Algebraic functional equations solved by a geometrical general method

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Abstract

In Section 1 we recall and state improved versions of some results of [16] concerning the equation

$$g(x) = g(f(x)) \quad \forall x \in A,$$

where f is the "unknown" function, while g is a given function satisfying some conditions. We also consider the operatorial case, when the variable x may "become" a self-adjoint operator acting on a Hilbert space. The existence of the function f is proved by constructing it effectively and without using the implicit function theorem (for details see [16] and [8]). This construction has a geometrical nature.

In Section 2 we apply the general type results of Section 1 to some concrete algebraic functional equations. The function f constructed by Theorem 2.1 has an interesting property vis-à-vis to prime integers. The operatorial case is also considered.

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1 General type results

In this section we recall and improve some results of [16].

1.1. Theorem. *Let* $u, v \in \bar{\mathbf{R}}$*,* $u < v$*,* $a \in]u, v[$ *and let* $g :]u, v[\to \mathbf{R}$ *be a continuous function. Assume that*

(a) $\lim_{x \searrow u} g(x) = \lim_{x \nearrow v} g(x) = w \in \bar{\mathbf{R}}$*,*

(b) g *is strictly decreasing on* $]u, a]$ *and strictly increasing on* $[a, v[$*.*

Then there exists $f :]u, v[\to]u, v[$ *such that*

(1) $$g(x) = g(f(x)) \quad \forall x \in]u, v[$$

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and f has the following properties:

(i) f is strictly decreasing on $]u,v[$ and we have

$$\lim_{x\searrow u} f(x) = v, \quad \lim_{x\nearrow v} f(x) = u;$$

(ii) a is the unique fixed point of f;

(iii) we have $f^{-1} = f$ on $]u,v[$;

(iv) f is continuous on $]u,v[$;

(v) if we assume in addition that $g \in C^n(]u,v[\setminus\{a\})$, $n \in \mathbf{N} \cup \{\infty\}$, $n \geq 1$, then $f \in C^n(]u,v[\setminus\{a\})$;

(vi) if g is derivable on $]u,v[\setminus\{a\}$, so is f;

(vii) if $g \in C^2(]u,v[)$, $g''(a) \neq 0$ and there exists

$$\rho_1 := \lim_{x\to a} f'(x) \in \bar{\mathbf{R}},$$

then $f \in C^1(]u,v[) \cap C^2(]u,v[\setminus\{a\})$ and $f'(a) = -1$;

(viii) if $g \in C^3(]u,v[)$, $g''(a) \neq 0$ and there exist

$$\rho_1 := \lim_{x\to a} f'(x) \in \bar{\mathbf{R}} \quad \textit{and} \quad \rho_2 := \lim_{x\to a} f''(x) \in \mathbf{R},$$

then $f \in C^2(]u,v[) \cap C^3(]u,v[\setminus\{a\})$ and

$$f''(a) = \rho_2 = -\frac{2}{3} \cdot \frac{g'''(a)}{g''(a)};$$

(ix) put $g_l = g|_{]u,a]}$, $g_r = g|_{[a,v[}$; then we have

$$f(x_0) = (g_r^{-1} \circ g_l)(x_0) = \sup\{x \in [a,v[\,; g_r(x) \leq g_l(x_0)\} \quad \forall x_0 \in]u,a]$$

and

$$f(x_0) = (g_l^{-1} \circ g_r)(x_0) = \inf\{x \in]u,a]\,; g_l(x) \leq g_r(x_0)\} \quad \forall x_0 \in [a,v[\,.$$

The proof of this theorem is similar to that of Theorem 1.1 [16]. The definition of f and some of its properties are clear if we use Figure 1 [16], p.62. A proof of the improved version stated in the present work will appear in [14].

Next we consider an operatorial version of Theorem 1.1. We shall use some of the notations of Theorem 1.10 [16]. So, X will be an order-complete vector lattice. Denote by $Izom_+(X)$ the set of all vector space izomorphisms $T : X \to X$ which apply X_+ onto itself.

