Poisson-gradient dynamical systems with convex potential

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Abstract

The basic aim is to extend some results and concepts of non-autonomous second order differential systems with convex potentials to the new context of multi-time Poisson-gradient PDE systems with convex potential. In this sense, we prove that minimizers of a suitable action functional are multiple periodical solutions of a Dirichlet problem associated to the Euler-Lagrange equations. Automatically, these are solutions of the associated multi-time Hamiltonian equations.

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1 Poisson-gradient PDEs

There are two methods to study the periodic solutions of boundary problems attached to some partial derivative equations (PDEs):

- the method of Fourier expansions in terms of eigenfunctions of a PDE operator (the method of separation of variables);
- the method of minimizers of suitable action functionals.

Our paper refers to the second method, continuing the ideas in the papers [11, 14, 19]. We start with the set $T_0 = \left[0, T^1\right] \times ... \times [0, T^p] \subset \mathbb{R}^p$ determined by the diagonal points $O = (0, ..., 0), T = \left(T^1, ..., T^p\right)$, and with the Sobolev space $W_T^{1,2}$ of the functions $u \in L^2 [T_0, \mathbb{R}^n]$, having weak derivatives $\dfrac{\partial u}{\partial t} \in L^2 [T_0, \mathbb{R}^n]$. The weak derivatives are defined using the space C_T^∞ of all indefinitely differentiable multiple T-periodic functions from $\mathbb{R}^p$ into $\mathbb{R}^n$.

We denote by H_T^1 the Hilbert space associated to the Sobolev space $W_T^{1,2}$. The euclidean structure on H_T^1 is given by the scalar product

$$\langle u, v\rangle = \int_{T_0} \left(\delta_{ij} u^i (t) v^j (t) + \delta_{ij}\delta^{\alpha\beta} \frac{\partial u^i}{\partial t^\alpha} (t) \frac{\partial v^j}{\partial t^\beta} (t)\right) dt^1 \wedge ... \wedge dt^p$$

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and the associated Euclidean norm. These are induced by the scalar product (Riemannian metric)

$$G = \begin{pmatrix} \delta_{ij} & 0 \\ 0 & \delta^{\alpha\beta}\delta_{ij} \end{pmatrix}$$

on $\mathbb{R}^{n+np}$ (see the jet space $J^1(T_0, \mathbb{R}^n)$).

Let $t = (t^1, ..., t^p)$ be a generic point in $\mathbb{R}^p$. Then the opposite faces of the parallelepiped T_0 can be described by the equations

$$S_i^- : t^i = 0, S_i^+ : t^i = T^i$$

for each $i = 1, ..., p$.

Suppose the function $u(t)$ has a weak Laplacian Δu and $u \to F(t,u)$ is a convex function. In these hypothesis, we formulate some conditions in which the Dirichlet problem (associated to a Poisson-gradient PDE system)

$$\Delta u(t) = \nabla F(t, u(t)) \tag{1}$$

$$u\,|_{S_i^-} = u\,|_{S_i^+}, \frac{\partial u}{\partial t}\,|_{S_i^-} = \frac{\partial u}{\partial t}\,|_{S_i^+}, i = 1, ..., p \tag{2}$$

has solution. To do that, we denote

$$\left|\frac{\partial u}{\partial t}\right|^2 = \delta^{\alpha\beta}\delta_{ij}\frac{\partial u^i}{\partial t^\alpha}\frac{\partial u^j}{\partial t^\beta}$$

and we use the Lagrangian

$$L\left(t, u(t), \frac{\partial u}{\partial t}\right) = \frac{1}{2}\left|\frac{\partial u}{\partial t}\right|^2 + F(t, u(t)) \tag{3}$$

and the action

$$\varphi(u) = \int_{T_0} L\left(t, u(t), \frac{\partial u}{\partial t}\right) dt^1 \wedge ... \wedge dt^p \tag{4}$$

Then, using minimizing sequences, we show that the action φ has a minimum point u (extremal, solution of the Poisson-gradient dynamical system (1), satisfying the boundary conditions (2)). Consequently the solution u is multiple periodical, with the reduced period $T = (T^1, ..., T^p)$. Our arguments extend those of the book [6], Theorems 1.4, 1.5, 1.7 and 1.8, which are dedicated to single-time problems.

