

An operational method for solving an integral differential equation

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Abstract

One-dimensional linear transport equation with a source term is considered. This is rewritten as a Cauchy problem: $\frac{dw}{dt} + Aw = F$, $w|_{t=0} = w_0$ where w belongs to a suitable subset of a Hilbert space, whose elements are pairs of real-valued functions depending on three variables: a space variable $z \in [0, a]$, an angle variable ν , with $\mu = \cos \nu \in [-1, 1]$ and a time variable $t \in [0, T]$.

The result of calculations from the numerical example show how the density depends on the time t and angle ν for a given value of the space variable z .

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1 Problem Formulation

Let us consider a transport equation in a plan - parallel geometry:

$$\frac{1}{\nu_c} \cdot \frac{\partial \varphi}{\partial t} + \mu \frac{\partial \varphi}{\partial z} + \sigma \cdot \varphi = \frac{\sigma_s}{2} \int_{-1}^1 \varphi d\mu + f(z, \mu, t) \quad (1)$$

with the following boundary conditions:

$$\begin{aligned} \varphi &= 0 & \text{if } z = 0, \mu > 0 \\ \varphi &= 0 & \text{if } z = H, \mu < 0 \end{aligned} \quad (2)$$

and the initial condition:

$$\varphi = \varphi_0 \quad \text{if } t = 0. \quad (3)$$

The unknown φ of the problem (1)-(3) is the density of the particles. These move with the speed ν_c , which makes an angle ν with the real axis Oz at moment t and $\mu = \cos \nu$. In the right-hand side of (1), f , is the radioactive source function, σ , σ_s are the continuous functions in the interval $[0, H]$ and satisfy the conditions:

$$0 < \sigma_0 \leq \sigma \leq \sigma_1 < \infty; \quad 0 < \sigma_s \leq \sigma'_s < \infty; \quad 0 < \sigma_{c0} \leq \sigma_c = \sigma - \sigma_s \quad (4)$$

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Further on, we consider for simplicity, $\nu_c = 1$. Using the notations:

$$\varphi^+ = \varphi(z, \mu, t); \quad \varphi^- = \varphi(z, -\mu, t), \quad \text{where } \mu > 0, \quad (5)$$

the equation (1) can be written in the form:

$$\begin{aligned} \frac{\partial \varphi^+}{\partial t} + \mu \frac{\partial \varphi^+}{\partial z} + \sigma \cdot \varphi^+ &= \frac{\sigma_s}{2} \int_0^1 (\varphi^+ + \varphi^-) d\mu + f^+ \\ \frac{\partial \varphi^-}{\partial t} - \mu \frac{\partial \varphi^-}{\partial z} + \sigma \cdot \varphi^- &= \frac{\sigma_s}{2} \int_0^1 (\varphi^+ + \varphi^-) d\mu + f^- \end{aligned} \quad (6)$$

Substituting: $\mu' = -\mu > 0$, we get:

$$\int_{-1}^0 \varphi(z, \mu, t) d\mu = - \int_1^0 \varphi(z, -\mu', t) d\mu' = \int_0^1 \varphi(z, -\mu', t) d\mu' = \int_0^1 \varphi^- d\mu.$$

The boundary value problem becomes:

$$\varphi^+(0, \mu, t) = 0; \quad \varphi^-(H, \mu, t) = 0, \quad \forall \mu \in [0, 1], \quad \forall t \in [0, T] \quad (7)$$

Adding and substrating the equations (6) and introducing the notations:

$$\begin{aligned} u &= \frac{1}{2}(\varphi^+ + \varphi^-) & g &= \frac{1}{2}(f^+ + f^-) \\ v &= \frac{1}{2}(\varphi^+ - \varphi^-) & r &= \frac{1}{2}(f^+ - f^-) \end{aligned} \quad (8)$$

we obtain the following system:

$$\begin{aligned} \frac{\partial u}{\partial t} + \mu \cdot \frac{\partial v}{\partial z} + \sigma + \mu &= \sigma_s \int_0^1 u du' + g \\ \frac{\partial v}{\partial t} + \mu \cdot \frac{\partial u}{\partial z} + \sigma + v &= r. \end{aligned} \quad (9)$$

The boundary-initial conditions are:

$$\begin{aligned} u + v &= 0 \quad \text{for } z = 0 \\ u - v &= 0 \quad \text{for } z = H \end{aligned} \quad (10)$$

and respectively:

$$u = u^0, \quad v = v^0 \quad \text{for } t = 0. \quad (11)$$

Let us define in a measurable set $D = [0, H] \times [0, 1]$ the Hilbert space $\Phi \subset L_2(D)$ of vector functions having two scalar components:

$$w = \begin{pmatrix} u \\ v \end{pmatrix}, \quad w^0 = \begin{pmatrix} u^0 \\ v^0 \end{pmatrix}, \quad F = \begin{pmatrix} g \\ r \end{pmatrix}. \quad (12)$$

with the scalar product for every fixed t :

$$(\alpha(t), \beta(t)) = \sum_{i=1}^2 \int_0^1 d\mu \int_0^H \alpha^i(z, \mu, t) \beta^i(z, \mu, t) dz \quad (13)$$

where α^i, β^i are the components of the vector functions α, β .