1.2. Theorem. *Let X be an order-complete vector lattice with its positive convex cone X_+, $a \in X$, A_l a convex subset such that*

$$a \in A_l \subset \{x \in X; x \leq a\}.$$

A_r a convex subset such that

$$a \in A_r \subset \{x \in X; x \geq a\}.$$

Let $g_l : A_l \to X$ be a convex operator such that

$$\partial g_l(x) \cap (-Izom_+(X)) \neq \Phi, \quad \forall x \in A_l \setminus \{a\}.$$

Let $g_r : A_r \to X$ be a convex operator such that

$$\partial g_r(x) \cap (Izom_+(X)) \neq \Phi, \quad \forall x \in A_r \setminus \{a\}.$$

We also assume that

$$g_l(a) = g_r(a) \quad and \quad R(g_l) = R(g_r),$$

where $R(g)$ is the range of g.

Let $g : A := A_l \cup A_r \to X$ be defined by

$$g(x) := \begin{cases} g_l(x), & x \in A_l, \\ g_r(x), & x \in A_r. \end{cases}$$

Then there exists $F : A \to A$ such that

$$g(x) = g(F(x)), \quad \forall x \in A,$$

F is strictly decreasing on A and has the following properties:

(i) a is the only fixed point of F;

(ii) there exists F^{-1} and $F^{-1} = F$ on A;

(iii)

$$F(x_0) = g_r^{-1}(g_l(x_0)) = \sup\{x \in A_r; g_r(x) \leq g_l(x_0)\} \quad \forall x_0 \in A_l,$$

$$F(x_0) = g_l^{-1}(g_r(x_0)) = \inf\{x \in A_l; g_l(x) \leq g_r(x_0)\} \quad \forall x_0 \in A_r.$$

The proof of this theorem is similar to that of Theorem 1.10 [16], p.72-74. Note that this proof is based on some results from [8] concerning the construction of the inverse of a convex "monotone" operator.

2 Applications

2.1. Theorem. *Let $\alpha, \lambda, \beta, \mu$ be positive real numbers such that $\alpha \neq \beta$, $\lambda \neq \mu$ and let $k \in \mathbf{N}$, $k \geq 1$. Then there exists a function $f : \mathbf{R} \to \mathbf{R}$ such that*

$$(1') \qquad \begin{aligned} -\alpha x^{2k+1} - \lambda x &= \beta(f(x))^{2k+1} + \mu f(x) \quad \forall x < 0, \\ \beta x^{2k+1} + \mu x &= -\alpha(f(x))^{2k+1} - \lambda f(x) \quad \forall x > 0, \end{aligned}$$

and f has the following properties:

(i) f is strictly decreasing on $\mathbf{R}$, $\lim_{x \searrow -\infty} f(x) = +\infty$, $\lim_{x \nearrow +\infty} f(x) = -\infty$;

(ii) 0 is the only fixed point of f;

(iii) $f^{-1} = f$ on $\mathbf{R}$;

(iv) f is continuous on $\mathbf{R}$;

(v) $f \in C^\infty(\mathbf{R} \setminus \{0\})$;

(vi) if there exists $\rho_1 = \lim_{x \to 0} f'(x) \in \bar{\mathbf{R}}$, then $f \in C^1(\mathbf{R})$ and $f'(0) = -1$;

(vii) the following constructive formulae for $f(x_0)$ hold:

$$f(x_0) = \sup\{x \geq 0; \beta x^{2k+1} + \mu x \leq -\alpha x_0^{2k+1} - \lambda x_0\} \quad \forall x_0 < 0;$$

$$f(x_0) = \inf\{x \leq 0; -\alpha x^{2k+1} - \lambda x \leq \beta x_0^{2k+1} + \mu x_0\} \quad \forall x_0 > 0;$$

(viii) the straight line

$$y = -\left(\frac{\alpha}{\beta}\right)^{\frac{1}{2k+1}} x$$

is an asymptote at $-\infty$ *for the graph of* f*, while*

$$y = -\left(\frac{\beta}{\alpha}\right)^{\frac{1}{2k+1}} x$$

is an asymptote at $+\infty$*;*

(ix) if there exists $\alpha_1, \beta_1 \in \mathbf{N} \setminus \{0\}$ *such that*

$$\alpha = \alpha_1^{2k+1},\ \beta = \beta_1^{2k+1}, \quad (\alpha_1, \beta_1) = 1,$$

then for any prime negative integer x *with* $|x|$ *sufficiently large,* $f(x)$ *cannot be a prime integer.*

Proof. We apply Theorem 1.1 to $u := -\infty$, $v := +\infty$ and $g : \mathbf{R} \to \mathbf{R}$ defined by

$$g(x) := -\alpha x^{2k+1} - \lambda x, \quad x \leq 0,$$

$$g(x) := \beta x^{2k+1} + \mu x, \qquad x \geq 0.$$

It is easy to see that g satisfies conditions (a), (b) of Theorem 1.1, where $w := +\infty$, $a := 0$. From Theorem 1.1, we infer that there exists $f : \mathbf{R} \to \mathbf{R}$ such that $(1')$ hold and f has the properties (i)-(vii) mentioned in the statement of Theorem 2.1.