2 Periodic solutions of Poisson-gradient PDEs

Let us show that some conditions upon the potential F ensure periodic solutions for the problem (1)+(2).

Theorem 1. *Let* $F : T_0 \times \mathbb{R}^n \to \mathbb{R}, (t,x) \to F(t,x)$ *and* $|x| = \sqrt{\delta_{ij}x^i x^j}$. *We consider that* $F(t,x)$ *is measurable in* t *for any* $x \in \mathbb{R}^n$ *and of class* C^1 *in* x *for any* $t \in T_0$.

If *there exist* $a \in C^1(\mathbb{R}^+, \mathbb{R}^+)$ *with the derivative* a' *bounded from above and* $b \in C\left(T_0, \mathbb{R}^+\right)$ *such that*

$$|F(t,x)| \le a(|x|)\, b(t), |\nabla_x F(t,x)| \le a(|x|)\, b(t),$$

for any $x \in \mathbb{R}^n$ *and any* $t \in T_0$*, then the action (4) is of class* C^1*.*

Proof. The reasons are similar to those in [15, Theorem 3].

Corollary 2. *The some hypothesis as in Theorem 1. If* $u \in H_T^1$ *is a solution of the equation* $\varphi'(u) = 0$ *(critical point), then the function* u *has a weak Laplacian* $\triangle u$ *(the Jacobian matrix* $\dfrac{\partial u}{\partial t}$ *has a weak divergence) and*

$$\triangle u = \nabla F(t, u(t))$$

a.e. on T_0 *and*

$$u \,|_{S_i^-} = u \,|_{S_i^+}, \frac{\partial u}{\partial t} \,|_{S_i^-} = \frac{\partial u}{\partial t} \,|_{S_i^+} . \tag{5}$$

Proof. We build the function

$$\Phi : [-1, 1] \to \mathbb{R},$$

$$\Phi(\lambda) = \varphi(u + \lambda v) =$$

$$\int_{T_0} \left[\frac{1}{2} \left| \frac{\partial}{\partial t} (u(t) + \lambda v(t)) \right|^2 + F(t, u(t) + \lambda v(t)) \right] dt^1 \wedge ... \wedge dt^p,$$

where $v \in C_T^\infty$. The point $\lambda = 0$ is a critical point of Φ if and only if the point u is a critical point of φ. Consequently

$$0 = \langle \varphi'(u), v \rangle = \int_{T_0} \left[\delta^{\alpha\beta} \delta_{ij} \frac{\partial u^i}{\partial t^\alpha} \frac{\partial v^j}{\partial t^\beta} + \delta_{ij} \nabla^i F(t, u(t))\, v^j(t) \right] dt^1 \wedge ... \wedge dt^p,$$

for all $v \in H_T^1$ and hence for all $v \in C_T^\infty$. Using the definition of the weak divergence,

$$\int_{T_0} \delta^{\alpha\beta} \delta_{ij} \frac{\partial u^i}{\partial t^\alpha} \frac{\partial v^j}{\partial t^\beta} dt^1 \wedge ... \wedge dt^p = - \int_{T_0} \delta^{\alpha\beta} \delta_{ij} \frac{\partial^2 u^i}{\partial t^\alpha \partial t^\beta} v^j dt^1 \wedge ... \wedge dt^p,$$

the Jacobian matrix $\dfrac{\partial u}{\partial t}$ has weak divergence (the function u has a weak Laplacian) and

$$\triangle u(t) = \nabla F(t, u(t))$$

a.e. on T_0. Also, the existence of the weak derivatives $\dfrac{\partial u}{\partial t}$ and weak divergence $\triangle u$ implies that

$$u \,|_{S_i^-} = u \,|_{S_i^+}, \frac{\partial u}{\partial t} \,|_{S_i^-} = \frac{\partial u}{\partial t} \,|_{S_i^+} .$$

Remark. If the function u is at least of class C^2, then the definition of the weak divergence of the Jacobian matrix $\frac{\partial u}{\partial t}$ (or of the weak Laplacian $\triangle u$) coincides with the classical definition. This fact is obvious if we have in mind the formula of *integration by parts*

$$\int_{T_0} \delta^{\alpha\beta}\delta_{ij}\frac{\partial u^i}{\partial t^\alpha}\frac{\partial v^j}{\partial t^\beta}dt^1\wedge...\wedge dt^p$$

$$=\int_{T_0}\delta^{\alpha\beta}\delta_{ij}\frac{\partial}{\partial t^\alpha}\left(\frac{\partial u^i}{\partial t^\alpha}v^j\right)dt^1\wedge...\wedge dt^p-\int_{T_0}\delta^{\alpha\beta}\delta_{ij}\frac{\partial^2 u^i}{\partial t^\alpha\partial t^\beta}v^j dt^1\wedge...\wedge dt^p.$$

