

Efficiency and duality in multiobjective nonsmooth programming

Ștefan Mititelu

Abstract

The aim of this paper is to establish efficiency necessary conditions of Kuhn-Tucker type in different forms and weak, direct and converse duality theorems in Wolfe sense for a multiobjective nonsmooth program involving real functions defined in a locally convex space.

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1 Introduction and preliminaries

Let X be a locally convex space and A be a nonempty open set in X . Let also the nonsmooth (nondifferentiable) vector functions $f = (f_1, \dots, f_p)' : A \rightarrow \mathbf{R}^p$, $g = (g_1, \dots, g_m)' : A \rightarrow \mathbf{R}^m$ and $h = (h_1, \dots, h_q)' : A \rightarrow \mathbf{R}^q$ be, where $p, m, q \in \mathbf{N}^*$. We consider the following multiobjective mathematical program:

$$(MP) \quad \begin{cases} \text{Minimize (Pareto)} & (f_1(x), \dots, f_p(x)) \\ \text{subject to} & g(x) \preceq 0, h(x) = 0, x \in A. \end{cases}$$

The domain of this program is the set

$$D = \{x \in A \mid g(x) \preceq 0, h(x) = 0\}.$$

The purpose of this paper is to establish efficiency necessary conditions of Kuhn-Tucker type under different forms for (MP) and to develop a Wolfe duality by weak, direct and converse theorems for the same program. The mathematical instrument used in study is the Clarke-extended subdifferential for real functions defined on X .

1. First, we present the Clarke-extended subdifferential [4,6] and some of his properties and also others notions that are used in this paper. Let X^* be the dual space of X . Let $F : A \rightarrow \mathbf{R}$ be a real function.

Definition 1.1. The *Clarke directional derivative* of F at the point $x \in A$ in direction $v \in X$, denoted $F^0(x; v)$, is defined by

$$F^0(x; v) = \lim_{\substack{x' \rightarrow x \\ \lambda \downarrow 0}} \frac{F(x' + \lambda v) - F(x')}{\lambda}.$$

This derivative was introduced by Clarke [1] in 1973 for Lipschitz functions. In 1989 [6] we defined the Clarke-extended subdifferential of F at x by

Definition 1.2 (Mititelu [4, 6, 7]). The set

$$\partial F(x) = \{\xi \in X^* \mid F^0(x; v) \geq \langle \xi, v \rangle, \quad \forall v \in X\},$$

where $\langle \xi, v \rangle = \xi(v)$, is said to be the *subdifferential* (or *generalized gradient*) of F at $x \in A$.

If $\partial F(x) \neq \emptyset$, then F is called *subdifferentiable* at x . The elements of $\partial F(x)$ are called *subgradients* of F at x .

The vector function $F = (F_1, \dots, F_m) : A \rightarrow \mathbf{R}^m$ is subdifferentiable when all its components F_1, \dots, F_m are subdifferentiable functions. The following properties of the subdifferential hold [7]:

(P1) *If F is continuously differentiable function at x , then $\partial F(x) = \{\nabla F(x)\}$.*

We quote by [7] other properties of ∂F , that will be used in this paper.

(P2) *If the direction function $F^0(x; \cdot)$ is finite, then the subdifferential $\partial F(x)$ is a nonempty, convex and compact set. Moreover,*

$$F^0(x; v) = \max\{\langle \xi, v \rangle \mid \xi \in \partial F(x)\}, \quad \forall x \in A.$$

(P3) *If F is subdifferentiable at x , then λF ($\lambda \in \mathbf{R}$) is subdifferentiable at x and*

$$\partial(\lambda F)(x) = \lambda \cdot \partial F(x).$$

(P4) *If the functions F_1, \dots, F_m are subdifferentiable at $x \in A$, then the function $\sum_{i=1}^m F_i$ is subdifferentiable at x and*

$$\partial \left(\sum_{i=1}^m F_i \right) (x) \subseteq \sum_{i=1}^m \partial F_i(x).$$

