

# Exact solutions in first-order differential equations with periodic inputs

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## Abstract

A method for solving differential equations with periodic inputs is provided by means of an adequate use of a computer algebra system.

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**Key words:** Periodic function, Fourier series, First-order differential equation

## 1 Introduction

Many engineering applications lead to the same kind of differential equations and thus the same mathematical tools enable their study and resolution (cf. [2, 4]). Among the most useful differential equations we find the linear ones  $L[y(t)] = f(t)$  where  $f$  is a known function and  $L$  is the  $n^{\text{th}}$  order linear differential operator

$$L = D^n + a_{n-1}(t)D^{n-1} + \dots + a_1(t)D + a_0, \quad D^i = \frac{d^i}{dt^i}, 1 \leq i \leq n.$$

These equations are easily solved when it has got constant coefficients  $a_i$ , the roots of the homogeneous associated equation are known and the function  $f$  is an adequate combination of exponential, cosine, sine and polynomial functions.

In fact if  $n = 1$  and we have got the initial condition  $y(t_0) = y_0$ , then the linear differential equation may formally be solved by means of

$$y(t) = y(t_0) e^{-A(t)+A(t_0)} + \int_{t_0}^t e^{A(z)} f(z) dz$$

where  $A(t)$  is any antiderivative of  $a_0(t)$ . However difficulties arise when the involved integrals are not easy to find. Then we may try to get a series solution by means of an adequate use of power series, or use some numerical method in order to get an approximate solution of the differential equation.

Another interesting case in many applications appears when the input function  $f$  is periodic. Then the usual method to handle the problem involves either the use

of Fourier series or Laplace transforms [1, 3]. In fact these methods are also applied to solve some partial differential equations. In this note we show how, in the case of periodic inputs, computer algebra systems (CAS) may avoid to follow this process by enabling to obtain the general solution of first-order differential equations or a particular solution of an initial value problem. In both cases the solutions may easily be evaluated and plotted.

## 2 Periodic functions

Let us recall that a real function  $f$  is said to be *periodic* if there is some real number  $p > 0$  called *period* such that  $f(x) = f(x + np)$  for each  $n \in \mathbf{N}$ . In particular, if  $f$  is  $2\pi$ -periodic its Fourier series is the trigonometric function given by

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

where the coefficients  $a_0, a_n, b_n$  are given by means of the Euler's formulae which in turn are obtained with the aid of the orthogonality relationships of the sine and cosine functions.

According to the Dirichlet conditions if  $f$  is a piecewise regular function and has a finite number of finite discontinuities and a finite number of extrema, then  $f$  can be expanded into a Fourier series which converges to  $f$  at its continuous points and the mean of the positive and negative limits at its discontinuity points.

Visualizing how the Fourier series tends to  $f$  by means of CAS requires plotting  $f$  as well as the approximating trigonometric polynomials. Since up to now standard programs do not incorporate an implemented function to generate periodic functions, in [7] we built up an easily implemented method that generates such a kind of functions.

For the sake of completeness let us recall the following result which is both easy and providential in order to generate a  $(b-a)$ -periodic function which coincides with a given function  $f$  defined on the real interval  $[a, b[$ . For each real number  $t$  let us denote by  $I(t)$  the integer number  $n$  such that  $n \leq t < n + 1$ , *i.e.*  $I$  will stand for the integer part function.

**Proposition 1.** *If  $f$  is a real-valued function defined over  $[a, b[$ , then*

$$g(t) = f\left(t - (b-a)I\left(\frac{t-a}{b-a}\right)\right)$$

*is a  $(b-a)$ -periodic function defined over  $\mathbf{R}$  that coincides with  $f$  in  $[a, b[$ .*

*Proof.* Since  $I\left(\frac{t-a}{b-a}\right) = 0$  for  $t \in [a, b[$ , it follows that  $g$  coincides with  $f$  in  $[a, b[$ . On the other hand,

$$g(t + (b-a)) = f\left(t + (b-a) - (b-a)I\left(1 + \frac{t-a}{b-a}\right)\right) = g(t),$$

and the conclusion follows. QED

**Remark 1.** CAS do usually have a command which provide  $t$  modulo  $k$ . For instance, the DERIVE™ program [5] generates the function  $g$  given by the above proposition by just substituting the variable  $t$  of  $f(t)$  by  $a + \text{MOD}(t - a, b - a)$ . MATHEMATICA [11] and MATLAB [9] programs enjoy similar capabilities by means of their corresponding commands (cf. [8]).

