

State space models for generalized linear control systems

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Abstract

Generalized linear systems with coefficient matrices of bounded variation and controls in the space of regulated function are considered. Controllability and observability of these systems are studied as well as the realizations of their weighting patterns. The connection between controllability, observability and minimality of a realization is emphasized.

Mathematics Subject Classification: 93C25, 93C05, 93B05, 93B07, 93B20.

Key words: functions of bounded variation, regulated functions, linear systems, controllability, observability, weighting patterns.

1 Introduction

The Perron-Stieltjes integral with respect to functions which are not necessarily of bounded variation was defined in [10]. This integral is equivalent to the Kurzweil integral (see [3] and [6]). In this paper, using the results of M.Tvrđý ([8], [9]) concerning the properties of the Perron-Stieltjes integral with respect to regulated functions and the differential equation in this space, a class of generalized linear systems is considered, having the controls in the space of regulated functions and the coefficient matrices of bounded variation. This allows us to extend in this framework the concepts of controllability, observability and weighting patterns (see for instance [1] and [2]). Finally, minimal realizations of given weighting patterns are obtained. Linear boundary value systems were studied in the same framework in [4].

A function $f : [a, b] \rightarrow \mathbf{R}$ which possesses finite one side limits $f(t-)$ and $f(t+)$ for any $t \in [a, b]$ (where by definition $f(a-) = f(a)$ and $f(b+) = f(b)$) is said to be *regulated* on $[a, b]$. The set of all regulated functions denoted by $G(a, b)$, endowed with the supremal norm, is a Banach space; the set $BV(a, b)$ of functions of bounded variation on $[a, b]$ with the norm $\|f\| = |f(a)| + var_a^b f$ is also a Banach space; the Banach space of n -vector valued functions belonging to $G(a, b)$ and $BV(a, b)$ respectively are denoted by $G^n(a, b)$ and $BV^n(a, b)$ (or simply G^n and BV^n); $BV^{n \times m}$ denotes the space of $n \times m$ matrices with entries in $BV(a, b)$. For a matrix function

$U : [a, b] \times [a, b] \rightarrow \mathbf{R}^{n \times m}$, let us denote by $\nu(U)$ its two-dimensional Vitali variation (see [7], Definition I.6.1).

A pair $D = (d, s)$ where $d = \{t_0, t_1, \dots, t_m\}$ is a division of $[a, b]$ (i.e. $a = t_0 < t_1 < \dots < t_m = b$) and $s = \{s_1, \dots, s_m\}$ verifies $t_{j-1} \leq s_j \leq t_j$, $j = 1, \dots, m$ is called a *partition* of $[a, b]$.

A function $\delta : [a, b] \rightarrow (0, +\infty)$ is called a *gauge* on $[a, b]$.

Given a gauge δ , the partition (d, s) is said to be δ -*fine* if

$$[t_{j-1}, t_j] \subset (s_j - \delta(s_j), s_j + \delta(s_j)), \quad j = 1, \dots, m.$$

Given the function $f, g : [a, b] \rightarrow \mathbf{R}$ and a partition $D = (d, s)$ of $[a, b]$ let us associate the integral sum

$$S_D(f\Delta g) = \sum_{j=1}^m f(s_j)(g(t_j) - g(t_{j-1})).$$

Definition 1.1. The number $I \in \mathbf{R}$ is said to be the *Perron-Stieltjes (Kurzweil) integral of f with respect to g from a to b* and it is denoted as $\int_a^b f dg$ or $\int_a^b f(t)dg(t)$ if for any $\varepsilon > 0$ there exists a gauge δ on $[a, b]$ such that

$$|I - S_D(f\Delta g)| < \varepsilon$$

for all δ -fine partitions D of $[a, b]$.

Given $f \in G(a, b)$ and $g \in G([a, b] \times [a, b])$ we define the differences $\Delta^+, \Delta^-, \Delta$ and $\Delta_s^+, \Delta_s^-, \Delta_s$ by $\Delta^+ f(t) = f(t+) - f(t)$, $\Delta^- f(t) = f(t) - f(t-)$, $\Delta f(t) = f(t+) - f(t-)$, $\Delta_s^+ g(t, s) = g(t, s+) - g(t, s)$, $\Delta_s^- g(t, s) = g(t, s) - g(t, s-)$; $\mathbf{D}^-(f)$, $\mathbf{D}^+(f)$ denote respectively the set of the left and right discontinuities of f in $[a, b]$ and similarly with respect to the argument t we can define $\mathbf{D}_t^-(g)$, $\mathbf{D}_t^+(g)$.

Let us recall some basic properties of the Perron-Stieltjes integral, by following [7] and [8]. The existence theorem of the Perron-Stieltjes integral $\int_a^b f dg$ for $f \in BV(a, b)$ and $g \in G(a, b)$, due to Tvrdý [8] is essential for our treatment.

Theorem 1.2. ([7], **Theorem I.4.19** and [8], **Theorem 2.8**). *If $f \in G(a, b)$ and $g \in BV(a, b)$ then the Perron-Stieltjes integrals $\int_a^b f dg$ and $\int_a^b g df$ exist.*

In the sequel we shall denote by \sum_t the sum $\sum_{t \in \mathbf{D}}$ where $\mathbf{D} = \mathbf{D}^-(f) \cup \mathbf{D}^+(f) \cup \mathbf{D}^-(g) \cup \mathbf{D}^+(g)$.