2. Also, we use Clarke's tangent cone and the normal cone in Clarke sense at a point x of a nonempty subset C of A .

Clarke's tangent cone to C at $x \in C$ is defined by (one of its equivalent forms) by the set [1]:

$$T_C(x) = \{v \in \mathbf{R}^n \mid \forall t_k \downarrow 0, \forall (x^k) \subset C : x^k \rightarrow x, \exists v^k \rightarrow v \text{ such that } x + tv \in C\}.$$

The normal cone to C at $x \in C$ is defined by the set [1]:

$$N_C(x) = \{\nu \in \mathbf{R}^n \mid \langle \nu, v \rangle \leq 0, \quad \forall v \in T_C(x)\}.$$

The cones $T_C(x)$ and $N_C(x)$ are nonempty, closed and convex sets [2], [14].

3. Consider the next scalar program

$$(P) \quad \begin{cases} \text{Minimize} & F(x) \\ \text{subject to} & g(x) \leq 0, h(x) = 0, x \in C. \end{cases}$$

We denote with \bar{D} the domain of (P). Let $x^0 \in \bar{D}$ and we define the sets $I^0 = \{i | g_i(x^0) = 0\}$ and $J^0 = \{1, \dots, m\} \setminus I^0$. Consider the following constraint qualification for \bar{D} at x^0 :

$$R(x^0) \quad \begin{cases} \exists v \in X : g_{I^0}^0(x^0; v) \leq 0, h^0(x^0; v) = 0, \\ \exists \varepsilon > 0 : g_{J^0}(x^0 + \varepsilon v) \leq 0, h(x^0 + \varepsilon v) = 0. \end{cases}$$

For $C \subseteq A$, Mititelu established the following necessary optimality conditions of Kuhn-Tucker type for (P) at x^0 :

Theorem 1.1 (Theorem 2.1 [9]). *Let x^0 be a local solution of the program (P), where the functions f, g and h are subdifferentiable, and $h^0(x^0; \cdot)$ is finite. Well, we suppose that (P) satisfies at x^0 the constraint qualification $R(x^0)$. Then there are vectors $u^0 = (u_1^0, \dots, u_m^0)' \in \mathbf{R}^m$ and $v^0 = (v_1^0, \dots, v_m^0)' \in \mathbf{R}^q$ such that the next Kuhn-Tucker conditions for (P) at x^0 are satisfied:*

$$(KT) \quad \begin{cases} \partial F(x^0) + \sum_{i=1}^m u_i^0 \partial g_i(x^0) + \sum_{j=1}^q v_j^0 \partial h_j(x^0) + N_C(x^0) \supset \{0\} \\ u^0' g(x^0) = 0, \quad u^0 \geq 0 \end{cases}$$

Theorem 1.2 (Corollary 2.1 [9]). *Let x^0 be a local solution of (P), where the functions f, g and h are subdifferentiable, and $h^0(x^0; \cdot)$ is finite. We also suppose that (P) satisfies at x^0 the constraint qualification $R(x^0)$. Then there are vectors $u^0 = (u_1^0, \dots, u_m^0)' \in \mathbf{R}^m$ and $v^0 = (v_1^0, \dots, v_m^0)' \in \mathbf{R}^q$ such that the following Kuhn-Tucker conditions for (P) at x^0 are satisfied:*

$$(KT_1) \quad \begin{cases} \partial F(x^0) + \sum_{i=1}^m u_i^0 \partial g_i(x^0) + \sum_{j=1}^q v_j^0 \partial h_j(x^0) \supset \{0\} \\ u^0' g(x^0) = 0, \quad u^0 \geq 0 \end{cases}$$