**Example 1.** Represent the  $2\pi$ -periodic function  $f$  such that

$$f(t) = |t|, t \in [-\pi, \pi].$$

*Solution.* Having in mind the above, if we are to use the DERIVE program we just have to introduce

$$\text{SUBST}(\text{ABS}(t), t, -\text{PI} + \text{MOD}(t + \text{PI}, 2\text{PI}))$$

Simplifying the above expression and plotting the generated function we obtain the graph that appears in Fig 1.

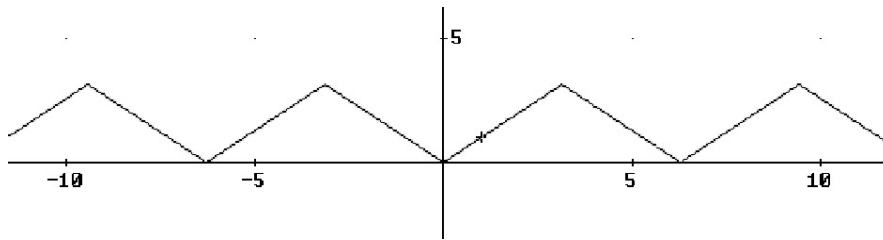


Figure 1: The  $2\pi$ -periodic function  $f$

This function can also be implemented within other CAS. For instance with MATHEMATICA we might introduce

```
Plot(Abs(MOD(x,2 Pi,- Pi)),{x,-12,12}, PlotRange {-3,6}),
```

and with MATLAB we might write

```
L = 'exp(-pi + mod(x+pi,2*pi))'
ezplot(L,[-12,12])
```

obtaining similar graphs to the one shown in Fig. 1.

### 3 Applications

Let us now recall some of the commands that enable to solve differential equations in one of the aforementioned CAS, namely with the DERIVE program. Given a differential equation having the form

$$p(x, y) + q(x, y) \frac{dy}{dx} = 0$$

we may obtain a general solution with DSOLVE1\_GEN( $p, q, x, y, c$ ). If we want to find a specific solution for symbolic or numeric initial conditions  $(x_0, y_0)$ , we may

either find a general solution, then substitute  $x_0$  and  $y_0$  and solve for  $c$  the equation obtained, or simply use  $\text{DSOLVE1}(p, q, x, y, x_0, y_0)$  which is similar to  $\text{DSOLVE1\_GEN}$  but simplifies to the specific solution for the initial condition  $y = y_0$  at  $x = x_0$ .

Let us also recall that  $\text{LINEAR1\_GEN}(p, q, x, y, c)$  provides the general solution of a linear differential equation written in the form

$$\frac{dy}{dx} + p(x) = q(x)$$

in terms of the symbolic constant  $c$ . The command  $\text{LINEAR1}(p, q, x, y, x_0, y_0)$  simplifies to the explicit solution for the initial condition  $y = y_0$  at  $x = x_0$ , there being other available commands for other specific kinds of differential equations (cf. [10]).