We consider that the functions $w \in \Phi$ satisfy the condition: $(Aw, w) < \infty$. Let Φ_0 be a subset of Φ , where the functions w are continuous, have $\frac{\partial w}{\partial z}$ continuous on D and the components u, v verify the boundary conditions (10).

We shall define the operators:

$$A = \begin{pmatrix} \sigma - \sigma_s \int_0^1 d\mu' & \mu \frac{\partial}{\partial z} \\ \mu \frac{\partial}{\partial z} & \sigma \end{pmatrix} \quad (14)$$

which is not dependent of t and $L = \frac{\partial}{\partial t} + A$. Certainly, the operator L will be defined on Φ_0 .

As a result, the problem (9)-(11) becomes:

$$\begin{aligned} \frac{\partial w}{\partial t} + Aw &= F, \quad (z, \mu, t) \in D \times [0, T], \quad D = [0, H] \times [0, 1] \\ w(z, \mu, 0) &= w^0, \quad \forall (z, \mu) \in D \end{aligned} \quad (15)$$

The operator A is symmetric, because for any α, β we have: $(A\alpha, \beta) = (\beta, A\alpha)$. Such an operator is called definite, if the following inequality is satisfied:

$$(A\alpha, \alpha) \geq \gamma \|\alpha\| \quad (16)$$

where γ is a positive constant, [5].

Certainly, if A is positive definite, then it is positive: $(A\alpha, \alpha) \geq 0$.

Also, if the operator L is positive definite, then the equation:

$$Lw = F \quad (17)$$

has only one solution. Indeed, let w_1 be an element such that

$$Lw_1 = F.$$

Hence, $L(w - w_1) = 0 \Rightarrow L\bar{w} = 0 \Rightarrow (L\bar{w}, \bar{w}) = 0$. But, the operator L is positive definite, such that $(L\bar{w}, \bar{w}) \geq 0, \forall \bar{w} \in D_L$ - the domain of $L \Rightarrow 0 \geq \gamma \|\bar{w}\|$ and hence, $\bar{w} = 0 \Rightarrow w = w_1$.

Now, we prove that A is a positive operator for each $w \neq 0$. We obtain:

$$\begin{aligned} (Aw, w) &= \int_0^1 d\mu \int_0^H \left[\sigma u^2 - \sigma_s \int_0^1 u^2 d\mu' + \mu u \frac{\partial v}{\partial z} + \mu v \frac{\partial u}{\partial z} + \sigma v^2 \right] dz = \\ &= \int_0^1 d\mu \int_0^H \left[\sigma(u^2 + v^2) + \mu \frac{\partial}{\partial z}(uv) - \sigma_s \int_0^1 u^2 d\mu' \right] dz \end{aligned} \quad (18)$$

The function u is continuous and by virtue mean value theorem on $[0,1]$, there exists

at least one point μ_0 in $(0,1)$ such that $\int_0^1 u^2 d\mu' = u^2(\mu_0)$. Then

$$\begin{aligned}
(Aw, w) &= \int_0^1 d\mu \int_0^H \left[\sigma u^2 - \sigma_s u^2(\mu_0) + \sigma v^2 + \mu \frac{\partial}{\partial z}(uv) \right] dz > \\
&> \int_0^1 d\mu \int_0^H \left[(\sigma - \sigma_s) u^2 + \sigma v^2 + \mu \frac{\partial}{\partial z}(uv) \right] dz \geq \int_0^1 \mu d\mu \int_0^H \frac{\partial}{\partial z}(uv) dz = \quad (19) \\
&= \frac{1}{4} \int_0^1 \mu [(\varphi^+)^2 - (\varphi^-)^2]_0^H d\mu = \frac{1}{4} \int_0^1 \mu ((\varphi^+)^2 + (\varphi^-)^2) d\mu > 0
\end{aligned}$$

according with (7).

In order to get a solution of the problem (15), we go through three stages.

First, a difference scheme is given in order to approximate the space derivatives which appearing in A . We consider on z -axis two points systems:

- a principal system, $\{z_k\}_k$, $k \in \{0, 1, \dots, N\}$ with $z_0 = 0$ and $z_N = H$;
- a secondary system, $\{z_{k+1/2}\}_k$, $k \in \{0, 1, \dots, N-1\}$ which verifies the inequality:

$$z_{k-1/2} < z_k < z_{k+1/2}.$$

Integrating the first equation (9) on the intervals: $(z_0, z_{1/2})$, $(z_{k-1/2}, z_{k+1/2})$, $k \in \{1, 2, \dots, N-1\}$, $(z_{N-1/2}, z_N)$ and the second equation on (z_{k-1}, z_k) , $k \in \{1, 2, \dots, N\}$, the system can be written in the form:

$$\begin{aligned}
&\frac{\partial}{\partial t} \int_{z_0}^{z_{1/2}} u dz + \mu \int_{z_0}^{z_{1/2}} \frac{\partial v}{\partial z} dz + \int_{z_0}^{z_{1/2}} \sigma u dz = \int_{z_0}^{z_{1/2}} \sigma_s dz \int_0^1 u d\mu' + \int_{z_0}^{z_{1/2}} g dz \\
&\frac{\partial}{\partial t} \int_{z_0}^{z_1} v dz + \mu \int_{z_0}^{z_1} \frac{\partial u}{\partial z} dz + \int_{z_0}^{z_1} \sigma v dz = \int_{z_0}^{z_1} r dz \\
&\dots\dots\dots \\
&\frac{\partial}{\partial t} \int_{z_{k-1/2}}^{z_{k+1/2}} u dz + \mu \int_{z_{k-1/2}}^{z_{k+1/2}} \frac{\partial v}{\partial z} dz + \int_{z_{k-1/2}}^{z_{k+1/2}} \sigma u dz = \int_{z_{k-1/2}}^{z_{k+1/2}} \sigma_s dz \int_0^1 u d\mu' + \int_{z_{k-1/2}}^{z_{k+1/2}} g dz \\
&\frac{\partial}{\partial t} \int_{z_k}^{z_{k+1}} v dz + \mu \int_{z_k}^{z_{k+1}} \frac{\partial u}{\partial z} dz + \int_{z_k}^{z_{k+1}} \sigma v dz = \int_{z_k}^{z_{k+1}} r dz \\
&\dots\dots\dots \\
&\frac{\partial}{\partial t} \int_{z_{N-1/2}}^{z_N} u dz + \mu \int_{z_{N-1/2}}^{z_N} \frac{\partial v}{\partial z} dz + \int_{z_{N-1/2}}^{z_N} \sigma u dz = \int_{z_{N-1/2}}^{z_N} \sigma_s dz \int_0^1 u d\mu' + \int_{z_{N-1/2}}^{z_N} g dz \\
&\frac{\partial}{\partial t} \int_{z_{N-1}}^{z_N} v dz + \mu \int_{z_{N-1}}^{z_N} \frac{\partial u}{\partial z} dz + \int_{z_{N-1}}^{z_N} \sigma v dz = \int_{z_{N-1}}^{z_N} r dz
\end{aligned} \tag{20}$$

With the following notations:

$$\begin{aligned}
\Delta z_0 &= z_{1/2} - z_0, & \Delta z_k &= z_{k+1/2} - z_{k-1/2}, & k &= 1, 2, \dots, N-1 \\
\Delta z_N &= z_N - z_{N-1/2}, & \Delta z_{k-1/2} &= z_k - z_{k-1}, & k &= 1, 2, \dots, N
\end{aligned} \tag{21}$$

we define the mean values:

$$\begin{aligned}\sigma_k &= \frac{1}{\Delta z_k} \int_{z_{k-1/2}}^{z_{k+1/2}} \sigma dz, & \sigma_{k+1/2} &= \frac{1}{\Delta z_{k+1/2}} \int_{z_k}^{z_{k+1}} \sigma dz, & \sigma_{s_k} &= \frac{1}{\Delta z_k} \int_{z_{k-1/2}}^{z_{k+1/2}} \sigma_2 dz \\ g_k &= \frac{1}{\Delta z_k} \int_{z_{k-1/2}}^{z_{k+1/2}} g dz, & r_{k+1/2} &= \frac{1}{\Delta z_{k+1/2}} \int_{z_k}^{z_{k+1}} \sigma dz\end{aligned}\quad (22)$$

Let $h = \max_{0 \leq k \leq N-1} (\Delta z_0, \Delta z_N, \Delta z_k, \Delta z_{k+1/2})$. Dividing the first equation (20) by Δz_0 and the second by $\Delta z_{1/2}$ we obtain:

$$\begin{aligned}\frac{\partial}{\partial t} \left[\frac{1}{\Delta z_0} \int_{z_0}^{z_{1/2}} u dz \right] + \mu \frac{1}{\Delta z_0} [v]_{z_0}^{z_{1/2}} + \frac{1}{\Delta z_0} \int_{z_0}^{z_{1/2}} \sigma(z) u(z, \mu, t) dz &= \\ = \int \left[\frac{1}{\Delta z_0} \int_{z_0}^{z_{1/2}} \sigma_s u dz \right] d\mu' + \frac{1}{\Delta z_0} \int_{z_0}^{z_{1/2}} g dz &\end{aligned}\quad (23)$$

$$\frac{\partial}{\partial t} \left[\frac{1}{\Delta z_{1/2}} \int_{z_0}^{z_1} v dz \right] + \mu \frac{1}{\Delta z_{1/2}} [u]_{z_0}^{z_1} + \frac{1}{\Delta z_{1/2}} \int_{z_0}^{z_1} \sigma v dz = \frac{1}{\Delta z_{1/2}} \int_{z_0}^{z_1} r dz.$$