Theorem 1.3 (Corollary 2.2 [9]). *Let x^0 be a local solution of (P), where the functions f, g and h are subdifferentiable. Also, we suppose that (P) satisfies at x^0 the constraint qualification $R(x^0)$. Then there are vectors $u^0 = (u_1^0, \dots, u_m^0)' \in \mathbf{R}^m$ and $v^0 = (v_1^0, \dots, v_m^0)' \in \mathbf{R}^q$ such that the following Kuhn-Tucker conditions at x^0 are satisfied:*

$$(KT_2) \quad \begin{cases} \partial F(x^0) + \sum_{i=1}^m u_i^0 \partial g_i(x^0) + \sum_{j=1}^q v_j^0 \partial h_j(x^0) + N_C(x^0) \supset \{0\} \\ u^0' g(x^0) = 0, \quad u^0 \geq 0, v^0 \geq 0. \end{cases}$$

Theorem 1.4 (Corollary 2.3 [9]). *Let x^0 be a local solution of (P), where the functions f, g and h are subdifferentiable. Moreover, we suppose that (P) satisfies at*

x^0 the constraint qualification $R(x^0)$. Then there are vectors $u^0 = (u_1^0, \dots, u_m^0)' \in \mathbf{R}^m$ and $v^0 = (v_1^0, \dots, v_m^0)' \in \mathbf{R}^q$ such that the following Kuhn-Tucker conditions for (P) at x^0 are satisfied:

$$(KT_2) \quad \begin{cases} \partial F(x^0) + \sum_{i=1}^m u_i^0 \partial g_i(x^0) + \sum_{j=1}^q v_j^0 \partial h_j(x^0) + N_C(x^0) \supset \{0\} \\ u^0 g(x^0) = 0, \quad u^0 \geq 0, v^0 \geq 0. \end{cases}$$

Remark. If $C = A$, then \bar{D} becomes D .

4. According to [10], for $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$ we shall use the following notations:

$$\begin{aligned} x = y &\Leftrightarrow x_i = y_i, \quad i = \overline{1, n}, \\ x > y &\Leftrightarrow x_i > y_i, \quad i = \overline{1, n}, \\ x \geq y &\Leftrightarrow x_i \geq y_i, \quad i = \overline{1, n}, \\ x \geq y &\Leftrightarrow x \geq y, \quad x \neq y. \end{aligned}$$

Definition 1.3 [Geoffrion[2]]. A point $x^0 \in D$ is said to be an *efficient solution* (Pareto minimum) for (MP) if there exists no other feasible point $x \in D$ such that $f(x) \leq f(x^0)$ and $f(x) \neq f(x^0)$ (or equivalently, $f(x) \leq f(x^0)$).

Lemma 1.1 (Kanniapan [3]). A point $x^0 \in D$ is an *efficient solution* to (MP) if and only if x^0 solves the scalar program

$$(P_k) \quad \begin{cases} \text{Minimize} & f_k(x) \\ \text{subject to} & f_s(x) \leq f_s(x^0), \quad \forall s \neq k \\ & g(x) \leq 0, h(x) = 0, x \in A \end{cases}$$

for each $k = \overline{1, p}$.

Definition 1.4 (Geoffrion [2]). A feasible point $x^0 \in D$ is said to be a *properly efficient solution* in (MP) if it is efficient solution in (MP) and there exists a scalar $M > 0$ such that, for each i , we have

$$\frac{f_i(x) - f_i(x^0)}{f_j(x^0) - f_j(x)} \leq M$$

for some j such that $f_j(x) < f_j(x^0)$, whenever $x \in D$ and $f_i(x) > f_i(x^0)$.

Geoffrion considered the following scalar parametric program:

$$(P_t) \quad \begin{cases} \text{Minimize} & t'f(x) \\ \text{subject to:} & g(x) \leq 0, h(x) = 0, x \in A, t > 0 \end{cases}$$

and he established the following:

Lemma 1.2 (Geoffrion [2]). Let $t > 0$ be fixed. If x^0 is an optimal solution of (P_t) , then x^0 is a properly efficient solution of (MP).

2 Kuhn-Tucker efficiency conditions for (MP)

In this section we establish necessary efficiency conditions of Kuhn-Tucker type for the multiobjective program (MP) under different forms.

