Using of stochastic Ito and Stratonovich integrals derived security pricing

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Abstract

We seek for good numerical approximations of solutions for stochastic differential equation (SDE) of stock price of underlying asset of an European call option. Using Ito and Stratonovich integrals and SDE approximate solution we predict the stock price on market.

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1 Introduction

In the most important problem on the financial market is to estimate the price of underlying asset of an European call option (see [5], [6]).

In the central form of SDE for evolution of a firm stock price is:

(1.1.1) $$dS = S\mu dt + S\sigma dW$$

where μ, σ and $W(t)$ express respectively the drift, volatility and the Wiener process.

Generaly (see [1]), a differential equation

(1.1.2) $$\frac{dX(t)}{dt} = b(X(t), t) + B(X(t), t)\xi(t),\ X(0) = X_0$$

with $X(\cdot) : [0, \infty) \to \Re^n$ random function, $b : \Re^n \times [0, T] \to \Re^n$, $B : \Re^n \times [0, T] \to \mathcal{M}_{n\times m}(\Re)$, $\xi : \Re \to \Re^m$, m - dimensional white noise, defines a stochastic differential equation (SDE):

(1.1.3) $$dX(t) = b(X(t), t)dt + B(X(t), t)dW(t),\ X(0) = X_0$$

if the white noise is solution of a m - dimensional Wiener process.

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<u>Remark 1</u>: For m=1, the random variable ξ is 1-dimensional *white noise* if $E(\xi(t)) = 0, \forall t \in \Re$ and $E(\xi(t) \cdot \xi(s)) = \begin{cases} 0, & t \neq s \\ \sigma^2, & t = s \end{cases}$. The m-dimensional Wiener process (see [6]) is solution of differential equation: $\dot{X}(t) = \xi(t),\ X(0) = 0$. In one dimensional case Wiener process has the following characteristics:

- $W(0) = 0$ a.s.
- $W(t) - W(s) \sim N(0, t-s),\ \forall t \geq s \geq 0$
- $\forall\ 0 < t_1 < t_2 < \ldots < t_n$ random variables $W(t_1), W(t_2) - W(t_1), \ldots, W(t_n) - W(t_{n-1})$ are independent
- $W(t)$ has continuous path.

2 Stochastic integrals

Integral form of solution of SDE (1.3) is:

(2.2.1) $$X(t) = X_0 + \int_0^t b(X(s), s)ds + \int_0^t B(X(s), s)dW(s),\ \forall t \geq 0.$$

The central problem is to calculate the third term of the right side of equation (2.1). We shall give then the general form of a stochastic integral.

2.1 Discretization and properties of Ito integral

<u>Definition 1</u>: Let be $[0, T] \subset \Re$ and $\Delta = \{t_0 = 0 < t_1 < \ldots < t_m = T\}$ partition of $[0, T]$, with norm $||\Delta|| = \max_{k \in \{0,1,\ldots,m-1\}} (t_{k+1} - t_k)$ and intermediary points $\tau_k = (1-\lambda)t_k - \lambda t_{k+1},\ \forall k \in \{0, 1, \ldots, m-1\},\ \lambda \in [0, 1]$ then

(2.2.2) $$\int_0^t B(X(s), s)dW(s) = \lim_{||\Delta|| \to 0} \sum_{k=0}^{m-1} B(X(\tau_k), \tau_k)(W(t_{k+1}) - W(t_k)).$$

Particular case:
i) Ito integral definition: taking in (2.2) $\lambda = 0 \Rightarrow \tau_k = t_k$,
ii)Stratonovich integral definition: taking in (2.2) $\lambda = 1/2 \Rightarrow \tau_k = (t_k + t_{k+1})/2$.

Let be $S(t)$ express the stock price at moment t satisfying (1.1) and $f(t_{k)}$ number of shares in a portofolio, buyed at time t_k, immediately after the announcing the price of underlying asset and before change the portofolio at time t_{k+1}, $f(t) = f(t_k)$, $\forall t \in [o, T]$ represent the total gain at moment t, from trading shares.