In accordance with the boundary conditions:

$$v_0 = v|_{z=0} = \frac{1}{2} (\varphi^+ - \varphi^-)|_{z=0} = -\frac{1}{2} \varphi^- \quad (24)$$

$$u_0 = u|_{z=0} = \frac{1}{2} (\varphi^+ + \varphi^-)|_{z=0} = -\frac{1}{2} \varphi^- \quad (25)$$

we have: $v_0 = -u_0$ and the relation (23) can be rewritten in the form:

$$\begin{aligned}\frac{\partial u_0}{\partial t} + \mu \frac{v_{1/2} + u_0}{\Delta z_0} + \sigma_0 u_0 = \sigma_{s_0} \int_0^1 u_0 d\mu' + g_0, & \quad g_0 = \frac{1}{\Delta z_0} \int_{z_0}^{z_{1/2}} g dz \\ \frac{\partial v_{1/2}}{\partial t} + \mu \frac{u_1 - u_0}{\Delta z_{1/2}} + \sigma_{1/2} v_{1/2} = r_{1/2}\end{aligned}\quad (26)$$

where the functions u, v are replaced by their values in the points: $z = 0, z = 1/2, z = 1$. Similarly, we get

$$\begin{aligned}\frac{\partial u_k}{\partial t} + \mu \frac{v_{k+1/2} - v_{k-1/2}}{\Delta z_k} + \sigma_k u_k = \sigma_{s_k} \int_0^1 u_k d\mu' + g_k \\ \frac{\partial v_{k+1/2}}{\partial t} + \mu \frac{u_{k+1} - u_k}{\Delta z_{k+1/2}} + \sigma_{k+1/2} v_{k+1/2} = r_{k+1/2}, \quad k = 1, 2, \dots, N-1\end{aligned}\quad (27)$$

$$\begin{aligned}\frac{\partial v_{N-1/2}}{\partial t} + \mu \frac{u_N - u_{N-1}}{\Delta z_{N-1/2}} + \sigma_{N-1/2} v_{N-1/2} = r_{N-1/2} \\ \frac{\partial u_N}{\partial t} + \mu \frac{v_{N-1/2}}{\Delta z_N} + \sigma_N u_N = \sigma_{s_N} \int_0^1 u_N d\mu' + g_N, \quad g_N = \frac{1}{\Delta z_N} \int_{z_{N-1/2}}^{z_N} g dz\end{aligned}\quad (28)$$

where $u_N = v_N$.

Let us consider $M(0, 2N)$, the Hilbert space of the vector functions $\alpha = (\alpha_0, \alpha_{1/2}, \alpha_1, \dots, \alpha_N)$ with the scalar product:

$$(\alpha, \beta) = \sum_{i=0}^{2N} \int_0^1 \Delta z_{i/2} \alpha_{i/2} \beta_{i/2} d\mu \tag{29}$$

and the norm:

$$\|\alpha\| = \sqrt{(\alpha, \alpha)}, \quad \alpha \in M_h(0, 2N). \tag{30}$$

We define the vector functions:

$$\begin{aligned} \varphi &= (u_0, v_{1/2}, u_1, \dots, u_{N-1}, v_{N-1/2}, u_N) \\ f &= (g_0, r_{1/2}, g_1, \dots, g_{N-1}, r_{N-1/2}, g_N) \\ \varphi^0 &= (u_0^0, v_{1/2}^0, u_1^0, \dots, u_{N-1}^0, v_{N-1/2}^0, u_N^0) \end{aligned} \tag{31}$$

and the operator: $A = L - S$, where

$$L = \begin{bmatrix} \frac{\mu}{\Delta z_0} + \sigma_0 & \frac{\mu}{\sigma_0} & 0 & 0 & \dots & 0 & 0 \\ -\frac{\mu}{\Delta z_{1/2}} & \sigma_{1/2} & \frac{\mu}{\Delta z_{1/2}} & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & \sigma_{N-1/2} & \frac{\mu}{\Delta z_{N-1/2}} \\ 0 & 0 & 0 & 0 & \dots & -\frac{\mu}{\Delta z_N} & \frac{\mu}{\Delta z_N} + \sigma_N \end{bmatrix} \tag{32}$$

$$S = \text{diag} \left(\sigma_{s_{i/2}} \int_0^1 \gamma_{i/2} d\mu \right), \quad i = 0, 1, \dots, 2N; \quad \gamma_{i/2} = \begin{cases} 1 & \text{if } i/2 \text{ is entier number} \\ 0 & \text{if } i/2 \text{ is rational number} \end{cases} \tag{33}$$

Then, the system (26)-(28) has the form:

$$\begin{aligned} \frac{d\varphi}{dt} + A\varphi &= f, \quad t \in [0, T] \\ \varphi(z, \mu, 0) &= \varphi^0 \end{aligned} \tag{34}$$

where φ, f, φ^0 were defined by (31). In spite of the fact that we use, for simplicity of the writing, the same notations as (1), (3), here φ is the numerical solution for our problem.