Theorem 1.3 (integration-by-parts, [8], Theorem 2.15). *If $f \in G(a, b)$ and $g \in BV(a, b)$ then*

$$\int_a^b f dg + \int_a^b g df = f(b)g(b) - f(a)g(a) + \sum_t [\Delta^- f(t)\Delta^- g(t) - \Delta^+ f(t)\Delta^+ g(t)]. \quad (1.1)$$

Theorem 1.4 ([8], Proposition 2.16). *If $\int_a^b f dg$ exists, then the function $h(t) = \int_a^t f dg$ is defined on $[a, b]$ and*

i) if $g \in G(a, b)$ then $h \in G(a, b)$ and, for any $t \in [a, b]$

$$\Delta^+ h(t) = f(t)\Delta^+ g(t), \quad \Delta^- h(t) = f(t)\Delta^- g(t) \tag{1.2}$$

ii) if $g \in BV(a, b)$ and f is bounded on $[a, b]$, then $h \in BV(a, b)$.

Theorem 1.5 (substitution, [8], Theorem 2.19). Let f, g, h be such that h is bounded on $[a, b]$ and the integral $\int_a^b f dg$ exists. Then the integral $\int_a^b h(t)f(t)dg(t)$ exists if and only if the integral $\int_a^b h(t)d\left[\int_a^t f(s)dg(s)\right]$ exists, and in this case

$$\int_a^b h(t)f(t)dg(t) = \int_a^b h(t)d\left[\int_a^t f(s)dg(s)\right]. \tag{1.3}$$

Theorem 1.6 (Dirichlet formula, [7], Theorem I.4.32). If $h : [a, b] \times [a, b] \rightarrow \mathbf{R}$ is a bounded function and $\text{var}_a^b h(s, \cdot) + \text{var}_a^b h(\cdot, t) < \infty, \forall t, s \in [a, b]$, then for any $f, g \in BV(a, b)$

$$\begin{aligned} \int_a^b dg(t) \left(\int_a^t h(s, t)df(s) \right) &= \int_a^b \left(\int_s^b dg(t)h(s, t) \right) df(s) + \\ &+ \sum_t [\Delta^- g(t)h(t, t)\Delta^- f(t) - \Delta^+ g(t)h(t, t)\Delta^+ f(t)]. \end{aligned} \tag{1.4}$$

2 Generalized differential equations

The symbol

$$dx = d[A]x + dg \tag{2.1}$$

where $A \in BV^{n \times n}$ and $g \in G^n(a, b)$ is said to be a *generalized linear differential equation (GLDE) in the space of regulated functions*.

Definition 2.1. A function $x : [a, b] \rightarrow \mathbf{R}^n$ is said to be a *solution* of GLDE (2.1) if for any $t, t_0 \in [a, b]$ it verifies the equality

$$x(t) = x(t_0) + \int_{t_0}^t d[A(s)]x(s) + g(t) - g(t_0). \tag{2.2}$$

If x satisfies the initial condition

$$x(t_0) = x_0 \tag{2.3}$$

for given $t_0 \in [a, b]$ and $x_0 \in \mathbf{R}^n$ then x is called the *solution of the initial value problem* (2.1), (2.3).

Theorem 2.2 ([7], Theorem III.2.10). Assume that for any $t \in [a, b]$ the matrix $A \in BV^{n \times n}$ verifies the conditions

$$\det[I + \Delta^+ A(t)] \neq 0 \tag{2.4}$$

and

$$\det[I - \Delta^- A(t)] \neq 0. \quad (2.5)$$

Then there exists a unique matrix valued function $U : [a, b] \times [a, b] \rightarrow \mathbf{R}^{n \times n}$ such that, for any $(t, s) \in [a, b] \times [a, b]$

$$U(t, s) = I + \int_s^t d[A(\tau)]U(\tau, s). \quad (2.6)$$

$U(t, s)$ is called the fundamental matrix solution of the homogeneous equation

$$dx = d[A]x \quad (2.1')$$

and has the following properties, for any $\tau, t, s \in [a, b]$:

$$U(t, s) = U(t, \tau)U(\tau, s); \quad (2.7)$$

$$U(t, t) = I; \quad (2.8)$$

$$U(t+, s) = [I + \Delta^+ A(t)]U(t, s), \quad U(t-, s) = [I - \Delta^- A(t)]U(t, s); \quad (2.9)$$

$$U(t, s+) = U(t, s)[I + \Delta^+ A(s)]^{-1}, \quad U(t, s-) = U(t, s)[I - \Delta^- A(s)]^{-1};$$

$$U(t, s)^{-1} = U(s, t); \quad (2.10)$$

there exists a constant $M > 0$ such that

$$|U(t, s)| + \text{var}_a^b U(t, \cdot) + \text{var}_a^b U(\cdot, s) + \nu(U) < M. \quad (2.11)$$

Some methods for the calculus of the fundamental matrix $U(t, s)$ were provided in [5].

From [7], Theorem III.3.1 and [9], Proposition 2.5, one obtains

Theorem 2.3 (Variation-of-constants formula). *If $A \in BV^{n \times n}$ satisfies the conditions (2.4) and (2.5), then the initial value problem (2.1), (2.3) has a unique solution given by*

$$x(t) = U(t, t_0)X_0 + g(t) - g(t_0) - \int_{t_0}^t d_s[U(t, s)](g(s) - g(t_0)). \quad (2.12)$$

If $g \in G^n$ ($g \in BV^n$) then $x \in G^n$ ($x \in BV^n$).