Corollary 3. *The some hypothesis as in Theorem 1. If* $|x|\to\infty$ *implies* $\int_{T_0} F(t,x)\,dt^1\wedge...\wedge dt^p\to\infty$ *and* $F(t,x)$ *is convex in* x *for any* $t\in T_0$*, then there exists a function* u *that is a solution of the boundary value problem* (5).

Proof. Let $G:\mathbb{R}^n\to\mathbb{R}$, $G(x)=\int_{T_0}F(t,x)\,dt^1\wedge...\wedge dt^p$. By assumptions, the convex function G has a minimum point $x=\overline{x}$. Consequently, $\nabla G(\overline{x})=\int_{T_0}\nabla F(t,\overline{x})\,dt^1\wedge...\wedge dt^p=0$.

Let (u_k) be a minimizing sequence for the action (4). We use the decomposition $u_k=\overline{u}_k+\widetilde{u}_k$, where $\overline{u}_k=\int_{T_0}u_k(t)\,dt^1\wedge...\wedge dt^p$. The convexity of F implies

$$F(t,u_k(t))\geq F(t,\overline{x})+(\nabla F(t,\overline{x}),u_k(t)-\overline{x}).$$

It follows

$$\varphi(u_k)\geq\frac{1}{2}\int_{T_0}\left|\frac{\partial u_k}{\partial t}\right|^2 dt^1\wedge...\wedge dt^p+$$

$$+\int_{T_0}F(t,\overline{x})\,dt^1\wedge...\wedge dt^p+\int_{T_0}(\nabla F(t,\overline{x}),u_k(t)-\overline{x})\,dt^1\wedge...\wedge dt^p$$

$$=\frac{1}{2}\int_{T_0}\left|\frac{\partial u_k}{\partial t}\right|^2 dt^1\wedge...\wedge dt^p+\int_{T_0}F(t,\overline{x})\,dt^1\wedge...\wedge dt^p+$$

$$+\int_{T_0}(\nabla F(t,\overline{x}),\widetilde{u}_k(t))\,dt^1\wedge...\wedge dt^p.$$

On the other hand, by Schwartz inequality, we can write

$$(\nabla F(t,\overline{x}),\widetilde{u}_k(t))\leq|\nabla F(t,\overline{u})|\,|\widetilde{u}_k(t)|\leq a(|\overline{x}|)\,b(t)\,|\widetilde{u}_k(t)|.$$

Consequently,

$$\varphi(u_k)\geq\frac{1}{2}\int_{T_0}\left|\frac{\partial u_k}{\partial t}\right|^2 dt^1\wedge...\wedge dt^p+$$

$$+\int_{T_0} F(t,\overline{x})\,dt^1\wedge...\wedge dt^p - a(|\overline{x}|)\int_{T_0} b(t)\,\widetilde{u}_k(t)\,dt^1\wedge...\wedge dt^p$$

$$\geq \frac{1}{2}\int_{T_0}\left|\frac{\partial u_k}{\partial t}\right|^2 dt^1\wedge...\wedge dt^p +$$

$$+\int_{T_0} F(t,\overline{x})\,dt^1\wedge...\wedge dt^p - a(|\overline{x}|)\,b_0\int_{T_0}|\widetilde{u}_k(t)|\,dt^1\wedge...\wedge dt^p,$$

where $b_0=\max\limits_{t\in T_0} b(t)$. Using the Wirtinger inequality for multiple integral, we find

$$\varphi(u_k)\geq \frac{1}{2}\int_{T_0}\left|\frac{\partial u_k}{\partial t}\right|^2 dt^1\wedge...\wedge dt^p +$$

$$+\int_{T_0} F(t,\overline{x})\,dt^1\wedge...\wedge dt^p - a(|\overline{x}|)\,b_0 C_1\left(\int_{T_0}\left|\frac{\partial u_k}{\partial t}\right|^2 dt^1\wedge...\wedge dt^p\right)^{\frac{1}{2}},$$