We shall prove that L is a positive operator. Indeed, let $w \in M$ and then

$$\begin{aligned}
(Lw, w) &= \int_0^1 \Delta z_0 \left(\mu \frac{w_{1/2} + w_0}{\Delta z_0} + \sigma_0 w_0 \right) w_0 d\mu + \\
&+ \int_0^1 \Delta z_{1/2} \left(\mu \frac{w_1 - w_0}{\Delta z_{1/2}} + \sigma_0 w_0 \right) w_{1/2} d\mu + \\
&+ \sum_{i=1}^{N-1} \int_0^1 \Delta z_i \left(\mu \frac{w_{i+1/2} + w_{i-1/2}}{\Delta z_i} + \sigma_i w_i \right) w_i d\mu + \\
&+ \sum_{i=1}^{N-1} \int_0^1 \Delta z_{i+1/2} \left(\mu \frac{w_{i+1} - w_i}{\Delta z_{i+1/2}} + \sigma_{i+1/2} w_{i+1/2} \right) w_{i+1/2} d\mu + \\
&+ \int_0^1 \Delta z_N \left(\mu \frac{w_N + w_{N-1}}{\Delta z_N} + \sigma_N w_N \right) w_N d\mu = \\
&= \int_0^1 \mu (w_0^2 + w_N^2) d\mu + \sum_{i=1}^{2N} \int_0^1 \Delta z_{i/2} \sigma_{i/2} w_{i/2}^2 d\mu > \\
&> \sigma_0 \sum_{i=1}^{2N} \int_0^1 \Delta z_{i/2} w_{i/2}^2 d\mu = \sigma_0 \|w\|^2 > 0
\end{aligned}$$

because continuous functions on $[0, H]$, σ_i , are bounded on $[0, H]$ and by hypothesis $\sigma \geq \sigma_0 > 0$.

In the second stage a difference scheme is given to approximate the time derivatives. This is used together with the bicycle splitting-up method, which writes the operator A as a sum:

$$A(t) = A_1(t) + A_2(t), \quad A_1(t) \geq 0, \quad A_2(t) \geq 0 \quad (35)$$

Let us divide the close interval $[0, T]$ into n subintervals by choosing points: $t_0 = 0, t_1, \dots, t_n = T$. Next, we take an arbitrary subinterval: $[t_{j-1}, t_{j+1}] = [t_{j-1}, t_{j-2/3}] \cup [t_{j-2/3}, t_{j-1/3}] \cup [t_{j-1/3}, t_{j+1/3}] \cup [t_{j+1/3}, t_{j+2/3}] \cup [t_{j+2/3}, t_{j+1}]$, which has the length equal to 5τ , where τ is the time step. Approximating the operators A_1, A_2 on this the subinterval by: $\Lambda_k^j = A_k(t_j)$, $k = 1, 2$, we shall obtain from (34) a difference system using the Krank-Nicholson scheme, [5]:

$$\begin{aligned}
\frac{\varphi^{j-2/3} - \varphi^{j-1}}{\tau} + \Lambda_1^j \frac{\varphi^{j-1} + \varphi^{j-2/3}}{2} &= 0 \\
\frac{\varphi^{j-1/3} - \varphi^{j-2/3}}{\tau} + \Lambda_2^j \frac{\varphi^{j-2/3} + \varphi^{j-1/3}}{2} &= 0 \\
\frac{\varphi^{j-2/3} - \varphi^{j-1}}{2\tau} &= f^j \\
\frac{\varphi^{j+2/3} - \varphi^{j+1/3}}{\tau} + \Lambda_2^j \frac{\varphi^{j+2/3} + \varphi^{j+1/3}}{2} &= 0 \\
\frac{\varphi^{j+1} - \varphi^{j+2/3}}{\tau} + \Lambda_1^j \frac{\varphi^{j+1} + \varphi^{j+2/3}}{2} &= 0
\end{aligned} \quad (36)$$

or in other form:

$$\begin{aligned}
\varphi^{j-2/3} &= \left(E + \frac{\tau}{2}\Lambda_1^j\right)^{-1} \left(E - \frac{\tau}{2}\Lambda_1^j\right) = T_1^j \varphi^{j-1} \\
\varphi^{j-1/3} &= T_2^j \varphi^{j-2/3} \\
\varphi^{j+1/3} &= \varphi^{j-1/3} + 2\tau f^j \\
\varphi^{j+2/3} &= T_2^j \varphi^{j+1/3} \\
\varphi^{j+1} &= T_1^j \varphi^{j+2/3}
\end{aligned} \tag{37}$$

Finally, we have the recurrence formula

$$\varphi^{j+1} = T_1^j T_2^j T_2^j T_1^j \varphi^{j-1} + 2\tau T_1^j T_2^j f^j$$

To find the solution of the system (36), we consider the first equation, for a fixed μ and the operators (32), (33). We get:

$$\begin{aligned}
& \begin{bmatrix} 1 + \frac{\tau}{2} \frac{\mu}{\Delta z_0} & \frac{\tau}{2} \frac{\mu}{\Delta z_0} & 0 & \dots & 0 & 0 & 0 \\ -\frac{\tau}{2} \frac{\mu}{\Delta z_{1/2}} & 1 & \frac{\tau}{2} \frac{\mu}{\Delta z_{1/2}} & \dots & 0 & 0 & 0 \\ 0 & -\frac{\tau}{2} \frac{\mu}{\Delta z_{1/2}} & 1 & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & -\frac{\tau}{2} \frac{\mu}{\Delta z_{N-1/2}} & 1 & \frac{\tau}{2} \frac{\mu}{\Delta z_{N-1/2}} \\ 0 & 0 & 0 & \dots & 0 & -\frac{\tau}{2} \frac{\mu}{\Delta z_N} & 1 + \frac{\tau}{2} \frac{\mu}{\Delta z_N} \end{bmatrix} \cdot \begin{bmatrix} u_0^{j-2/3} \\ v_{1/2}^{j-2/3} \\ u_1^{j-2/3} \\ \vdots \\ v_{N-1/2}^{j-2/3} \\ u_N^{j-2/3} \end{bmatrix} = \\
& = \begin{bmatrix} 1 - \frac{\tau}{2} \frac{\mu}{\Delta z_0} & -\frac{\tau}{2} \frac{\mu}{\Delta z_0} & 0 & \dots & 0 & 0 & 0 \\ \frac{\tau}{2} \frac{\mu}{\Delta z_{1/2}} & 1 & -\frac{\tau}{2} \frac{\mu}{\Delta z_{1/2}} & \dots & 0 & 0 & 0 \\ 0 & \frac{\tau}{2} \frac{\mu}{\Delta z_1} & 1 & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & \frac{\tau}{2} \frac{\mu}{\Delta z_{N-1/2}} & 1 & -\frac{\tau}{2} \frac{\mu}{\Delta z_{N-1/2}} \\ 0 & 0 & 0 & \dots & 0 & \frac{\tau}{2} \frac{\mu}{\Delta z_N} & 1 - \frac{\tau}{2} \frac{\mu}{\Delta z_N} \end{bmatrix} \cdot \begin{bmatrix} u_0^{j-1} \\ v_{1/2}^{j-1} \\ u_1^{j-1} \\ \vdots \\ v_{N-1/2}^{j-1} \\ u_N^{j-1} \end{bmatrix} =
\end{aligned}$$

We obtain an analogous relation from the last equation. For the second and the fourth

equation we have:

$$\begin{bmatrix} 1 + \frac{\tau}{2} \left(\sigma_0 - \sigma_{s_0} \int_0^1 d\mu \right) & 0 & 0 & \dots & 0 \\ 0 & 1 + \frac{\tau}{2} \sigma_{1/2} & 0 & \dots & 0 \\ 0 & 0 & \frac{\tau}{2} \left(\sigma_0 - \sigma_{s_1} \int_0^1 d\mu \right) & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 + \frac{\tau}{2} \left(\sigma_N - \sigma_{s_N} \int_0^1 d\mu \right) \end{bmatrix} \cdot$$

$$\begin{bmatrix} u_0^{j-1/3} \\ v_{1/2}^{j-1/3} \\ u_1^{j-1/3} \\ \vdots \\ v_N^{j-1/3} \\ u_N^{j-1} \end{bmatrix} =$$

$$\begin{bmatrix} 1 - \frac{\tau}{2} \left(\sigma_0 - \sigma_{s_0} \int_0^1 d\mu \right) & 0 & 0 & \dots & 0 \\ 0 & 1 - \frac{\tau}{2} \sigma_{1/2} & 0 & \dots & 0 \\ 0 & 0 & 1 - \frac{\tau}{2} \left(\sigma_1 - \sigma_{s_1} \int_0^1 d\mu \right) & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 - \frac{\tau}{2} \left(\sigma_N - \sigma_{s_N} \int_0^1 d\mu \right) \end{bmatrix} \cdot$$

$$\begin{bmatrix} u_0^{j-2/3} \\ v_{1/2}^{j-2/3} \\ u_1^{j-2/3} \\ \vdots \\ v_N^{j-2/3} \end{bmatrix} =$$

We obtain the following relations for the numerical solution:

$$\varphi^{j-1/3} = \left(E + \frac{\tau}{2} A_2^j \right)^{-1} \left(E - \frac{\tau}{2} A_2^j \right) \varphi^{j-2/3}$$

$$\varphi^{j-1/3} = \begin{bmatrix} \frac{1}{1 + \sigma_{c_0}\tau/2} & 0 & \dots & 0 \\ 0 & \frac{1}{1 + \sigma_{1/2}\tau/2} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \frac{1}{1 + \sigma_{c_N}\tau/2} \end{bmatrix}.$$

$$\begin{bmatrix} \left(1 - \frac{\sigma_0\tau}{2}\right) u_0^{j-2/3} + \frac{\sigma_{s0}\tau}{2} \int_0^1 u_0^{j-2/3} d\mu & 0 & \dots & 0 \\ 0 & \left(1 - \frac{\tau\sigma_{1/2}}{2}\right) v_{1/2}^{j-2/3} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \left(1 - \frac{\sigma_N\tau}{2}\right) u_N^{j-2/3} + \frac{\sigma_{sN}\tau}{2} \int_0^1 u_N^{j-2/3} d\mu \end{bmatrix}$$