Now we prove (viii). We have to prove that $y = -\left(\frac{\alpha}{\beta}\right)^{\frac{1}{2k+1}} x$ is an asymptote at $-\infty$ for the graph of f (the corresponding assertion of (viii) at $+\infty$ can be proved in a similar way). The first equality $(1')$ yields

$$(1'') \qquad -\alpha - \frac{\lambda}{x^{2k}} = \beta \left(\frac{f(x)}{x}\right)^{2k+1} + \mu \frac{f(x)}{x} \cdot \frac{1}{x^{2k}},\ x < 0$$

Let $(x_n)_n$ be an arbitrary sequence such that

$$x_n \to -\infty.$$

Then the corresponding sequence $\left(\frac{f(x_n)}{x_n}\right)_n$ can be decomposed into its subsequences which converge in $\bar{\mathbf{R}}$. Let $\left(\frac{f(x_{l_n})}{x_{l_n}}\right)_n$ be such a subsequence, and put

$$s := \lim_n \frac{f(x_{l_n})}{x_{l_n}} \in \bar{\mathbf{R}}$$

Equality $(1'')$ written for $x = x_{l_n}$ becomes

$$(1''') \qquad -\alpha - \frac{\lambda}{x_{l_n}^{2k}} = \beta \left(\frac{f(x_{l_n})}{x_{l_n}} \right)^{2k+1} + \mu \frac{f(x_{l_n})}{x_{l_n}} \cdot \frac{1}{x_{l_n}^{2k}}, \quad n \in \mathbf{N}.$$

Passing to the limit when $n \to \infty$, we obtain

$$-\alpha = \beta s^{2k+1} + \mu s \cdot 0 = s(\beta s^{2k} + 0).$$

It follows easily that we have

$$s = -\left(\frac{\alpha}{\beta} \right)^{\frac{1}{2k+1}}.$$

Thus for any subsequence $\left(\frac{f(x_{l_n})}{x_{l_n}} \right)_n$ which converges in $\bar{\mathbf{R}}$, its limit is the same, namely $-\left(\frac{\alpha}{\beta} \right)^{\frac{1}{2k+1}}$. It follows that

$$x_n \to -\infty \Rightarrow \lim_n \frac{f(x_n)}{x_n} = -\left(\frac{\alpha}{\beta} \right)^{\frac{1}{2k+1}}.$$

This proves that

$$\lim_{x \to -\infty} \frac{f(x)}{x} = -\left(\frac{\alpha}{\beta} \right)^{\frac{1}{2k+1}}.$$

Now we compute

$$\lim_{x \to -\infty} \left[f(x) + \left(\frac{\alpha}{\beta} \right)^{\frac{1}{2k+1}} x \right] = \lim_{x \to -\infty} \frac{\frac{f(x)}{x} + \left(\frac{\alpha}{\beta} \right)^{\frac{1}{2k+1}}}{x^{-1}} = \frac{0}{0} =$$

$$= \lim_{x \to -\infty} \frac{[f'(x) \cdot x - f(x)]x^{-2}}{-x^{-2}} = \lim_{x \to -\infty} [f(x) - x f'(x)] =$$

$$\overset{(1')}{=} \lim_{x \to -\infty} \left[f(x) - x \cdot \frac{-(2k+1)\alpha x^{2k} - \lambda}{(2k+1)\beta(f(x))^{2k} + \mu} \right] =$$

$$= \lim_{x \to -\infty} \frac{(2k+1)[\beta(f(x))^{2k+1} + \alpha x^{2k+1}] + \mu f(x) + \lambda x}{(2k+1)\beta(f(x))^{2k} + \mu} =$$

$$\overset{(1')}{=} \lim_{x \to -\infty} \frac{(2k+1)[-\mu f(x) - \lambda x] + \mu f(x) + \lambda x}{(2k+1)\beta(f(x))^{2k} + \mu} =$$

$$= -2k \lim_{x \to -\infty} \frac{\mu f(x) + \lambda x}{(2k+1)\beta(f(x))^{2k} + \mu} =$$

$$= -2k \lim_{x \to -\infty} \frac{\mu + \lambda \frac{x}{f(x)}}{(2k+1)\beta(f(x))^{2k-1} + \frac{\mu}{f(x)}} =$$

$$= -2k \cdot \frac{\mu + \lambda \left[-\left(\frac{\beta}{\alpha} \right)^{\frac{1}{2k+1}} \right]}{(+\infty)} = 0.$$