3 Generalized linear systems

Let us consider the *generalized linear system* (GLS) Σ with the state space representation

$$dx(t) = d[A(t)]x(t) + B(t)du(t), \quad (3.1)$$

$$y(t) = C(t)x(t) + D(t)u(t), \quad t \in [a, b] \quad (3.2)$$

$x \in G^n$, $u \in G^m$, $y \in G^p$ are the state, the input and the output of the system, respectively.

Proposition 3.1. *If*

$$\det[(I - \Delta^- A(t))(I + \Delta^+(t))] \neq 0 \tag{3.3}$$

for any $t \in [a, b]$, then the input-output map of Σ is given by

$$\begin{aligned} y(t) &= C(t)U(t, a)x_0 + \int_a^t C(t)U(t, s)B(s)du(s) + \\ &+ \sum_{a \leq s < t} C(t)U(t, a)\Delta_s^+(U(a, s))B(s)\Delta^+u(s) - \\ &- \sum_{a < s \leq t} C(t)U(t, a)\Delta_s^-(U(a, s))B(s)\Delta^-u(s) + D(t)u(t) \end{aligned} \tag{3.4}$$

where $x_0 = x(a)$ is the initial state of the system.

Proof. The equation (3.1) is of the form (2.2) where $g(t) = \int_a^t B(s)du(s)$. It follows from [7, Corollary I 4.12] and from Theorem 2.2 respectively that

$$\Delta^+g(t) = \lim_{s \downarrow t} \int_a^s B(r)du(r) - \int_a^t B(r)du(r) = B(t)\Delta^+u(t),$$

$$\Delta^-g(t) = B(t)\Delta^-u(t)$$

and

$$\begin{aligned} \Delta_s^+U(t, s) &= U(t, s+) - U(t, s) = U(t, s)((I + \Delta^+A(s))^{-1} - I) = \\ &= U(t, a)U(a, s)((I + \Delta^+A(s))^{-1} - I) = \\ &= U(t, a)(U(a, s+) - U(a, s)) = U(t, a)\Delta_s^+U(a, s); \end{aligned}$$

similarly

$$\Delta_s^-U(t, s) = U(t, a)\Delta_s^-U(a, s).$$

If A satisfies the assumption (3.3), the solution of (3.1) is

$$x(t) = x(a) + \int_a^t B(s)du(s) - \int_a^t d_s[U(t, s)] \int_a^s B(r)du(r).$$

By use of the integration-by-parts formula (1.1) we obtain

$$\begin{aligned} x(t) &= U(t, a)x_0 + \int_a^t U(t, s)d \left[\int_a^s B(r)du(r) \right] + \\ &+ \sum_{a \leq s < t} \Delta_s^+U(t, s) \cdot \Delta^+ \left(\int_a^s B(r)du(r) \right) - \\ &- \sum_{a < s \leq t} \Delta_s^-U(t, s)\Delta^- \left(\int_a^s B(r)du(r) \right) \end{aligned}$$

and, by Theorem 1.5, the state of the system Σ is given by

$$\begin{aligned}
 x(t) &= U(t, a)x_0 + \int_a^t U(t, s)B(s)du(s) + \\
 &+ \sum_{a \leq s < t} U(t, a)\Delta_s^+(U(a, s))B(s)\Delta^+u(s) - \\
 &- \sum_{a < s \leq t} U(t, a)\Delta_s^-(U(a, s))B(s)\Delta^-u(s).
 \end{aligned}
 \tag{3.5}$$

Hence, the input-output map of the GLS (3.1), (3.2) has the form (3.4). □

Remark 3.2. By Theorem 1.4, x and y are regulated vector functions. If $u \in BV^n$, then x is of bounded variation and so is y . Given the matrix $A \in BV^{n \times n}$ we consider the set of *admissible controls*

$$\mathcal{U}(a, b) = \{u \in G^m(a, b) \mid \mathbf{D}^+(A) \cap \mathbf{D}^+(u) = \emptyset, \quad \mathbf{D}^-(A) \cap \mathbf{D}^-(u) = \emptyset\}.$$

Then the formulae (3.5) and (3.4) have a simplified form and we obtain

Corollary 3.3. *If $u \in \mathcal{U}(a, b)$ then the state equation and the input-output map of the system Σ respectively have the form*

$$x(t) = U(t, a)x_0 + \int_a^t U(t, s)B(s)du(s)
 \tag{3.6}$$

and

$$y(t) = C(t)U(t, a)x_0 + \int_a^t C(t)U(t, s)B(s)d(s) + D(t)u(t).
 \tag{3.7}$$

In the sequel we shall denote $\mathcal{U}(a, b)$ by \mathcal{U} if there is no confusion and sometimes we shall consider for simplicity only admissible controls $u \in \mathcal{U}$.

Remark 3.3. Classical dynamical systems with the state representation $x'(t) = \tilde{A}(t)x(t) + B(t)u(t)$ are particular cases of GLS with the absolutely continuous state matrix $A(t) = \int_a^t \tilde{A}(s)ds$ and controls $u(t) = \int_a^t \tilde{u}(s)ds$. In this, case (3.1) becomes the usual state equation $x'(t) = \tilde{A}(t)x(t) + B(t)\tilde{u}(t)$ and $U(t, s) = \Phi_{\tilde{A}}(t, s)$ where $\Phi_{\tilde{A}}(t, s)$ is the fundamental matrix of the system $x'(t) = \tilde{A}(t)x(t)$, and the input-output map (3.7) becomes the usual one.