Elements of the product are:

$$\varphi_i^{j-1/3} = \frac{1}{1 + \frac{\tau\sigma_{ci}}{2}} \left[\left(1 - \frac{\tau\sigma_i}{2}\right) \varphi_i^{j-2/3} + \frac{\tau\sigma_{si}}{2} \int_0^1 \varphi_i^{j-2/3} d\mu \right], \quad i = 0, 1, \dots, N \quad (38)$$

$$\varphi_{i-1/2}^{j-1/3} = \frac{1 - \sigma_{i-1/2}\tau/2}{1 + \sigma_{i-1/2}\tau/2} \cdot \varphi_i^{j-2/3}, \quad i = 0, 1, \dots, N \quad (39)$$

Analogously, we have

$$\varphi_i^{j+1/3} = \frac{1}{1 + \frac{\tau\sigma_{ci}}{2}} \left[\left(1 - \frac{\tau\sigma_i}{2}\right) \varphi_i^{j+1/3} + \frac{\tau\sigma_{si}}{2} \int_0^1 \varphi_i^{j+1/3} d\mu \right], \quad i = 0, 1, \dots, N \quad (40)$$

$$\varphi_{i-1/2}^{j+2/3} = \frac{1 - \sigma_{i-1/2}\tau/2}{1 + \sigma_{i-1/2}\tau/2} \cdot \varphi_i^{j+1/3}, \quad i = 0, 1, \dots, N \quad (41)$$

At the third stage, we consider the points: $\mu_0 = 0, \mu_1, \dots, \mu_m = 1$, in the interval $[0, 1]$ and compute the integrals with respect to μ , using a numerical integration (trapezoidal approximation):

$$\int_0^1 \psi(\mu) d\mu \approx \sum_{l=1}^m S_l \psi_l, \quad \psi_l = \psi(\mu_l) \quad (42)$$

Then, the system (36) can be written in the form:

$$\begin{aligned} \left(E + \frac{\tau}{2} A_{1,l}^j\right) \varphi_l^{j-2/3} &= \left(E - \frac{\tau}{2} A_{1,l}^j\right) \varphi_l^{j-1} \\ \left(E + \frac{\tau}{2} A_{2,l}^j\right) \varphi_l^{j-1/3} &= \left(E - \frac{\tau}{2} A_{2,l}^j\right) \varphi_l^{j-2/3} \\ \varphi_l^{j+1/3} &= \varphi_l^{j-1/3} + 2\tau f_l^j \\ \left(E + \frac{\tau}{2} A_{2,l}^j\right) \varphi_l^{j+2/3} &= \left(E - \frac{\tau}{2} A_{2,l}^j\right) \varphi_l^{j+1/3} \\ \left(E + \frac{\tau}{2} A_{1,l}^j\right) \varphi_l^{j+1} &= \left(E - \frac{\tau}{2} A_{1,l}^j\right) \varphi_l^{j+2/3} \end{aligned} \quad (43)$$

In this choice of the steps, which correspond to the variables z, t , we use the condition:

$$\tau \leq \min_i(\Delta z_{i/2}) \tag{44}$$

2 Numerical Example

We wish to find the solution of the problem:

$$\frac{d\varphi(z, \mu, t)}{dt} + A(t)\varphi(z, \mu, t) = f(z, \mu, t), (z, \mu, t) \in [0, 4] \times [0, 1] \times [0, 2] \tag{45}$$

$$\varphi(z, \mu, 0) = \varphi^0(z, \mu) \tag{46}$$

Considering the partition of $[0,4]$ into four subintervals of equal length by points:

$$z_0 = 0 < z_{1/2} < z_1 < z_{3/2} < z_2 = 4$$

with

$$\begin{aligned} \Delta z_0 &= z_{1/2} - z_0 = 1; \Delta z_{1/2} = z_1 - z_0 = 2; \Delta z_1 = z_{3/2} - z_{1/2} = 2; \\ \Delta z_{3/2} &= z_2 - z_1 = 2; \Delta z_2 = z_2 - z_{3/2} = 1. \end{aligned}$$

The partition of the interval $[0,1]$ is: $\mu_0 = 0 < \mu_1 = 1/2 < \mu_2 = 1$.

For the variable t , we consider the regular partition of the interval $[0,2]$ by the points: $t_0 = 0 < t_{1/3} < t_{2/3} < t_1 < t_{4/3} < t_{5/3} < t_2 = 2$. The initial value problem is defined by:

$$\varphi^0 = (u_0^0, v_{1/2}^0, u_1^0, v_{3/2}^0, u_2^0) = (1, 1, 1, 1, 1) \tag{47}$$

The functions $\sigma(z)$, $\sigma_s(z)$ and f , which here depends only of μ are defined with the help of the fig.1 and fig.2. The values of φ_i , $i \in \{0, 1/2, 1, 3/2, 2\}$ with respect to μ_i and t_j are presented in table 1.