Theorem 2.1 (Necessary efficiency conditions). *Let $x^0 \in D$ be a local efficient solution of (MP), where the functions f , g and h are subdifferentiable, but $h^0(x^0; \cdot)$ is finite on X . Also, we suppose that (MP) satisfies at x^0 the constraint qualification $R(x^0)$. Then there are vectors $t^0 = (t_1^0, \dots, t_p^0)' \in \mathbf{R}^n$, $u^0 = (u_1^0, \dots, u_m^0)' \in \mathbf{R}^m$ and $y^0 = (y_1^0, \dots, y_q^0) \in \mathbf{R}^q$ such that the following Kuhn-Tucker type conditions at x^0 are satisfied:*

$$(KTe) \quad \begin{cases} \sum_{k=1}^p t_k^0 \partial f_k(x^0) + \sum_{i=1}^m u_i^0 \partial g_i(x^0) + \sum_{j=1}^q v_j^0 \partial h_j(x^0) + N_A(x^0) \supset \{0\} \\ u^{0'} g(x^0) = 0, \quad u^0 \geq 0 \\ t^0 > 0 \text{ or } t^0 \geq 0, \quad t^{0'} e = 1, \quad e = (1, \dots, 1)' \in \mathbf{R}^p \end{cases}$$

Proof. If $\partial f(x^0) = \{0\}$, then (KTe) exist with $u^0 = 0$, $v^0 = 0$ and an arbitrary $t \in \mathbf{R}^p$, satisfying $t > 0$ or $t \geq 0$, $t'e = 1$. Now, we suppose that $\partial f(x^0) \supset \{0\}$. After Lemma 1.1, x^0 is optimal for each f_k , $k = \overline{1, p}$ on the domain

$$D_k = \{x \in A \mid f_s(x) - f_s(x^0) \leq 0, \quad s \neq k\}.$$

Then, according to Theorem 1.1, where $C = A$, there are vectors $u^0 \in \mathbf{R}^m$ and $v^0 \in \mathbf{R}^q$ such that the following Kuhn-Tucker optimality conditions are satisfied:

$$(KTo) \quad \begin{cases} \partial f_k(x^0) + \sum_{i=1}^m u_i^0 \partial g_i(x^0) + \sum_{j=1}^q v_j^0 \partial h_j(x^0) + N_A(x^0) \supset \{0\} \\ u^{0'} g(x^0) = 0, \quad u^0 \geq 0. \end{cases}$$

Consider the arbitrary numbers $t_k > 0$ or $t_k \geq 0$, $k = \overline{1, p}$, where $\sum_{k=1}^p t_k \neq 0$.

Multiplying the first relation of (KTo) by $t_k^0 = t_k / \sum_{k=1}^p t_k > 0$ or ≥ 0 and summing by k it results

$$\sum_{k=1}^p t_k^0 \partial f_k(x^0) + \sum_{i=1}^m u_i^0 \partial g_i(x^0) + \sum_{j=1}^q v_j^0 \partial h_j(x^0) \supset \{0\},$$

where $t^0 > 0$ or $t^0 \geq 0$, $t^{0'} e = 1$.

Proceeding similarly as in the proof of Theorem 2.1 we obtain the following corollaries:

COROLLARY 2.2 (Necessary efficiency conditions). *Let x^0 be a local efficient solution of (MP), where the functions f , g and h are subdifferentiable, but $h^0(x^0; \cdot)$ is finite. We also suppose that (MP) satisfies at x^0 the constraint qualification $R(x^0)$. Then there are vectors $t^0 = (t_1^0, \dots, t_p^0)' \in \mathbf{R}^n$, $u^0 = (u_1^0, \dots, u_m^0)' \in \mathbf{R}^m$ and $v^0 =$*