(2.2.3) $$I(t) = \sum_{k=0}^{n-1} f(t_k)[S(t_k) - S(t_k)].$$

<u>Definition 2</u>: Let be $\Delta = \{t_0 = 0 < t_1 < \ldots < t_m = T\}$ partition of $[0, T]$, with norm $||\Delta|| = \max_{k \in \{0,1,\ldots,m-1\}} (t_{k+1} - t_k)$

i) $FV_{[0,T]}(f) = \lim_{||\Delta|| \to 0} \sum_{k=0}^{m-1} \mid f(t_{k+1}) - f(t_k) \mid$ is *first variation* of function f,

ii) $< f > (T) = \lim_{||\Delta|| \to 0} \sum_{k=0}^{m-1} \mid (f(t_{k+1}) - f(t_k))^2 \mid$ is *quadratic variation* of f.

The most important Ito integral properties (see [6], and [1]), used to solve stochastic differential equations, could be enumerated as follows:

i) linearity: $\forall a,\ b \in \Re$ if $I(t) = \int_0^t f(s)\, dW(s),\ J(t) = \int_0^t g(s)\, dW(s)$ then: $aI(t) + bJ(t) = \int_0^t (af(s) + bg(s))\, dW(s)$,

ii) martingale: •$\{I(t)\}_{t \in [0,T]}$ is a martingale with respect to the filtration generated by standard Brownian motion,
•$E(I(t)) = 0,\ \forall t \in [0,T]$,

iii) continuity: $I(t)$ is continuous,

iv) Ito isometry: if $I(t) = \int_0^t f(s)\, dW(s)$ then $E(I(t)^2) = \int_0^t f^2(s) ds$

v) $< I > (T) = T$ or $dI(t)dI(t) = f^2(t)dt$.

2.2 Existence and uniqueness of solution for SDE (Ito approach)

For the following stochastic differential equation:

$$dX(t) = b(X(t), t)dt + B(X(t), t)dW(t),\ X(0) = X_0 \tag{2.2.4}$$

where $X : [0, T] \to \Re^n$, $b : \Re^n \times [0, T] \to \Re^n$, $B : \Re^n \times [0, T] \to \mathcal{M}_{n\times m}(\Re)$, continuous with the property:
$\exists C > 0$ such that: $||b(x, t) - b(x', t)|| \leq C||x - x'||, \forall x, x' \in \Re^n$,
$||B(x, t) - B(x', t)|| \leq C||x - x'||, \forall x, x' \in \Re^n$,
$||b(x, t)|| \leq C(1 + ||x||), \forall x \in \Re^n, \forall t \in [0, T]$,
$||B(x, t)|| \leq C(1 + ||x||), \forall x \in \Re^n, \forall t \in [0, T]$,
X_0 a $\Re^n$ valued random variable such that: $E(||X_0^2||) < \infty$ and independent of $\mathcal{W}^+(0) = \sigma(W(s) - W(t)) \mid s \geq t$ future of Brownian motion beyond time $t = 0$,

there *exist a unique solution* $X \in L^2(0, T)$ of (2.4).

Remark2: The uniqueness is in the sense that if we find two such solutions for SDE (2.4), $X(t)$ and $X(t')$, then it is necessary that $P(X(t) = X'(t), \forall t \in [0, T]) = 0$ a.s. The solution will be construct using a sequence of random variable: $X^0 := X_0$;
$X^{n+1}(t) = X^0(t) + \int_0^t b(X^n(s), s)ds + \int_0^t B(X^n(s), s)dW(s)$.