This proves that $y = -\left(\frac{\alpha}{\beta}\right)^{\frac{1}{2k+1}} x$ is an asymptote at $-\infty$ for the graph of f. Similarly, $y = -\left(\frac{\beta}{\alpha}\right)^{\frac{1}{2k+1}} x$ is an asymptote at $+\infty$.

Now we prove (ix). Assume that $\alpha, \beta, \alpha_1, \beta_1$ are as in hypothesis of (ix). We have already proved that

$$0 = \lim_{x \to -\infty} \left[f(x) + \left(\frac{\alpha}{\beta}\right)^{\frac{1}{2k+1}} x \right].$$

Assuming that $x < 0$ with $|x|$ arbitrary large is a prime integer such that $f(x)$ is also a prime integer, one obtains:

$$0 = \lim_{x \to -\infty} \left[f(x) + \left(\frac{\alpha}{\beta}\right)^{\frac{1}{2k+1}} x \right] = \lim_{x \to -\infty} \left[f(x) + \frac{\alpha_1}{\beta_1} x \right] = \frac{1}{\beta_1} \lim_{x \to -\infty} [\beta_1 f(x) + \alpha_1 x],$$

which is equivalent to

$$0 = \lim_{x \to -\infty} [\beta_1 f(x) + \alpha_1 x].$$

If $\alpha_1, \beta_1, x, f(x)$ are integers, the last equality leads to

$$\beta_1 f(x) + \alpha_1 x = 0$$

for $|x|$ sufficiently large. This may be rewritten as

$$\beta_1 f(x) = -\alpha_1 x \tag{2}$$

Since we have assumed that $(\alpha_1, \beta_1) = 1$, we deduce that

$$\alpha_1 | f(x) \quad \text{and} \quad \beta_1 | x.$$

But x and $f(x)$, are primes such that $x < 0$, $f(x) > 0$, so that we must have

$$\alpha_1 \in \{1, f(x)\} \quad \text{and} \quad \beta_1 \in \{1, -x\}.$$

For $\alpha_1 = 1$, $\beta_1 = 1$, (2) becomes

$$f(x) = -x.$$

Using $(1')$, this yields

$$-x^{2k+1} - \lambda x = -x^{2k+1} - \mu x, \quad \text{i.e.} \quad \lambda = \mu,$$

which is impossible because of hypothesis $\lambda \neq \mu$.

The other cases, when $\alpha_1 = f(x)$ or $\beta_1 = -x$ cannot occur for $|x|$ sufficiently large, since α_1, β_1 are given constants and $\lim_{x \to -\infty} f(x) = +\infty$.

The proof is complete. □

2.2. Remark. An assertion which is similar to (ix) Theorem 2.1., but for positive prime integers sufficiently large holds.

Next we apply Theorem 1.2 to an operatorial version of Theorem 2.1, when x "becomes" a self-adjoint operator U acting on a Hilbert space.

Let H be a Hilbert space. Denote by $\mathcal{A}(H)$ the real vector space of all self-adjoint operators acting on H. Let T be a fixed element of $\mathcal{A}(H)$. Put

$$\mathcal{A}_1 = \mathcal{A}_1(T) := \{U \in \mathcal{A}(H); UT = TU\},$$
$$X := \{U \in \mathcal{A}_1; UV = VU \quad \forall V \in \mathcal{A}_1\}$$
$$\text{(see [5], p.303} - 305)$$
$$X_+ := \{U \in X; \langle U(h), h\rangle \geq 0, \quad \forall h \in H\}.$$

It is known that X is an order-complete vector lattice and a commutative algebra. With these notations and definitions, we prove the following result.