$(v_1^0, \dots, v_q^0) \in \mathbf{R}^q$ such that the following Kuhn-Tucker type conditions for (MP) at x^0 are satisfied:

$$(KTe1) \quad \begin{cases} \sum_{k=1}^p t_k^0 \partial f_k(x^0) + \sum_{i=1}^m u_i^0 \partial g_i(x^0) + \sum_{j=1}^q v_j^0 \partial h_j(x^0) \supset \{0\} \\ u^{0'} g(x^0) = 0, \quad u^0 \geq 0 \\ t^0 > 0 \text{ or } t^0 \geq 0, \quad t^{0'} e = 1. \end{cases}$$

Proof. See the proof of Theorem 2.1 and Theorem 1.2.

COROLLARY 2.3 (Necessary efficiency conditions). *Let x^0 be a local efficient solution of (MP), where the functions f , g and h are subdifferentiable. Also, we suppose that (MP) satisfies at x^0 the constraint qualification $R(x^0)$. Then there are vectors $t^0 = (t_1^0, \dots, t_p^0)' \in \mathbf{R}^p$, $u^0 = (u_1^0, \dots, u_m^0)' \in \mathbf{R}^m$ and $v^0 = (v_1^0, \dots, v_q^0)' \in \mathbf{R}^q$ such that the following Kuhn-Tucker type conditions at x^0 for (MP) are satisfied:*

$$(KTe2) \quad \begin{cases} \sum_{k=1}^p t_k^0 \partial f_k(x^0) + \sum_{i=1}^m u_i^0 \partial g_i(x^0) + \sum_{j=1}^q v_j^0 \partial h_j(x^0) + N_A(x^0) \supset \{0\} \\ u^{0'} g(x^0) = 0, \quad u^0 \geq 0, \quad v^0 \geq 0 \\ t^0 > 0 \text{ or } t^0 \geq 0, \quad t^{0'} e = 1. \end{cases}$$

Proof. See the proof of Theorem 2.1 and Theorem 1.3.

COROLLARY 2.4 (Efficiency necessary conditions). *Let x^0 be a local efficient solution of (MP), where the functions f , g and h are subdifferentiable. Moreover, we suppose that (MP) satisfies at x^0 the constraint qualification $R(x^0)$. Then there are vectors $t^0 = (t_1^0, \dots, t_p^0)' \in \mathbf{R}^p$, $u^0 = (u_1^0, \dots, u_m^0)' \in \mathbf{R}^m$ and $v^0 = (v_1^0, \dots, v_q^0)' \in \mathbf{R}^q$ such that the following Kuhn-Tucker type conditions at x^0 for (MP) are satisfied:*

$$(KTe3) \quad \begin{cases} \sum_{k=1}^p t_k \partial f_k(x^0) + \sum_{i=1}^m u_i \partial g_i(x^0) + \sum_{j=1}^q v_j \partial h_j(x^0) \supset \{0\} \\ u^{0'} g(x^0) = 0, \quad u^0 \geq 0, \quad v^0 \geq 0 \\ t^0 > 0 \text{ or } t^0 \geq 0, \quad t^{0'} e = 1. \end{cases}$$

Proof. See also the proof of Theorem 2.1 and Theorem 1.3.

3 Wolfe duality for the multiobjective program (MD)

In this section we shall give an extension to the Wolfe duality by the differentiable multiobjective case, using subdifferentiable nonsmooth functions. The vector Lagrangian associated to (MD) is the vector function:

$$L(x, u, v) = f(x) + [u'g(x) + v'h(x)]e.$$

The dual program in Wolfe's sense, associated to the nonsmooth vector program (MP), is the following multiobjective nonsmooth program:

$$(MD) \begin{cases} \text{Maximize} & L(y, u, v) = f(y) + [u'g(y) + v'h(y)]e \\ \text{subject to:} & \sum_{k=1}^p t_k \partial f_k(y) + \sum_{i=1}^m u_i \partial g_i(y) + \sum_{j=1}^q v_j \partial h_j(y) + N_A(y) \supset \{0\} \\ & y \in A, t \geq 0, t'e = 1, u \geq 0. \end{cases}$$