with $C_1>0$. Thus

$$\varphi(u_k)\geq \frac{1}{2}\int_{T_0}\left|\frac{\partial u_k}{\partial t}\right|^2 dt^1\wedge...\wedge dt^p + C_2 - C_3\left(\int_{T_0}\left|\frac{\partial u_k}{\partial t}\right|^2 dt^1\wedge...\wedge dt^p\right)^{\frac{1}{2}},$$

and, consequently, the function of degree two in the right hand member must be a decreasing restriction, i.e., there exists $C_4>0$, such that
$\int_{T_0}\left|\frac{\partial u_k}{\partial t}\right|^2 dt^1\wedge...\wedge dt^p < C_4$. It follows

$$\int_{T_0}\left|\frac{\partial \widetilde{u}_k}{\partial t}\right|^2 dt^1\wedge...\wedge dt^p < C_4$$

and so $\|\widetilde{u}_k\|<C_5$.

Again, the convexity of F leads to

$$F\left(t,\frac{\overline{u}_k}{2}\right)=F\left(t,\frac{1}{2}\left(u_k(t)-\widetilde{u}_k(t)\right)\right)\leq \frac{1}{2}F(t,u_k(t))+\frac{1}{2}F(t,-\widetilde{u}_k(t)),$$

$\forall t\in T_0$, $\forall k\in N$, so

$$F(t,u_k(t))\geq 2F\left(t,\frac{\overline{u}_k(t)}{2}\right)-F(t,-\widetilde{u}_k(t)).$$

Consequently

$$\varphi(u_k)=\frac{1}{2}\int_{T_0}\left|\frac{\partial u_k}{\partial t}\right|^2 dt^1\wedge...\wedge dt^p+\int_{T_0}F(t,u_k(t))\,dt^1\wedge...\wedge dt^p\geq$$

$$\geq \frac{1}{2}\int_{T_0}\left|\frac{\partial u_k}{\partial t}\right|^2 dt^1\wedge ...\wedge dt^p +,$$

$$+2\int_{T_0} F\left(t,\frac{\overline{u}_k}{2}\right)dt^1\wedge ...\wedge dt^p - \int_{T_0} F\left(t,-\widetilde{u}_k\left(t\right)\right)dt^1\wedge ...\wedge dt^p$$

and hence $\varphi\left(u_k\right)\geq 2\int_{T_0} F\left(t,\frac{\overline{u}_k}{2}\right)dt^1\wedge ...\wedge dt^p - c_6$.

This means that $||\overline{u}_k||\not\to\infty$. So the sequence $(\overline{u}_k)$ is bounded and implicitly the sequence (u_k) is bounded in H^1_T. The Hilbert space H^1_T is reflexive. By consequence, the sequence (u_k) (or one of his subsequence) is weakly convergent in H^1_T with the limit u. The Mazur's theorem assure that there exists a sequence (v_k) with the general term $v_k=\sum_{j=1}^{k}\alpha_{kj}u_j, \sum_{j=1}^{k}\alpha_{kj}=1, \alpha_{kj}\geq 0$, which is strongly converges to u in H^1_{ST}.

Now we consider $c>\underline{\lim}\ \varphi\left(u_k\right)$. Going if necessary to a subsequence, we can assume that $c>\varphi\left(u_k\right)$ for all $k\in N^*$. Since φ is lower semi-continuous in H^1_T and φ is convex, we obtain

$$\varphi\left(u\right)\leq \underline{\lim}\varphi\left(v_k\right)\leq \underline{\lim}\left(\sum_{j=1}^{k}\alpha_{kj}\varphi\left(u_j\right)\right)\leq\left(\sum_{j=1}^{k}\alpha_{kj}\right)c=c.$$

Because $c>\underline{\lim}\ \varphi\left(u_k\right)$ is arbitrary, we have $\varphi\left(u\right)\leq\underline{\lim}\ \varphi\left(u_k\right)$.

Thus, the action $\varphi\left(u\right)$ has a minimum point u in H^1_T, and so u is a solution of the problem (5).

Thanks to the properties of the strictly convex functions, we can reinforce the previous theorem. For that, we recall two equivalent properties of a strictly convex function $G\in C^1\left(\mathbb{R}^n,\mathbb{R}\right)$:

1) G has a critical point $\overline{x}\in\mathbb{R}^n$;

2) $G\left(x\right)\to\infty$ when $|x|\to\infty$.