2.3. Theorem. *Let $k \in \mathbf{N}$, $k \geq 1$ and let $\alpha, \lambda, \beta, \mu$ be positive real numbers such that $\alpha \neq \beta$, $\lambda \neq \mu$. Let*

$$A_l := \{U \in X;\ \sigma(U) \subset]-\infty, 0[\} \cup \{0\},$$
$$A_r := \{U \in X;\ \sigma(U) \subset]0, \infty[\} \cup \{0\},$$

where $\sigma(U)$ is the spectrum of U.

Put

$$A := A_l \cup A_r.$$

Then there exists a strictly decreasing map

$$F : A \to A$$

such that

$$(1^{IV}) \qquad \begin{aligned} -\alpha U^{2k+1} - \lambda U &= \beta(F(U))^{2k+1} + \mu F(U) \quad \forall U \in A_l, \\ \beta U^{2k+1} + \mu U &= -\alpha(F(U))^{2k+1} - \lambda F(U) \quad \forall U \in A_r \end{aligned}$$

and F has the following properties

(i) 0 is the only fixed point of F;

(ii) F is invertible and $F^{-1} = F$ on A;

(iii) F can be constructed by formulae

$$F(U_0) = \sup\{U \in A_r;\ \beta U^{2k+1} + \mu U \leq -\alpha U_0^{2k+1} - \lambda U_0\} \quad \forall U_0 \in A_l,$$
$$F(U_0) = \inf\{U \in A_l;\ -\alpha U^{2k+1} - \lambda U \leq \beta U_0^{2k+1} + \mu U_0\} \quad \forall U_0 \in A_r.$$

Proof. One applies Theorem 1.2 to X, A_l, A_r, A defined above and to

$$g_l(U) := -\alpha U^{2k+1} - \lambda U, \quad U \in A_l,$$
$$g_r(U) := \beta U^{2k+1} + \mu U, \quad U \in A_r.$$

Using the fact that the operators

$$P(U) = U^n, \quad n \in \mathbf{N} \setminus \{0\},\ U \in X_+$$

are convex on X_+ (see [16]), it is easy to see that g_l and g_r are convex on A_l, respectively A_r. On the other hand, we have

$$g_l'(U)(V) = [-(2k+1)\alpha U^{2k} - \lambda I] \cdot V \leq 0$$

and

$$[g_l'(U)]^{-1}(V) = [-(2k+1)\alpha U^{2k} - \lambda I]^{-1} \cdot V \leq 0, \ \forall U \in A_l \setminus \{0\}, \ \forall V \in X_+$$

(the product of two permutable self-adjoint operators, one of which being negative and the other one being positive, is a negative operator). It follows that

$$g_l'(U) \in -Izom_+(X) \quad \forall U \in A_l \setminus \{0\}.$$

Similarly,

$$g_r'(U) \in Izom_+(X) \quad \forall U \in A_r \setminus \{0\}.$$

The last condition of Theorem 1.2 that we have to verify is

$$R(g_l) = R(g_r).$$

Let $g_l(U_1) \in R(g_l)$ be such that $U_1 \in A_l \setminus \{0\}$. Let $f : \mathbf{R} \to \mathbf{R}$ be the function constructed in Theorem 2.1. Let

$$U_2 := F(U_1),$$

where $F(U_1)$ is as in Lemma 3.3.1 [4], p.227. Then

$$\sigma(U_2) = \sigma(F(U_1)) = f(\sigma(U_1)) \subset]0, +\infty[$$

$$(U_1 \in A_l \setminus \{0\} \Rightarrow \sigma(U_1) \subset]-\infty, 0[\Rightarrow f(\sigma(U_1)) \subset]0, +\infty[$$

since f applies $]-\infty, 0[$ onto $]0, +\infty[$). This leads to the fact that $U_2 \in A_r \setminus \{0\}$. On the other hand, the construction of f implies

$$g(t_1) = g(f(t_1)) \quad \forall t_1 < 0.$$

If we integrate this equality on the spectrum $\sigma(U_1) \subset]-\infty, 0[$, with respect to the spectral measure attached to U_1, one obtains:

$$g_l(U_1) = g_r(F(U_1)) = g_r(U_2),$$

where we have seen that $U_2 \in A_r \setminus \{0\}$. It follows that for any $U_1 \in A_r \setminus \{0\}$, we have

$$g_l(U_1) \in R(g_r).$$

Hence $R(g_l) \subset R(g_r)$. The converse relation can be proved similarly, so that one obtains

$$R(g_l) = R(g_r).$$

Now the conclusion follows from Theorem 1.2. The proof is complete. □

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