1/2\cdot 2^{\frac{1}{2}}-1/2\cdot I\cdot 2^{\frac{1}{2}}],[1/2\cdot I\cdot 2^{\frac{1}{2}}-1/2\cdot 2^{\frac{1}{2}}+1,1/2\cdot 2^{\frac{1}{2}}+1/2\cdot I\cdot 2^{\frac{1}{2}}+1,-1/2\cdot I\cdot 2^{\frac{1}{2}}+1/2\cdot 2^{\frac{1}{2}}+1,-1/2\cdot 2^{\frac{1}{2}}-1/2\cdot I\cdot 2^{\frac{1}{2}}+1],1/z),\mathit{hypergeom}([1,-I,I],[1-I,1+I],1/z)-\mathit{hypergeom}([1,1/2\cdot I\cdot 2^{\frac{1}{2}}-1/2\cdot 2^{\frac{1}{2}},-1/2\cdot I\cdot 2^{\frac{1}{2}}+1/2\cdot 2^{\frac{1}{2}},1/2\cdot 2^{\frac{1}{2}}+1/2\cdot I\cdot 2^{\frac{1}{2}},-1/2\cdot 2^{\frac{1}{2}}-1/2\cdot I\cdot 2^{\frac{1}{2}}],[1/2\cdot I\cdot 2^{\frac{1}{2}}-1/2\cdot 2^{\frac{1}{2}}+1,1/2\cdot 2^{\frac{1}{2}}+1/2\cdot I\cdot 2^{\frac{1}{2}}+1,-1/2\cdot I\cdot 2^{\frac{1}{2}}+1/2\cdot 2^{\frac{1}{2}}+1,-1/2\cdot 2^{\frac{1}{2}}-1/2\cdot I\cdot 2^{\frac{1}{2}}+1],1/z)\}$

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