4 Controllability of Generalized Linear Systems

Let us consider the system

$$dx(t) = d[A(t)]x(t) + B(t)du(t)
 \tag{4.1}$$

where $A \in BV^{n \times n}$, $B \in BV^{n \times m}$ and $\det(I - \Delta^- A(t))(I + \Delta^+(A(t))) \neq 0$, $t \in \mathbf{R}$.

The couple $(x_0, t_0) \in \mathbf{R}^n$ is called a *phase* of the system if x_0 is the state of the system at time t_0 .

Definition 4.1. A phase (x, t) is *G-controllable (controllable)* if there exists some $s, s > t$ and some control $u \in G^m(t, s)$ ($u \in \mathcal{U}(t, x)$) which transfers (x, t) to the phase

$(0, s)$. A phase (x, t) is *G-reachable* (*reachable*) if there exists some $s, s < t$ and some control $u \in G^m(s, t), (u \in \mathcal{U}(s, t))$ which transfers the phase $(0, s)$ to (x, t) .

From (3.5) and (3.6) we obtain:

Proposition 4.2. *A phase (x, t) is G-controllable (controllable) iff (4.2) ((4.3)) holds for some $s > t$ and $u \in G^m(t, s) (u \in \mathcal{U}(t, s))$:*

$$\begin{aligned}
 x &= - \int_t^s U(t, r)B(r)du(r) - \sum_{t \leq r < s} \Delta_r^+(U(t, r))B(r)\Delta^+u(r) + \\
 &+ \sum_{t < r \leq s} \Delta_r^-(U(t, r))B(r)\Delta^-u(r);
 \end{aligned}
 \tag{4.2}$$

$$x = - \int_t^s U(t, r)B(r)du(r).
 \tag{4.3}$$

Proof. We replace in (3.5) ((3.6)) $x(t)$ by 0, x_0 by x , t by s , a by t , s by r , we premultiply the obtained relation by $U(t, s)$ and (4.2) ((4.3)) results by using (2.7) and (2.10). □

In a similar way we can prove

Proposition 4.3. *A phase (x, t) is G-reachable (reachable) iff (4.4) ((4.5)) holds for some $s < t$ and some control $u \in G^m(s, t) (u \in \mathcal{U}(s, t))$:*

$$\begin{aligned}
 x &= \int_s^t U(t, r)B(r)du(r) + \sum_{s \leq r < t} U(t, s)\Delta_r^+(U(s, r))B(r)\Delta^+u(r) - \\
 &- \sum_{s < r \leq t} U(t, s)\Delta_r^-(U(s, r))B(r)\Delta^-u(r);
 \end{aligned}
 \tag{4.4}$$

$$x = \int_s^t U(t, r)B(r)du(r).
 \tag{4.5}$$

Now let us consider the symmetric, non-negative matrices

$$\mathcal{C}(t, s) = \int_t^s U(t, r)B(r)B(r)^T U(t, r)^T dr, \quad t < s
 \tag{4.6}$$

$$\mathcal{A}(s, t) = \int_s^t U(t, r)B(r)B(r)^T U(t, r)^T dr, \quad s < t
 \tag{4.7}$$

called the *controllable Gramian* and *reachable Gramian* of Σ respectively.

We denote by $\mathcal{R}(M)$ and $\mathcal{N}(M)$ the range and the kernel of a linear operator M .

Theorem 4.4. *It is possible to transfer the phase (x_0, t_0) to (x_1, t_1) iff the vector $U(t_0, t_1)x_1 - x_0$ belongs to $\mathcal{R}(\mathcal{C}(t_0, t_1))$.*

Proof. Sufficiency. By hypothesis, $\exists v \in X = \mathbf{R}^n$ such that $U(t_0, t_1)x_1 - x_0 = \mathcal{C}(t_0, t_1)v$ hence by using (2.10) $U(t_0, t_1)^{-1} = U(t_0, t_1)$ we get

$$\begin{aligned}
x_1 &= U(t_0, t_1)x_0 + U(t_1, t_0)\mathcal{C}(t_0, t_1)v = \\
&= U(t_1, t_0)x_0 + \int_{t_0}^{t_1} U(t_0, s)B(s)B(s)^T U(t_0, s)^T v ds;
\end{aligned}$$

therefore, if we consider the control $u(t) = \int_{t_0}^t B(s)^T U(t_0, s)^T v ds$ we have $u \in \mathcal{U}(t_0, t_1)$ and by using (3.6) one obtains

$$x(t_1) = U(t_1, t_0)x_0 + \int_{t_0}^{t_1} U(t_1, s)B(s)du(s) = x_1.$$

Necessity. Let us suppose that it is possible to transfer the phase (x_0, t_0) to (x_1, t_1) ; again by (3.6) this is equivalent to

$$0 = U(t_1, t_0)(U(t_0, t_1)x_1 - x_0) + \int_{t_0}^{t_1} U(t_1, s)B(s)du(s),$$

for some $u \in \mathcal{U}(t_0, t_1)$ hence it is possible to transfer (x_2, t_0) (where $x_2 \stackrel{\text{def}}{=} U(t_0, t_1)x_1 - x_0$) to $(0, t_1)$ using the control $u_1 = -u$.

Since $\mathcal{C}(t_0, t_1)$ is symmetrical, we have the following orthogonal direct-sum decomposition of the state space: $X = \mathcal{R} \otimes \mathcal{N}$ where $\mathcal{R} = \mathcal{R}(\mathcal{C}(t_0, t_1))$ and $\mathcal{N} = \mathcal{N}(\mathcal{C}(t_0, t_1))$; then $x_2 = x_3 + x_4$, where $x_3 \in \mathcal{R}$ and $x_4 \in \mathcal{N}$.