2.3 The relationship between Ito and Stratonovich integrals

As defined in (2.2) we can write in general case, for any $\lambda \in [0,1]$ and partition Δ of $[0,T]$ the stochastic integral:

$$\int_0^T W(s)dW(s) = \lim_{||\Delta|| \to 0} \sum_{k=0}^{m-1} W(\tau_k)(W(t_{k+1}) - W(t_k)) \tag{2.2.5}$$

with $\tau_k = (1-\lambda)t_k - \lambda t_{k+1}$, $\forall k \in \{0, 1, \ldots, m-1\}$.

If we denote: $R_m = \sum_{k=0}^{m-1} W(\tau_k)(W(t_{k+1}) - W(t_k))$ then

$$\lim_{m \to \infty} R_m = \frac{W(t)}{2} + (\lambda - \frac{1}{2})T \tag{2.2.6}$$

From (2.5) and (2.6) one could find the relations:

i) for Ito stochastic integral: $\int_0^T W(s)dW(s) = \frac{1}{2}W(t) - \frac{1}{2}T$,

ii) for Stratonovich stochastic integral (denoted with " $\circ$"): $\int_0^T W(s) \circ dW(s) = \frac{1}{2}W(t)$,

that express the relationship between the two integrals.

More generaly, one dimensional Ito-Stratonovich conversion formula is given by:

$$\int_0^T b(W,t) \circ dW = \int_0^T b(W,t)dW + \frac{1}{2}\int_0^T \frac{\partial b}{\partial x}(W,t)dt. \tag{2.2.7}$$

3 Application for stock prices

We shall use a sample set of data that represents the evolution of a firm stock prices precised in Table 1., in order to approximate the solution of (1.1).

Table 1. Evolution of a firm stock prices and rentabilities estimated:

Day of the week	Data	Stock S_i	R_i	Day of the week	Data	Stock S_i	R_i
We	01.03.95	2.11		Fr	24.03.95	2.73	0.0706
Th	02.03.95	1.9	-0.0995	Mo	27.03.95	2.91	0.0659
Fr	03.03.95	2.18	0.1474	Tu	28.03.95	2.92	0.0034
Mo	06.03.95	2.16	-0.0092	We	29.03.95	2.92	0.0000
Tu	07.03.95	1.91	-0.1157	Th	30.03.95	3.12	0.0685
We	08.03.95	1.86	-0.0262	Fr	31.03.95	3.14	0.0064
Th	09.03.95	1.97	0.0591	Mo	03.04.95	3.13	-0.0032
Fr	10.03.95	2.27	0.1523	Tu	04.04.95	3.24	0.0351
Mo	13.03.95	2.49	0.0969	We	05.04.95	3.25	0.0031
Tu	14.03.95	2.76	0.1084	Th	06.04.95	3.28	0.0031
We	15.03.95	2.61	-0.0543	Fr	07.04.95	3.21	-0.0213
Th	16.03.95	2.67	0.0230	Mo	10.04.95	3.02	-0.0592
Fr	17.03.95	2.64	-0.0112	Tu	11.04.95	3.08	0.0199
Mo	20.03.95	2.6	-0.0152	We	12.04.95	3.19	0.00357
Tu	21.03.95	2.59	-0.0038	Mo	17.04.95	3.21	0.0063
We	22.03.95	2.59	0.0000	Tu	18.04.95	3.17	-0.0125
Th	23.03.95	2.55	-0.0154	We	19.04.95	3.24	0.0221

We estimate the drift μ and volatility σ using unbiased estimators. In discrete time the rentability of stock S over an time interval (t_{k-1}, t_k) is: $R(t_k) = \dfrac{S(t_k) - S(t_{k-1})}{S(t_{k-1})}$, $k \geq 1$, and in continuous time the rentability stock at time t is: $R(t) = dS(t)/S(t)$. We use a Kolmogorov - Smirnov test to verify if values of R follow a normal probability distribution. We find: $E(R) = 0.01474$; $VAR(R) = 0.0035084$; $\sigma = \sqrt{VAR(R)} = 0.05923196$.