We denote by $\Omega = \{(t, y, u, v) | \dots\}$ the domain of the dual (MD) and consider the set

$$\Omega_t = \{(y, u, v) | (t, y, u, v) \in \Omega\}.$$

The set $\Omega_0 = \bigcup_{\substack{t \geq 0 \\ t'e=1}} \Omega_t$ is the domain of the Lagrangian objective $L(y, u, v)$. Therefore, the domain Ω of the constraints of (MD) is different by the domain Ω_0 of the objective of this program.

Definition 3.1. A point $(t^0, x^0, u^0, v^0) \in \Omega$ is said to be t^0 -efficient solution of (MD) if (x^0, u^0, v^0) is an efficient point of maximum type for $L(y, u, v)$.

THEOREM 3.1 (Weak duality). *We suppose that the domains D and Ω of the dual multiobjective programs (MP) and (MD) are nonempty and for every $(t, y, u, v) \in \Omega$, the inequality $L(y, u, v) \geq L(x, u, v)$, $\forall x \in A$, is false. Then for $\forall x \in D$ and $\forall (t, y, u, v) \in \Omega$ the inequality $f(x) \leq L(y, u, v)$ is false.*

Proof. For each $(t, y, u, v) \in \Omega$ and $\forall x \in A$ the inequality

$$(3.1) \quad L(y, u, v) \leq f(x) + [u'g(x) + v'h(x)]e$$

is false. Taking into account the relations:

$$\begin{aligned} u'g(x) &\leq 0, \quad \forall x \in D, \quad \forall u \geq 0, \\ v'h(x) &= 0, \quad \forall x \in D, \quad \forall v \in \mathbf{R}^q, \end{aligned}$$

from relation (3.1) we infer that for each $(t, y, u, v) \in \Omega$ and $\forall x \in D$ the inequality $f(x) \leq L(y, u, v)$ is false. But the point (t, y, u, v) being arbitrarily taken in Ω , it results that the relation $f(x) \leq L(y, u, v)$, for $\forall x \in D$ and $\forall (t, y, u, v) \in \Omega$, is false.

THEOREM 3.2 (Direct duality). *Let x_0 be a local efficient solution of the primal (MP), where the functions f , g and h are subdifferentiable, while $h^0(x^0; \cdot)$ is finite. Also, we suppose the next hypotheses:*

(d1) *The domain D satisfies the constraint qualifications $R(x^0)$.*

(d2) *For every $(t, y, u, v) \in \Omega$, the inequality $L(y, u, v) \geq L(x, u, v)$, $\forall x \in A$ is false.*

Then there are vectors $t^0 \in \mathbf{R}^p$, $u^0 \in \mathbf{R}^m$ and $v^0 \in \mathbf{R}^q$ such that (t^0, x^0, u^0, v^0) is a t^0 -efficient solution of the dual (MD) and $f(x^0) = L(x^0, u^0, v^0)$.

Proof. Since x^0 is a local efficient solution of (MP), which satisfies the constraint qualification $R(x^0)$, the program (MP) verifies the conditions (KTe) by Theorem 2.1. It result that $(t^0, x^0, u^0, v^0) \in \Omega$ and $u^{0'}g(x^0) = 0$, $v^{0'}h(x^0) = 0$. Then,

$$f(x^0) = f(x^0) + [u^{0'}g(x^0) + v^{0'}h(x^0)]e = L(x^0, u^0, v^0).$$

After (d2), according to Theorem 3.1, it results that $f(x^0) \leq L(y, u, v)$ is false. Therefore $L(x^0, u^0, v^0) \leq L(y, u, v)$, $\forall (t, y, u, v) \in \Omega$ is false. Then for (t^0, x^0, u^0, v^0) , the component (x^0, u^0, v^0) is an efficient point of maximum type for L .