From sufficiency it results (with $0, x_3$ instead of x_1, x_0) that (x_3, t_0) can be transferred to $(0, t_1)$ by a control u_3 . It results by linearity that it is possible to transfer (x_4, t_0) to $(0, t_1)$ using the control $u_4 = u_1 - u_3$; indeed, by (3.6) we have

$$\begin{aligned}
x(t_1) &= U(t_1, t_0)x_4 + \int_{t_0}^{t_1} U(t_1, s)B(s)d(u_1(s) - u_2(s)) = \\
&= U(t_1, t_0)x_2 + \int_{t_0}^{t_1} U(t_1, s)B(s)du_1(s) - \\
&\quad - \left(U(t_1, t_0)x_3 + \int_{t_0}^{t_1} U(t_1, s)B(s)du_3(s) \right) = 0
\end{aligned}$$

hence

$$x_4 = - \int_{t_0}^{t_1} U(t_0, s)B(s)du_4(s).$$

Since $x_4 \in \mathcal{N}$ we have that

$$\begin{aligned}
0 &= x_4^T \mathcal{C}(t_0, t_1)x_4 = \int_{t_0}^{t_1} x_4^T U(t_0, s)B(s)B(s)^T U(t_0, s)^T x_4 ds = \\
&= \int_{t_0}^{t_1} \|B(s)^T U(t_0, s)^T x_4\|^2 ds;
\end{aligned}$$

the integrand is non-negative, hence $B(s)^T U(t_0, s)^T x_4 = 0$ almost everywhere on $[t_0, t_1]$. It implies

$$\|x_4\|^2 = x_4^T x_4 = - \int_{t_0}^{t_1} u_4(s)^T B(s)^T U(t_0, s)^T x_4 ds = 0$$

(no state $0 \leq x_4 \in \mathcal{N}$ is controllable), hence $x_4 = 0$ and $x_2 = x_3 \in \mathcal{R}$. □

If we replace in Theorem 4.4 (x_0, x_1) by $(0, x)$ and $(x, 0)$ respectively, we obtain:

Corollary 4.5. *A phase (x, t) is reachable iff there exists some moment $t_0 \leq t$ such that*

$$x \in U(t, t_0)\mathcal{R}(\mathcal{C}(t_0, t)).$$

A phase (x, t_0) is controllable iff there exists $t_f \geq t_0$ such that $x \in \mathcal{R}(\mathcal{C}(t_0, t_f))$. In a similar manner we can prove

Theorem 4.6. *A phase (x, t) is reachable iff $x \in \mathcal{R}(\mathcal{A}(t_0, t))$ for some $t_0 < t$.*

Corollary 4.7. *If $t_1(t_0)$ is any value of t_1 for which $\mathcal{C}(t_0, t_1)$ has maximal rank, then the set $X_c(t_0)$ of states which are controllable at time t_0 is $X_c(t_0) = \mathcal{R}(\mathcal{C}(t_0, t_1(t_0)))$.*

Proof. Clearly,

$$X_c(t_0) = \bigcup_{t \geq t_0} \mathcal{R}(\mathcal{C}(t_0, t)).$$

If $\mathcal{C}(t_0, t_1(t_0))$ has the maximal rank then $\mathcal{R}(\mathcal{C}(t_0, t) = \mathcal{R}(\mathcal{C}(t_0, t_1(t_0)))$ for any $t \geq t_0$, hence $X_c(t_0) = \mathcal{R}(\mathcal{C}(t_0, t_1(t_0)))$. □

Let $t_0, t, t_f \in \mathbf{R}$ be some fixed moments $t_0 < t < t_f$.

Definition 4.8. The system Σ is said to be *completely controllable* on $[t, t_f]$ (*completely reachable* on $[t_0, t]$) if every phase (x, t) is controllable (reachable) during the period $[t, t_f]$ ($[t_0, t]$).

Theorem 4.9. *Σ is completely controllable (completely reachable) on $[a, b]$ iff $\text{rank}\mathcal{C}(a, b) = n$ (i.e. the matrix $\mathcal{C}(a, b)$ is positive definite).*

Proof. The above condition is equivalent to $X = \mathcal{R}(\mathcal{C}(a, b))$, hence any phase $(x, a) \in \mathbf{R} \times X$ (where $X = \mathbf{R}^n$) is controllable (on $[a, b]$), i.e. Σ is controllable on $[a, b]$. Since $U(b, a)$ is nonsingular, it results that $U(b, a)\mathcal{R}(\mathcal{C}(a, b)) = U(b, a)X = X$; therefore any state x gives a reachable phase (x, b) and Σ is completely reachable on $[a, b]$. □