So the rentability folow a normal distribution with the estimated mean $\mu = 0.01474$ expressing the drift and square root variance $\sigma = 0.05923$ expressing the volatility.

Equation (1.1) could be written in a first case:

$$\frac{dS}{S} = \mu dt + \sigma dW(t), \quad S(0) = S_0 \tag{3.3.1}$$

with

$$\begin{aligned} &\mu = E(R) = 0.01474,\ \sigma = \sqrt{VAR(R)} = 0.05923, \\ &S_0 = 2.11,\ t_0 = T = 33,\ W(t) \sim N(0,t). \end{aligned} \tag{3.3.2}$$

Using Ito stochastic integral the solution of (3.1) - (3.2) has the form:

$$S(t) = S_0\, e^{\sigma W(t) + (\mu - \frac{\sigma^2}{2})t}. \tag{3.3.3}$$

In a second case we consider in (3.1) that the drift is a function of t and we estimate the values $\mu_k = \mu(t_k)$, $\forall k \in \{1, \dots, m\}$. In this case the solution of (3.1) is:

$$S(t) = S_0\, e^{\sigma W(t) + \int_0^t (\mu(s) - \frac{\sigma^2(s)}{2}) ds}. \tag{3.3.4}$$

Because of the difficulty to estimate the values $\sigma_k = \sigma(t_k), \forall k \in \{1, \dots, m\}$ we shall consider only the first and second case. But also the difficulty increase when the volatility is considered as random variable.

In order to obtain approximative values of solution (3.3) we must generate the Wiener process. We can do that either using 12 independent uniform random variables $U(0,1)$, using the algorithm RJNORM, using the polar method (from Box - Muller theorem, see [[3]]), or using *randn* MATLAB function for variables $N(0,1)$. The polar method is convenient to generate the Wiener process, as it could be seen in Fig. 1 rentability values are mean of the Wiener process generated values. Then we shall use the average of the generated values of $N(0,1)$ for Wiener process such that finally to find a stable generation process.

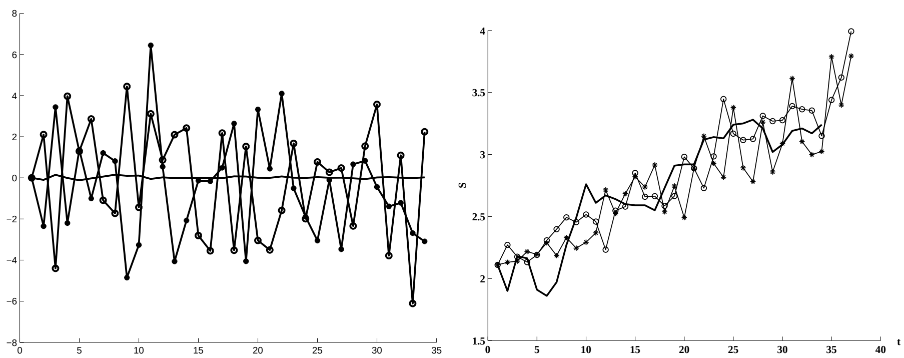

Figure 1: Generation of Wiener process and estimation of stock prices using polar method, $\mu = const$, $\sigma = const$

The calculus for integral $\int_0^t \mu(s)ds$ is made by using Particular case:
i) Simpson composed formula:

$$\int_0^{t_k} \mu(s)ds \cong \frac{1}{3}[\mu_k + 4\sum_{i=1}^{k} \mu_{2i-1} + 2\sum_{i=1}^{k-1} \mu_{2i}], \text{ k even,}$$

$$\int_0^{t_k} \mu(s)ds \cong \frac{1}{3}[\mu_k + 4\sum_{i=1}^{k+1} \mu_{2i-1} + 2\sum_{i=1}^{k} \mu_{2i}], h = 1, \text{ k odd,}$$

ii)Trapezoidal method:

$$\int_0^{t_k} \mu(s)ds \cong \frac{1}{2}[\mu_k + 2\sum_{i=1}^{k-1} \mu_i], h = 1,$$

the obtained values are presented in Fig.2.