**Example 2.** Considering as input the function  $f$  of Example 1, solve

$$\frac{dx}{dt} + x = f(t).$$

*Solution.* Combining the aforementioned implemented functions with the periodic function generator of Section 2 we may obtain the general integral with the  $\text{DERIVE}$  program by just simplifying

$$\text{DSOLVE1\_GEN}(x - \text{SUBST}(\text{ABS}(t), t, -\text{PI} + \text{MOD}(t + \text{PI}, 2 \text{PI})), 1, t, x, c).$$

Doing so, we obtain

$$\begin{aligned} & (\hat{e}^{\frac{\pi}{2}} + 1) \text{SIGN} \left( 2 \cdot \pi \cdot \text{FLOOR} \left( \frac{t}{2 \cdot \pi} + \frac{1}{2} \right) - t \right) \cdot \left( 2 \cdot \pi \cdot \hat{e}^t \cdot \text{FLOOR} \left( \frac{t}{2 \cdot \pi} + \frac{1}{2} \right) - \hat{e}^{2 \cdot \pi \cdot \text{FLOOR}(t/(2 \cdot \pi) + 1/2)} \cdot \hat{e}^t \cdot (1 - t) \right) + \\ & \hat{e}^{2 \cdot \pi \cdot \text{FLOOR}(t/(2 \cdot \pi) + 1/2)} \cdot (\hat{e}^{\frac{\pi}{2}} - 1) - \pi \cdot \hat{e}^t \cdot (\hat{e}^{\frac{\pi}{2}} + 1) = -c \cdot \hat{e}^t - c \end{aligned}$$

Assigning the arbitrary constant  $c$  a finite set of values we get the corresponding particular solutions, *e.g.* with all the integer values between -10 and 10 we obtain the 21 particular solutions whose graphs are depicted in Fig. 2. This figure also includes the input function  $f$ .

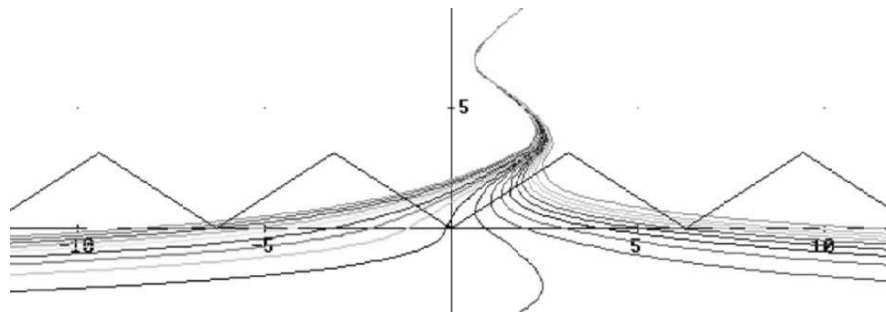


Figure 2: Particular integrals of an ODE with  $f$  as input

It is worth while noting that solving the differential equation of the above example by means of Fourier series implies calculating the Fourier coefficients (cf. [6, p. 363]) and the tedious fact of dealing with function series.

**Example 3.** An electrical circuit is modelled by the differential equation

$$\frac{di}{dt} + 1000i = 1000e(t)$$

where  $i(t)$  is the current in the circuit,  $e(t)$  is the applied voltage and  $t$  is the time variable in seconds. Find out the general expression of the current when  $e(t)$  is a  $2 \cdot 10^{-3}$ -periodic function such that

$$e(t) = \begin{cases} 10, & t \in [0, 10^{-3}[ \\ -10, & t \in [10^{-3}, 2 \cdot 10^{-3}[. \end{cases}$$

Next find the current at time  $2.5 \cdot 10^{-3}$  if, prior to closing the switch,  $i$  is zero.

*Solution.* Let us combine the LINEAR1 command with an adequate use of the CHI( $a, x, b$ ) function of the DERIVE program which simplifies to the characteristic function of the interval  $[a, b]$  in order to obtain the general solution. Writing

$$F(t) := 10 \text{CHI}(0, t, 0.001) - 10 \text{CHI}(0.001, t, 0.002)$$

then the (periodic) input voltage is given by  $e(t) = F(\text{MOD}(t, 0.002))$ .