From the relations (8) and using the mean values for $u_{1/2}, v_1, u_{3/2}$ we obtain the density, φ^+ , for $\mu > 0$ and the density, φ^- , for $\mu < 0$ for each value of z_i and t_j :

$$\begin{aligned} \varphi^+(1/2, \mu, t) &= v_{1/2} + \frac{u_0 + 2 \cdot u_1}{3} & \varphi^-(0, -\mu, t) &= 2u_0, \varphi^-(2, -\mu, t) = 0 \\ \varphi^+(1, \mu, t) &= u_1 + \frac{v_{1/2} + v_{3/2}}{2} & \varphi^-(1/2, \mu, t) &= \frac{u_0 + 2 \cdot u_1}{3} - v_{1/2} \\ \varphi^+(3/2, \mu, t) &= v_{3/2} + \frac{2 \cdot u_1 + u_2}{3} & \varphi^-(1, \mu, t) &= u_1 - \frac{v_{1/2} + v_{3/2}}{2} \\ \varphi^+(2, \mu, t) &= 2u_N, \varphi^+(0, \mu, t) = 0 & \varphi^-(3/2, \mu, t) &= \frac{2 \cdot u_1 + u_2}{3} - v_{3/2} \end{aligned}$$

Table 1

φ	t=1/3			t=2/3			t=4/3		
	$\mu = 0$	$\mu = \frac{1}{2}$	$\mu = 1$	$\mu = 0$	$\mu = \frac{1}{2}$	$\mu = 1$	$\mu = 0$	$\mu = \frac{1}{2}$	$\mu = 1$
u_0	1	0.69	0.44	0.92	0.67	0.46	1.25	0.97	0.69
$v_{1/2}$	1	0.98	0.95	0.62	0.62	0.59	0.95	0.9	0.82
u_1	1	0.99	0.99	0.87	0.85	0.87	1.2	1.17	1.09
$v_{3/2}$	1	1.	1.	0.56	0.56	0.56	0.89	0.86	0.79
u_2	1	1.	1.	0.69	0.69	0.69	0.99	0.99	0.92

φ	t=5/3			t=2		
	$\mu = \frac{1}{2}$	$\mu = 0$	$\mu = 1$	$\mu = 0$	$\mu = \frac{1}{2}$	$\mu = 1$
u_0	1.17	0.94	0.71	1.17	0.73	0.38
$v_{1/2}$	0.59	0.56	0.5	0.59	0.54	0.43
u_1	1.05	1.03	0.98	1.05	1.04	0.98
$v_{3/2}$	0.5	0.48	0.44	0.5	0.5	0.5
u_2	0.68	0.68	0.64	0.68	0.65	0.59

Table 2

φ^+	t=1/3			t=2/3			t=4/3		
	$\mu = 0$	$\mu = \frac{1}{2}$	$\mu = 1$	$\mu = 0$	$\mu = \frac{1}{2}$	$\mu = 1$	$\mu = 0$	$\mu = \frac{1}{2}$	$\mu = 1$
0	0	0	0	0	0	0	0	0	0
1/2	2	1.88	1.76	1.52	1.4	1.32	2.17	2	1.77
1	2	1.99	1.97	1.5	1.44	1.44	2.12	2.05	1.9
3/2	2	2	2	1.37	1.36	1.37	2	1.97	1.83
2	2	2	2	1.38	1.38	1.38	1.98	1.98	1.84

φ^+	t=5/3			t=2		
	$\mu = 0$	$\mu = 1/2$	$\mu = 1$	$\mu = 0$	$\mu = \frac{1}{2}$	$\mu = 1$
0	0	0	0	0	0	0
1/2	1.7	1.56	1.35	1.7	1.47	1.2
1	1.6	1.55	1.45	1.6	1.6	1.45
3/2	1.4	1.39	1.31	1.4	1.4	1.35
2	1.36	1.36	1.3	1.36	1.3	1.2

Table 3

φ^-	t=1/3			t=2/3			t=4/3		
	$\mu = 0$	$\mu = \frac{1}{2}$	$\mu = 1$	$\mu = 0$	$\mu = \frac{1}{2}$	$\mu = 1$	$\mu = 0$	$\mu = \frac{1}{2}$	$\mu = 1$
0	2	1.4	0.87	1.85	1.34	0.92	2.5	1.94	1.38
1/2	0	0	0	0.27	0.18	0.14	0.27	0.2	0.14
1	0	0.4	0.02	0.28	0.26	0.29	0.3	0.28	0.29
3/2	0	0.06	0	0.25	0.24	0.24	0.24	0.25	0.25
2	0	0	0	0	0	0	0	0	0

φ^-	t=5/3			t=2		
	$\mu = 0$	$\mu = 1/2$	$\mu = 1$	$\mu = 0$	$\mu = \frac{1}{2}$	$\mu = 1$
0	2.34	1.88	1.42	2.34	1.46	1.96
1/2	0.5	0.44	0.39	0.5	0.29	0.35
1	0.51	0.51	0.51	0.51	0.52	0.52
3/2	0.41	0.43	0.43	0.42	0.4	0.35
2	0	0	0	0	0	0

We mention that, the numerical values of φ^+ and φ^- were obtained for the neutron speed, $\nu_c = 1$. Also, in accordance with [3], we multiply with a factor 2 the number of the particles, which pass through a surface with unitary area.

Finally, to obtain the true values of φ^+ and φ^- we will multiply these with $2\nu_c$.

The results of this numerical example prove its importance: how depends the density in a point z at the time t for different values of angle ν .

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