Theorem 2. *We consider the problem (1)+(2). Suppose* $F:T_0\times\mathbb{R}^n\to\mathbb{R},(t,x)\to F(t,x)$ *has the properties*:

1) $F\left(t,x\right)$ *is measurable in* t *for any* $x\in\mathbb{R}^n$ *and of class* C^1 *in* x *for any* $t\in T_0$.

2) *There exist* $a\in C^1(\mathbb{R}^+,\mathbb{R}^+)$ *with the derivative* a' *bounded from above and* $b\in C\left(T_0,\mathbb{R}^+\right)$ *such that*

$$\left|F\left(t,x\right)\right|\leq a\left(|x|\right)b\left(t\right),\left|\nabla_x F\left(t,x\right)\right|\leq a\left(|x|\right)b\left(t\right),$$

for any $x\in\mathbb{R}^n$ *and any* $t\in T_0$.

3) *The function* $F\left(t,\cdot\right)$ *is strictly convex for any* $t\in T_0$.

Then, the following statements are equivalent:

1) *The problem* $(1)+(2)$ *has solutions;*

2) *There exists* $\overline{x}\in\mathbb{R}^n$ *so that* $\int_{T_0}\nabla F\left(t,\overline{x}\right)dt^1\wedge ...\wedge dt^p=0$;

3) $\int_{T_0} F\left(t,x\right)dt^1\wedge ...\wedge dt^p\to\infty$ *as* $|x|\to\infty$.

Proof. (see single-time case in [6, Theorem 1.8]). Let us prove that 1) implies 2):

We suppose that $u(t)$ is a solution of the problem (1)+(2). By integration we obtain

$$\sum_{i=1}^{p} \int_{T_0} \frac{\partial^2 u^j}{\partial t^{i^2}} dt^1 \wedge ... \wedge dt^p = \int_{T_0} \frac{\partial F}{\partial u^j} (t, u(t))\, dt^1 \wedge ... \wedge dt^p.$$

From the boundary conditions it results

$$\int_{T_0} \nabla F (t, u(t))\, dt^1 \wedge ... \wedge dt^p = 0. \tag{6}$$

On the other hand, the function $G(x) = \int_{T_0} F(t,x) dt^1 \wedge ... \wedge dt^p$ is strictly convex, because the function $F(t,\cdot)$ is strictly convex.

We suppose $u = \widetilde{u} + \overline{u}$, $\overline{u} = \int_{T_0} u(t)\, dt^1 \wedge ... \wedge dt^p$,

$$\widetilde{G}(x) = \int_{T_0} F(t, x + u(t))\, dt^1 \wedge ... \wedge dt^p.$$

From (6) we have $\nabla \widetilde{G}(\overline{u}) = 0$. From the properties of a strictly convex function, mentioned above, $\widetilde{G}(x)$ tends to ∞ when $|x|$ tends to ∞. Because the function $F(t,\cdot)$ is strictly convex, we obtain:

$$\widetilde{G}(x) \le \frac{1}{2} \int_{T_0} F(t, 2x)\, dt^1 \wedge ... \wedge dt^p + \frac{1}{2} \int_{T_0} F(t, 2u(t))\, dt^1 \wedge ... \wedge dt^p = \frac{1}{2} G(2x) + C.$$

For $|x| \to \infty$, $\widetilde{G}(x) \to \infty$ and consequently $G(2x) \to \infty$ and $G(x) \to \infty$. According to the properties of G, there exists $\overline{x}$ so that $\nabla G(\overline{x}) = 0$, i.e., $\int_{T_0} \nabla F(t, \overline{x})\, dt^1 \wedge ... \wedge dt^p = 0$.

Let us show that 2) implies 3):

The properties of G show that if $\overline{x}$ exists so that $\nabla G(\overline{x}) = 0$, then $G(x) \to \infty$ when $|x| \to \infty$, so $\int_{T_0} F(t,x)\, dt^1 \wedge ... \wedge dt^p \to \infty$ when $|x| \to \infty$.

Now, 3) implies 1). Indeed, the required implication is motivated by Theorem 1.

Remark. The previous results can be extended to PDEs produced in [12]-[19].

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