Once  $e(t)$  is simplified the general solution is obtained by just simplifying

$$\text{LINEAR1.GEN}(1000, 1000 \cdot e(t), t, i, c).$$

In order to seek the solution  $i(t)$  of the initial value problem corresponding to  $i(0) = 0$  we solve for the unknown  $c$  the equation that results of taking  $t = 0, i = 0$  in the general solution. By using the SOLVE command we immediately get the function  $i(t)$  whose graph is plotted in Fig. 3 together with the voltage  $e(t)$ .

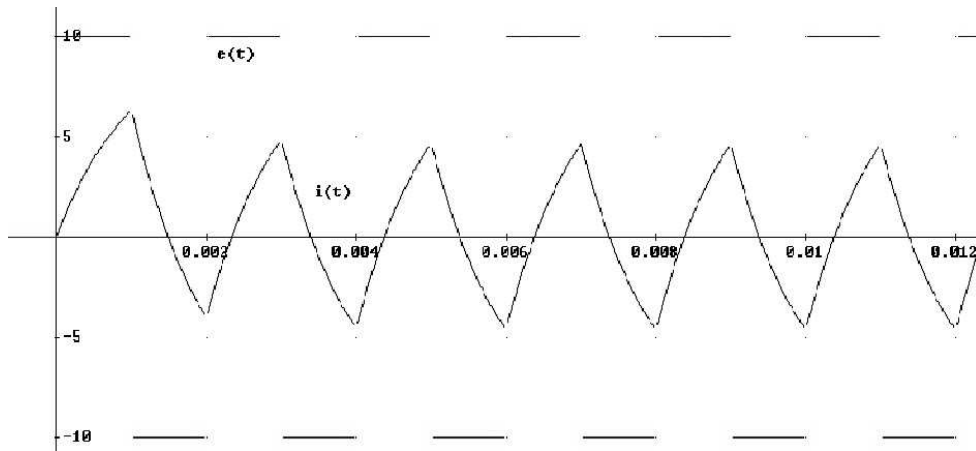


Figure 3: Input and solution of Example 3

Finally, the program easily provides that  $i(0.0025) = 1.51113$ .

## 4 Conclusion

Classical techniques to solve differential equations with periodic inputs usually involve the use of Laplace Transforms or Fourier Series which may carry out an important burden of work to be developed.

In this paper we have seen how CAS may avoid this in a very simple way, by working directly with the periodic function. CAS enable to get the general solution as well as the exact solution of an initial value problem. Once obtained, the same CAS facilitates its plotting or evaluation at a given point if desired.

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## References

- [1] R.J. Beerends, H.G. ter Morsche, J.C. van den Berg and E.M. van de Vrie, *Fourier and Laplace Transforms*, Cambridge University Press 2003.
- [2] P. Bugl, *Differential equations, Matrices and models*, Prentice Hall 1995.
- [3] J.W. Brown and R.V. Churchill, *Fourier Series and Boundary Value Problems*, McGraw-Hill 1993.
- [4] T.M. Creese and R.M. Haralick, *Differential equations for engineers*, McGraw-Hill 1978.
- [5] DERIVE™ 5. *The Mathematical Assistant for Your Personal Computer*, Texas Instruments 2000.
- [6] G. James, *Advanced Modern Engineering Mathematics*, Addison-Wesley 1994.
- [7] M.P. Legua, J.A. Morano, and L.M. Sánchez Ruiz, *Generating Periodic Functions*, WSEAS Trans. Syst. 1, 3 (2004), 37-39.
- [8] M.P. Legua, J.A. Morano, and L.M. Sánchez Ruiz, *Sine and cosine series representations*, WSEAS Trans. Math. 3, 3 (2004), 543-548.
- [9] MATLAB © . *The language of technical computing, Help Browser, Version 6.5*. The MATHWORKS 2002.
- [10] L.M. Sánchez Ruiz, M.P. Legua and J.A. Morano, *Matemáticas con DERIVE*, Ed. Universidad Politécnica de Valencia 2001.
- [11] S. Wolfram, *The MATHEMATICA © Book*, Wolfram Media/Cambridge University Press, 1999.

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