Consider the general case, when the maximal rank is $\text{rank}\mathcal{C}(a, t_1(a)) = r < n$. We consider the orthogonal direct-sum decomposition of X defined by $X = X_1(t) \otimes X_2(t)$, where $X_1(t) = U(t, a)\mathcal{R}(\mathcal{C}(a, t_1(a)))$ and $X_2(t) = U^T(a, t)\mathcal{N}(\mathcal{C}(a, t_1(a)))$ (using the notation $(M\mathcal{V} = \{Mv | v \in \mathcal{V}\})$ for any matrix M and linear space \mathcal{V}). Let us denote $\mathcal{C} = \mathcal{C}(a, t_1(a))$. Since $U(t, a)$ is nonsingular for any t, t_0 , it results that

$$\dim X_1(t) = \dim \mathcal{R}(\mathcal{C}(a, t_1(a))) = \text{rank}\mathcal{C}(a, t_1(a)) = r$$

$$\dim X_2(t) = \dim \mathcal{N}(\mathcal{C}(a, t_1(a))) = n - r,$$

hence

$$\dim X_1(t) + \dim X_2(t) = n = \dim X.$$

Let x_1, x_2 be such that $x_1 \in X_1(t)$, $x_2 \in X_2(t)$, that is $x_1 = U(t, a)\mathcal{C}\bar{x}$, $x_2 = U^T(a, t)\hat{x}$ for some \bar{x} and \hat{x} with $\bar{x} \in X$ and $\mathcal{C}\hat{x} = 0$. Then $x_1^T x_2 = \bar{x}^T \mathcal{C}^T U^T(t, a) U^T(a, t) \hat{x} = \bar{x}^T \mathcal{C} \hat{x} = 0$, $(x_1 \perp x_2)$, hence $X_1(t) \perp X_2(t)$ and $X = X_1(t) \oplus X_2(t)$. Therefore, any $x \in X$ can be uniquely written as $x = x_1(t) + x_2(t)$ with $x_1(t) \in X_1(t)$, $x_2(t) \in X_2(t)$, or $x = \begin{bmatrix} \tilde{x}_1(t) \\ \tilde{x}_2(t) \end{bmatrix}$ and $X_1(t) = \left\{ \begin{bmatrix} \tilde{x}_1(t) \\ 0 \end{bmatrix} \middle| \tilde{x}_1 \in \mathbf{R}^r \right\}$, $X_2(t) = \left\{ \begin{bmatrix} 0 \\ \tilde{x}_2(t) \end{bmatrix} \middle| \tilde{x}_2 \in \mathbf{R}^{n-r} \right\}$ in a basis $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2$, where $\mathcal{B}_1, \mathcal{B}_2$ are two bases in $X_1(t), X_2(t)$.

Theorem 4.10. *In terms of the above direct-sum decomposition, the fundamental equation of the linear system Σ have the form*

$$\begin{aligned} dx_1(t) &= d[\tilde{A}_{11}(t)]x_1(t) + d[\tilde{A}_{12}(t)]x_2(t) + \tilde{B}_1(t)du(t) \\ dx_2(t) &= d[\tilde{A}_{22}(t)]x_2(t) \end{aligned} \quad (4.8)$$

Proof. If $x_1(t) \in X_1(t)$, then $x_1(t) \in U(t, t_0)\mathcal{C}\bar{x}$ for some $\bar{x} \in X$; then, by Theorem 2.2

$$d[A(t)]x_1(t) = d[A(t)]U(t, a)\mathcal{C}\bar{x} = d[U(t, a)\mathcal{C}\bar{x}] = d[x_1(t)].$$

In a basis of the above direct-sum decomposition it results (for the block representation of A as $\tilde{A} = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{bmatrix}$)

$$d[\tilde{A}(t)] \begin{bmatrix} \tilde{x}_1(t) \\ 0 \end{bmatrix} = d \left[\begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{bmatrix} \right] \cdot \begin{bmatrix} \tilde{x}_1(t) \\ 0 \end{bmatrix} = \begin{bmatrix} d\tilde{x}_1(t) \\ 0 \end{bmatrix}, \quad \text{hence } \tilde{A}_{21} = 0.$$

If $x_2 \in X_2(t)$, there exists $\hat{x} \in \mathcal{N}(\mathcal{C})$ such that $x_2 = U^T(a, t)\hat{x}$. It was proved in Theorem 4.4 (with x_4 instead of \hat{x}) that $B^T(t)U^T(a, t)\hat{x} = 0$ a.e. hence $B^T(t)x_2 = 0$ for any $x_2 \in X_2(t)$. It results that any column of $B(t)$ belongs to $X_1(t)$, hence the matrix $B(t)$ in the above direct-sum decomposition has the form $\tilde{B}(t) = \begin{bmatrix} \tilde{B}_1(t) \\ 0 \end{bmatrix}$. \square

By Corollary 4.5, $X_1(t)$ is the linear space of reachable states of X .

5 Minimum energy transfer

We can prove some results which are similar to those given in Section 4 by replacing the controllability Gramian $\mathcal{C}(a, b)$ by the reachability Gramian $\mathcal{A}(a, b)$, for instance:

Theorem 5.1. *The system (4.1) is completely reachable on $[a, b]$ iff the matrix $\mathcal{A}(a, b)$ is positive definite.*

Proof. Sufficiency: if $\mathcal{A} = \mathcal{A}(a, b) > 0$ then its inverse \mathcal{A}^{-1} exists the control

$$\tilde{u}(t) = \int_a^t B(t)^T U(b, t)^T \mathcal{A}^{-1}(x_1 - U(b, a)x_0) dt \quad (5.1)$$

transfers the phase (x_0, a) to (x_1, b) ; indeed, by (3.6) and (1.3) the state provided by the control \tilde{u} is

$$x(b) = U(b, a)x_0 + \int_a^b U(b, t)B(t)B(t)^T U(b, t)^T dt \cdot \mathcal{A}^{-1}(x_1 - U(b, a)x_0) = x_1.$$

Necessity: Let us suppose that \mathcal{A} is not positive definite, hence there exists a vector $x \in \mathbf{R}^n$, $x \neq 0$, such that $x^T \mathcal{A} x = 0$; since $U(b, s)B(s) \in BV^{n \times m}$, from $\int_a^b x^T U(b, s)B(s)B(s)^T U(b, s)^T x ds = 0$, it results that

$$x^T U(b, s)B(s) = 0 \quad \text{a.e. in } [a, b]. \tag{5.2}$$

If the phase $(0, a)$ can be transferred to (x, b) , then from $x = \int_a^b U(b, s)B(s)du(s)$ we obtain by use of (5.2) $0 < x^T x = \int_a^b x^T U(b, s)B(s)du(s) = 0$, contradiction. □

Definition 5.2. For any $u \in \mathcal{U}(a, b)$ the number $E_u = \int_a^b u(t)^T du(t)$ is said to be the energy associated to the control u .