THEOREM 3.3 (Converse duality). *Let (t^0, x^0, u^0, v^0) be a t^0 -efficient solution of (MD), where $t^0 > 0$. We suppose that:*

(c1) *The primal program (MP) admits the efficient solution \bar{x} , where D verifies the constraint qualification $R(\bar{x})$ and $h^0(\bar{x}; \cdot)$ is finite.*

(c2) *The function $L(x, u^0, v^0)$ admits at $x = x^0$ an unic efficient minimum on A . Then $x^0 = \bar{x}$, where x^0 is a properly efficient solution of (MP) and $f(x^0) = L(x^0, u^0, v^0)$.*

Proof. We suppose that $x^0 \neq \bar{x}$ and we shall reach a contradiction. Because \bar{x} is an efficient solution of (MP) which satisfies the constraint qualification $R(\bar{x})$, there are vectors $\bar{t} \in \mathbf{R}^p$, $\bar{u} \in \mathbf{R}^m$ and $\bar{v} \in \mathbf{R}^q$ such that (MP) satisfies at \bar{x} efficiency Kuhn-Tucker conditions similarly to (KTe) by Theorem 2.1, with \bar{x} instead of x^0 ; consequently, $(\bar{t}, \bar{x}, \bar{u}, \bar{v}) \in \Omega$ and

$$(3.2) \quad \bar{u}'g(\bar{x}) = 0.$$

Moreover,

$$(3.3) \quad \bar{v}'h(\bar{x}) = 0.$$

Obviously, the relation $L(x^0, u^0, v^0) \leq L(\bar{x}, \bar{u}, \bar{v})$ is false, because (t^0, x^0, u^0, v^0) is an t^0 -efficient solution of (MD). Then, for $t^0 > 0$ it results that the relation $t^{0'}L(x^0, u^0, v^0) \leq t^{0'}L(\bar{x}, \bar{u}, \bar{v})$ is false. Therefore,

$$(3.4) \quad t^{0'}L(x^0, u^0, v^0) > t^{0'}L(\bar{x}, \bar{u}, \bar{v}).$$

For every $x \in D$ the relation $L(x, u^0, v^0) \leq L(x^0, u^0, v^0)$ is false. Then it results that the relation $t^{0'}L(x, u^0, v^0) \leq t^{0'}L(x^0, u^0, v^0)$ is false. Consequently we have

$$(3.5) \quad t^{0'}L(x, u^0, v^0) > t^{0'}L(x^0, u^0, v^0), \quad \forall x \in D$$

and for $x = \bar{x}$ it results

$$(3.6) \quad t^{0'}L(\bar{x}, u^0, v^0) > t^{0'}L(x^0, u^0, v^0).$$

From relations (3.4) and (3.6) it results

$$t^{0'}L(\bar{x}, u^0, v^0) > t^{0'}L(\bar{x}, \bar{u}, \bar{v}),$$

or equivalently, $u^{0'}g(x^0) > 0$, which is false. Therefore, $x^0 = \bar{x}$.

We have

$$(3.7) \quad t^{0'}f(x) \geq t^{0'}f(x) + u^{0'}g(x) + v^{0'}h(x) = t^{0'}L(x, u^0, v^0), \quad \forall x \in D$$

and from (3.5) and (3.7) it results $t^{0'}f(x) > t^{0'}L(x^0, u^0, v^0) = t^{0'}f(x^0)$. Therefore, $t^{0'}f(x) > t^{0'}f(x^0)$, $\forall x \in D$. Taking into account Lemma 1.2 it results that x^0 is a properly efficient solution of (MP).

Remark. The Wolfe duality, defined by Theorems 3.1-3.1, can be developed using any of Theorems 1. 2-1. 4 instead of Theorem 1. 1.

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Ștefan Mititelu
Technical University of Civil Engineering
Department of Mathematics and Informatics
Bd. Lacul Tei 124, sector 2
RO-020396 Bucharest, Romania
e-mail: st_mititelu@yahoo.com