In case with a variable drift we consider also for values of $S(t)$ a fit with a five

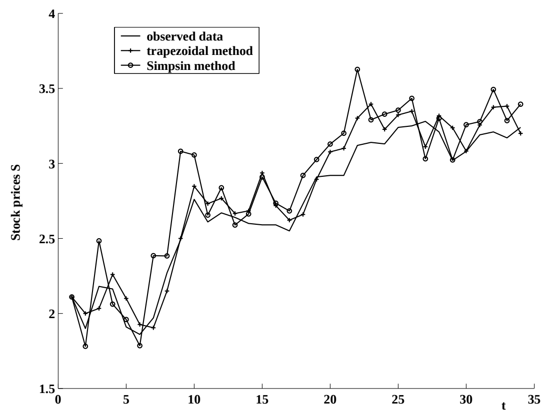

Figure 2: Estimation of stock prices using polar method for Wiener process, for $\mu = \mu(t)$, $\sigma = const$

degrees polynom that has the form:

$$y = -2.3e - 007x^5 + 1.9e - 005x^4 - 0.0006x^3 + 0.0084x^2 - 0.018x + 2.1$$

The approximate and predicted values of stock prices are depicted in Fig. 3.

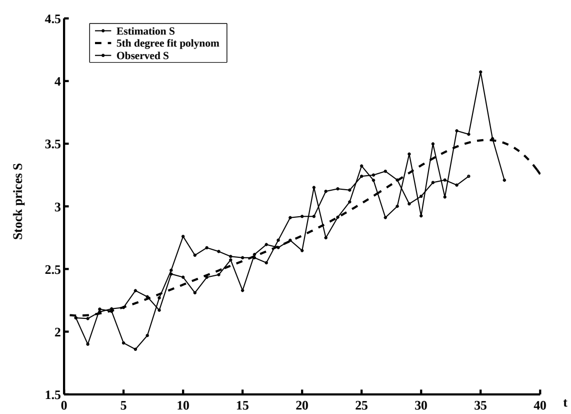

Figure 3: Estimated and predicted values of stock prices

We conclude that with the mean values obtained by the polar method for generating the Wiener process and trapezoidal composed formula for the integral we obtain a good approximation of given data and a local good prediction (for one or two days only).

The goodness of approximation is given by the coefficient

$$R^2 = 1 - \frac{\sum(S_{est} - S_{obs})^2}{\sum(S_{obs} - \overline{S})^2}$$

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For the values of stock prices estimated in case $\mu = const$, $\sigma = const$ in Fig. 1 we find for the coefficient R=0.8722, respectively R=0.8401. In case when $\mu = \mu(t)$, $\sigma = const$ in Fig. 3 we obtain for the coefficient R=0.9540 for the trapezoidal method in order to calculate the integral, respectively R=0.8811 for the case of using Simpson method.

References

[1] L.C. Evans, *Introduction to stochastic differential equations*, U.S.Berkely, 2003.

[2] P. Wilmott, *Derivative. Inginerie financiara. Teorie si practica*, Ed. Economica, Bucuresti, 2002.

[3] I. Vaduva, *Modele de simulare*,Ed. Univ. Buc., Bucharest, 2004.

[4] T. Cocs, A. Murphy, *An introduction to the mathematics of financial derivatives*, (course online), 2000.

[5] T. Bjork, *Arbitrage theory in continuous time*,Oxford University Press, Oxford, 1998.

[6] J. C. Hull, *Options, futures and other derivatives*, Prentice Hall, London, 2000.

[7] T. Mikosch, *Elementary stochastic calculus with finance in view*, World Scientific Publishing, Singapore, 1998.

[8] P. Wilmott, *Derivatives: The theory and practice of financial engineering*, John Wiley and Sons, Chichester, 1998.

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