Theorem 5.3. If the system Σ is completely reachable on $[a, b]$ then the control \tilde{u} given by (5.1) transfers the phase (x_0, t_0) to (x_1, t_1) with minimum expenditure of energy, i.e. $\int_a^b \tilde{u}(t)^T d\tilde{u}(t) \leq \int_a^b u(t)^T du(t)$ for all $u \in \mathcal{U}(a, b)$ which transfers x_0 to x_1 (\tilde{u} is the "minimum energy" control).

Proof. If u is a control which transfers x_0 to x_1 we denote by u_1 the control $u_1 = u - \tilde{u}$; then we have

$$\begin{aligned} x_1 &= U(b, a)x_0 + \int_a^b U(b, t)B(t)d\tilde{u}(t) = U(b, a)x_0 + \int_a^b U(b, t)B(t)du(t) = \\ &= U(b, a)x_0 + \int_a^b U(b, t)B(t)d\tilde{u}(t) + \int_a^b U(b, t)B(t)du_1(t), \end{aligned}$$

hence

$$\int_a^b U(b, t)B(t)du_1(t) = 0;$$

it follows that

$$\int_a^b \tilde{u}(t)^T du_1(t) = \int_a^b x_0^T (x_1^T - x_0^T U(b, a)^T) \mathcal{A}^{-1} U(b, t)B(t)du_1(t) = 0.$$

Consequently

$$\begin{aligned} E_u &= \int_a^b u(t)^T du(t) = \int_a^b (\tilde{u}(t)^T + u_1(t)^T)d(\tilde{u}(t) + u_1(t)) = \\ &= \int_a^b \tilde{u}(t)^T d\tilde{u}(t) + \int_a^b u_1(t)^T du_1(t) + 2 \int_a^b \tilde{u}(t)^T du_1(t) \geq \\ &\geq \int_a^b \tilde{u}(t)^T d\tilde{u}(t) = E_{\tilde{u}} \end{aligned}$$

6 Observability

Let us consider the GLS Σ with the state representation (3.1)(3.2).

Definition 6.1. The system Σ is *completely observable* over the time interval $[a, b]$ if for any pair of distinct initial states $x_{01}, x_{02} \in \mathbf{R}^n$ and for any input $u \in \mathcal{U}[a, b]$, the outputs $y_1(t)$ and $y_2(t)$ are distinct, that is $y_1(t) \neq y_2(t)$ for $t \in [a, b]$.

Theorem 6.2. *The system Σ is completely observable over $[a, b]$ iff the observability Gramian*

$$\mathcal{O} = \mathcal{O}(a, b) = \int_a^b U(t, a)^T C(t)^T C(t) U(t, a) dt \quad (6.1)$$

is positive definite.

Proof. By (3.7) for $i = 1, 2$,

$$y_i = C(t)U(t, a)x_{0i} + \int_a^t C(t)U(t, s)B(s)du(s) + D(t)u(t).$$

If Σ is completely observable, then for $x_{01} \neq x_{02}$

$$\begin{aligned} 0 &< \int_a^b |y_1(t) - y_2(t)|^2 dt = \int_a^b (y_1^T(t) - y_2^T(t))(y_1(t) - y_2(t)) dt = \\ &= (x_{01}^T - x_{02}^T) \left(\int_a^b U(t, a)^T C(t)^T C(t) U(t, a) dt \right) (x_{01} - x_{02}), \end{aligned}$$

hence $x^T \mathcal{O} x > 0$ for any $x \in \mathbf{R}^n$, $x \neq 0$.

Conversely, if $\mathcal{O} > 0$ and the system Σ is not completely controllable, then there exist $x_{01} \neq x_{02}$ such that $y_1(t) \equiv y_2(t)$, hence $C(t)U(t, a)(x_{01} - x_{02}) = 0$ for any $t \in [a, b]$. It follows that

$$(x_{01} - x_{02})^T \left(\int_a^b U(t, a)^T C(t)^T C(t) U(t, a) dt \right) (x_{01} - x_{02}) = 0$$

this contradicts the hypothesis that $\mathcal{O} > 0$. □

Corollary 6.3. *The following statements are equivalent:*

- i) *The system Σ is completely observable over $[a, b]$.*
- ii) $\text{rank} \mathcal{O}(a, b) = n$.
- iii) $\mathcal{O}(a, b) > 0$.
- iv) $\text{Ker} \{C(t)U(t, a), t \in [a, b]\} = \{0\}$.

Canonical forms from the point of view of the concept of observability, similar to (4.8) as well as the reachability and observability together can be derived.

7 Weighting patterns

As it was noticed in [1], in the design of systems for control or communication purposes it is much natural to specify a weighting pattern that it is to specify the complete systems Σ . We consider the system $\Sigma = (A_1(t), B(t), C(t))$ (hence $D(t) = 0$).

Definition 7.1. The $p \times m$ matrix W defined by

$$W = W(t, s) = C(t)U(t, s)B(s) \quad (7.1)$$

is called the *weighting pattern* of Σ .

From the input-output map formula (3.7), we obtain for $x_0 = 0$ the general response of Σ :

$$y(t) = \int_a^t C(t)U(t, s)B(s)du(s). \quad (7.2)$$

It can be expressed by the means of the weighting patterns as

$$y(t) = \int_{t_0}^t W(t, s)du(s) \quad (7.3)$$

hence, for zero initial condition, the weighting pattern completely determines the input-output behaviour of the system.

Definition 7.2. A $p \times m$ matrix W defined on $\mathbf{R} \times \mathbf{R}$ is said to be *realizable as a weighting pattern* if there exists a system Σ such that (7.1) holds. Σ is called a *realization* of the weighting pattern W .

If $\dim \Sigma \leq \dim \tilde{\Sigma}$ for any realization $\tilde{\Sigma}$ of W then Σ is said to be a *minimal realization* of W .

Theorem 7.3. A matrix W is realizable as a weighting pattern iff there exist two matrix functions G and H defined on \mathbf{R} such that, for any $t, s \in \mathbf{R}$ W has the factorization

$$W(t, s) = G(t)H(s) \quad (7.4)$$

Proof. *Sufficiency.* If there exist G and H , let us consider the system $\Sigma^0 = (0, H, G)$ (i.e. $A_1 = 0$, $B = H$, $C = G$). Then by (2.6) $U(t, s) = I$ and the equality (7.1) follows from (7.4)

Necessity. If $\Sigma = (A(t), B(t), C(t))$ is a realization of W then, for fixed $s_0 \in \mathbf{R}$, W can be written as

$$W(t, s) = C(t)U(t, t_0)U(t_0, s)B(s),$$

hence (7.4) holds with $G(t) = C(t)U(t, t_0)$ and $H(s) = U(t_0, s)B(s)$. \square

Definition 7.4. The weighting pattern W factorized as in (7.4) is said to be in *reduced form* if the columns of G and the rows of H are both linearly independent sets of functions on \mathbf{R} . Otherwise W is called *reducible*.

If W is in reduced form, the number of columns of G (equal to the number of rows of H) is called the *order* of W .

Proposition 7.5. If W is in reduced form and it has the order n , then $n = \text{rank } W(t, s)$.

Proof. By Sylvester's Inequality we get from (7.4):

$$\text{rank } G(t) + \text{rank } H(s) - n \leq \text{rank } W(t, s) \leq \min(\text{rank } G(t), \text{rank } H(s)).$$

Since $\text{rank } G(t) = \text{rank } H(s) = n$ it results $n \leq \text{rank } W(t, s) \leq n$. \square

Proposition 7.6. If the weighting pattern W is in reduced form then the dimension of any minimal realization is the same as the order of W .

Proof. Let n be the order of W and let us assume that W has a realization Σ with $\dim \Sigma = \tilde{n} < n$. Then W has the factorization determined in Theorem 7.3 in which the number of columns of $G(t)$ and the number of rows of $H(s)$ is at most \tilde{n} ; this contradicts the assumption that W has the order n . □

Theorem 7.7. A weighting pattern W always admits a minimal realization.

Proof. Let $W = GH$ be a factorization of W . By elementary row operations we can transform the matrix H into a matrix $\tilde{H} = MH = \begin{bmatrix} H_1 \\ 0 \end{bmatrix}$ where the rows of H_1 are linearly independent and M is a unimodular matrix. Let us partition the matrix $\tilde{G} = GM^{-1}$ as $\tilde{G} = \begin{bmatrix} G_1 & G_{12} \end{bmatrix}$ correspondingly. Then we have $W = GH = \tilde{G}\tilde{H} = G_1H_1$. Now G_1 can be transformed by elementary column operations into a matrix $\tilde{G}_1 = G_1P = \begin{bmatrix} G_2 & 0 \end{bmatrix}$ where the columns of G_2 are linearly independent and P is unimodular. Let us partition the matrix $\tilde{H}^1 = P^{-1}H_1$ as $\tilde{H}_1 = \begin{bmatrix} H_2 \\ H_{21} \end{bmatrix}$. We obtain $W = G_1H_1 = \tilde{G}_1\tilde{H}_1 = G_2H_2$, hence $W = G_2H_2$ is now in reduced form. By Theorem 7.3 and Proposition 7.6 it results that W has the minimal realization $\Sigma^0 = (0, H_2, G_2)$. □

The following theorem emphasizes the connection between controllability, observability and minimality. This result is similar to that concerning "classical" systems, see [1, §15, Theorem 2] and the proof is omitted.

Theorem 7.8. The system $\Sigma = (A_1(t), B(t), C(t))$ is a minimal realization of the weighting pattern $W(t, s)$ iff Σ is both completely controllable and completely observable on some interval $[t_0, t_1]$.

Conclusion. State space representation studied in this paper, with coefficient matrices of bounded variation and controls over spaces of regulated functions seems to be the most general framework in which linear control systems can be studied.

Basic concepts as controllability, observability and minimum energy control were studied in this framework, as well as the representation of a systems by means of a weighting pattern. Some realizations of a weighting pattern were provided and the connection between controllability, observability and minimality of a realization was emphasized.

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